On Cordial Labelings of Wheels with Other Graphs

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Abstract. A graph is said to be cordial if it has a 0-1 labeling that satisfies certain properties. A wheel W_n is the graph obtained from the join of the cycle C_n $(n \geq 3)$ and the null graph N_1. In this paper we investigate the cordiality of the join and the union of pairs of wheels and graphs consisting of a wheel and a path or a cycle.

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1 Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology communication, psychology, sociology, economics, engineering, operations research, and especially computer science.

One area of graph theory of considerable recent research is that of graph labeling. In a labeling of a particular type, the vertices are assigned values from a given set, the edges have a prescribed induced labeling, and the labelings must satisfy certain properties. An excellent reference on this subject is the survey by Gallian [6].

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [9] in 1966 and Golomb[7] in 1972, while harmonious labelings were first studied by Graham and Sloane [8] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [1] in 1990. Whereas the label of an edge vw for graceful and harmonious labeling is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced edge label $(f(v) + f(w))$ (mod2), which of course equals $|f(v) - f(w)|$. Because arithmetic modulo2 is an integral part of computer science, cordial labelings have close connections with that field.

More precisely, cordial graphs are defined as follows.

Let $G = (V, E)$ be a graph, let $f : V \to \{0, 1\}$ be a labeling of its vertices, and let $f^* : E \to \{0, 1\}$ is the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w)$ (mod 2). (Thus, for any edge $e, f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edge. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called cordial if it has a cordial labeling.

A wheel W_n is the graph obtained from the join of the cycle C_n $(n \geq 3)$ and the null graph N_1. So the order of the wheel W_n is n+1 and its size is 2n for all $n \geq 3$, in particular W_3=K_4, where K_4 is a complete graph of order 4.

As stated in [1], a wheel W_n is cordial if and only if n is not congruent to 3(mod 4). In this paper we extend this result to investigate the cordiality of the join and the union of pairs of wheels and graphs consisting of a wheel and a path or a cycle. In section 3, we show that the join $W_n + W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) (n, m) = (3,3),
(ii) $n = 3$ and $m \equiv 1$(mod 4(or vice versa),
(iii) $n \equiv 1$(mod 4) and $m \equiv 3$(mod 4)(or vice versa).
Also, we prove that the union $W_n \cup W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) $n = 3$ and $m \equiv 1$(mod 4(or vice versa),
(ii) $n \equiv 1$(mod 4) and $m \equiv 3$(mod 4)(or vice versa).
In section 4, we show that the join $W_n + P_m$ of wheels W_n and paths P_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) (n, m) = (3,1),(3,2) and (3,3),
(ii) $n \equiv 3$(mod 4) and m = 1.
Also, we show that the union $W_n \cup P_m$ of wheels W_n and paths P_m is cordial for all n and all m. In section 5, we show that the join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all n and all m if and only if $(m, n) \neq$ (3,3) and (3,4). Also, we prove that the union $W_n \cup C_m$ of wheels W_n and cycles C_m is cordial for all n and all m.

2 Terminology and notations

We introduce some notation and terminology for a graph with $4r$ vertices [2,3,4,5] , we let L_{4r} denote the labeling 00110011...0011, S_{4r} denote the labeling 11001100...1100 and O_r denotes the labeling 0000...0000 (zero repeated r- times) , I_r denotes the labeling 111...1111 (one repeated r-times), M_r denote the labeling 0101...01 (zero-one repeated r-times) if r is even and 0101 ... 010 (zero-one repeated r-times) if r is odd and M'_r denote the labeling 1010...010 (zero-one repeated r-times) if r is even and 10101 ... 101 (zero-one repeated r-times) if r is odd. In most cases, we then modify this by adding symbols at one end or the other (or both). Thus $01L_{4r}$ denotes the labeling 0100110011...0011 of either W_{4r+2}, C_{4r+2} or P_{4r+2}(It should be to remark that for the labeling of the wheel W_{4r+2}, we label the center of the wheel W_{4r+2} by the first label which is 0 in $01L_{4r}$ and other labelings for the vertices of C_{4r+1} which are $1L_{4r}$).

One exception to this is the labeling L'_{4r} obtained from L_{4r} by adding an initial 0 and deleting the last 1: that is, L'_{4r} is 000110011...11001. For specific labeling L and M of $G \cup H$, where G and H are paths or cycles or stars or null graphs, we let $[L; M]$ denote the joint labeling.

Additional notation that we use is the following.
For a given labeling of the join $G + H$, we let v_i and e_i (for i = 0,1) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0, v_1 = x_1 + y_1, e_0 = a_0 + b_0 + x_0y_0 + x_1y_1$ and $e_1 = a_1 + b_1 + x_0y_1 + x_1y_0$, thus, $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3 Joins of Pairs of Wheels

As stated in [1], a wheel W_n is cordial if and only if n is not congruent to 3(mod 4). In this section, we extend this result to show that the join $W_n + W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) (n, m) = (3,3),
(ii) $n = 3$ and $m \equiv 1$(mod 4(or vice versa),
(iii) $n \equiv 1$(mod 4) and $m \equiv 3$(mod 4)(or vice versa).
Also, we prove that the union $W_n \cup W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) $n = 3$ and $m \equiv 1$(mod 4(or vice versa),
(ii) $n \equiv 1$(mod 4) and $m \equiv 3$(mod 4)(or vice versa).
Lemma 3.1. The join $W_n + W_m$ of two wheels W_n and W_m is cordial for all $n > 3$ and all $m > 3$ except for $n \equiv 1$(mod 4) and $m \equiv 3$(mod 4)(or vice versa).
Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i or A''_i for the wheel W_n and B_j or B'_j or B''_j for the wheel W_m as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$,

we can compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of W_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r+1$	$4r$	$4r$
	$A_0' = 0L_{4r}$	$2r+1$	$2r$	$4r$	$4r$
	$A"_0 = 1L'_{4r}$	$2r+1$	$2r$	$4r-1$	$4r+1$
$i = 1$	$A_1 = 01L_{4r}$	$2r+1$	$2r+1$	$4r+1$	$4r+1$
	$A_1' = 11L_{4r}$	$2r$	$2r+2$	$4r+2$	$4r$
$i = 2$	$A_2 = 011L_{4r}$	$2r+1$	$2r+2$	$4r+2$	$4r+2$
	$A_2' = 001L_{4r}$	$2r+2$	$2r+1$	$4r+1$	$4r+3$
	$A"_2 = 110L_{4r}$	$2r+1$	$2r+2$	$4r+3$	$4r+1$
$i = 3$	$A_3 = 0011L_{4r}$	$2r+2$	$2r+2$	$4r+2$	$4r+4$
	$A_3' = 1100L_{4r}$	$2r+2$	$2r+2$	$4r+4$	$4r+2$
	$A"_3 = 1011L_{4r}$	$2r+1$	$2r+3$	$4r+3$	$4r+3$

$m = 4s + j$, $j = 0, 1, 2, 3$	Labeling of W_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = 1L_{4s}$	$2s$	$2s+1$	$4s$	$4s$
	$B_0' = 0L_{4s}$	$2s+1$	$2s$	$4s$	$4s$
	$B"_0 = 1L'_{4s}$	$2s+1$	$2s$	$4s-1$	$4s+1$
$j = 1$	$B_1 = 01L_{4s}$	$2s+1$	$2s+1$	$4s+1$	$4s+1$
$j = 2$	$B_2 = 011L_{4s}$	$2s+1$	$2s+2$	$4s+2$	$4s+2$
	$B_2' = 110L_{4s}$	$2s+1$	$2s+2$	$4s+3$	$4s+1$
$j = 3$	$B_3 = 0011L_{4s}$	$2s+2$	$2s+2$	$4s+2$	$4s+4$
	$B_3' = 1100L_{4s}$	$2s+2$	$2s+2$	$4s+4$	$4s+2$

Table 3.1. Labelings of Wheels.

$n=4r+i$, $i=0,1,2,3$	$m=4s+j$, $j=0,1,2,3$	W_n	W_m	v_0-v_1	e_0-e_1
0	0	A_0	B'_0	0	-1
0	1	A_0	B_1	-1	0
0	2	A'_0	B'_2	0	1
0	3	$A"_0$	B'_3	1	0
1	0	A_1	B_0	-1	0
1	1	A_1	B_1	0	0
1	2	A_1	B_2	-1	0
2	0	$A"_2$	B'_0	0	1
2	1	A_2	B_1	-1	0
2	2	A'_2	B'_2	0	-1
2	3	$A"_2$	B_3	-1	0
3	0	A'_3	$B"_0$	1	0
3	2	A_3	B'_2	-1	0
3	3	A'_3	B_3	0	0

Tabel 3.2. Combinations of labelings.

Lemma 3.2. If $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$(or vice versa), then the join $W_n + W_m$ of two wheels W_n and W_m is not cordial.
Proof. The proof follows directly from the following theorem [6], which states that if G is a graph with n vertices and m edges, and every vertex has odd degree, then G is not cordial when $n+m \equiv 2 \pmod 4$, the lemma follows.
Example 3.1. The graph $W_3 + W_3$ is not cordial.
Solution. The solution follows directly from the fact that $W_3 + W_3 =$ $K_4 + K_4 \equiv K_8$ and the complete graph K_8 is not cordial [1].
Lemma 3.3. The join $W_3 + W_m$ is cordial for all $m > 3$ except for $m \equiv 1(\text{mod } 4)$.
Proof. The following labelings is suffice:
$W_3 + W_{4s}$: [0001;1L_{4s}], $W_3 + W_{4s+2}$: [0001;011L_{4s}] and $W_3 + W_{4s+3}$: [0011;1100L_{4s}],the lemma follows.
Lemma 3.4. If $m \equiv 1(\text{mod } 4)$, then the join $W_3 + W_m$ is not cordial.
Proof. The proof follows directly similar to the proof of lemma 3.2 above, the lemma follows.
Theorem 3.1. The join $W_n + W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) $(n, m) =$ (3,3),
(ii) $n = 3$ and $m \equiv 1$(mod 4(or vice versa),
(iii) $n \equiv 1(\text{mod } 4)$ and $m \equiv 3$(mod 4)(or vice versa).
Proof. The proof follows directly from lemma 3.1, lemma 3.2, lemma 3.3, lemma 3.4 and example 3.1, the theorem follows.

Lemma 3.5. The union $W_n \cup W_m$ of two wheels W_n and W_m is cordial for all n and all m except for $n \equiv 1 (\text{mod } 4)$ and $m \equiv 3(\text{mod } 4)$(or vice versa).

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i or $A"_i$ for the wheel W_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the wheel W_m , where $m > 3$ as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 3.3. Since these are all 0,1, or -1. For n = m = 3, the following labeling is suffice $W_3 \cup W_3$: [0001;1110], the lemma follows.

$n = 4r + i$, $i = 0,1,2,3$	$m = 4s + j$, $j = 0,1,2,3$	W_n	W_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B'_0	0	0
0	1	A_0	B_1	-1	0
0	2	A_0	B_2	0	0
0	3	$A"_0$	B'_3	1	0
1	0	A_1	B_0	-1	0
1	1	A_1	B_1	0	0
1	2	A_1	B_2	-1	0
2	0	A_2	B'_0	0	0
2	1	A_2	B_1	-1	0
2	2	A'_2	B'_2	0	0
2	3	$A"_2$	B_3	-1	0
3	0	A'_3	$B"_0$	1	0
3	2	A_3	B'_2	-1	0
3	3	A_3	B'_3	0	0

Tabel 3.3. Combinations of labelings.

Lemma 3.6. If $n \equiv 1$ (mod 4) and $m \equiv 3$ (mod 4)(or vice versa), then the union $W_n \cup W_m$ of two wheels W_n and W_m is not cordial.

Proof. The proof follows directly similar to the proof of lemma 3.2 above, the lemma follows.

Lemma 3.7. The union $W_3 \cup W_m$ is cordial for all m except for $m \equiv 1(\text{mod } 4)$.

Proof. The following labelings is suffice:
$W_3 \cup W_{2s}$: [0001;1L_{4s}], $W_3 \cup W_{4s+2}$: [0011;110L_{4s}] and $W_3 \cup W_{4s+3}$: [0011;1100L_{4s}]. For n = m = 3 the following labeling is suffice $W_3 \cup W_3$: [0001;1110], the lemma follows.

Lemma 3.8. If $m \equiv 1(\text{mod } 4)$, then the union $W_3 \cup W_m$ is not cordial.

Proof. The proof follows directly similar to the proof of lemma 3.2 above, the lemma follows.

Theorem 3.2. The union $W_n \cup W_m$ of two wheels W_n and W_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) $n = 3$ and $m \equiv 1(\text{mod } 4($ or vice versa),
(ii) $n \equiv 1(\text{mod } 4)$ and $m \equiv 3(\text{mod } 4)($ or vice versa).
Proof. The proof follows directly from lemma 3.5, lemma 3.6, lemma 3.7 and lemma 3.8, the theorem follows.

4 Joins and Unions of Wheels and Paths

In this section we show that the join $W_n + P_m$ of wheels W_n and paths P_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) $(n, m) =$ (3,1),(3,2) and (3,3),
(ii) $n \equiv 3(\text{mod } 4)$ and m = 1.
Also, we show that the union $W_n \cup P_m$ of wheels W_n and paths P_m is cordial for all n and all m.
Lemma 4.1. The join $W_n + P_m$ of wheels W_n and paths P_m is cordial for all $n > 3$ and all $m > 3$.
Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i or $A"_i$ for the wheel W_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the path P_m, where $m > 3$ as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4.2. Since these are all 0,1, or -1. For $n = 4r + 3$ and $m = 5$, the following labeling is suffice $W_{4r+3} + P_5$: $[0011L_{4r}; 00011], the\ lemma\ follows.$

$n = 4r + i$, $i = 0,1,2,3$	Labeling of W_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r+1$	$4r$	$4r$
	$A'_0 = 0L_{4r}$	$2r+1$	$2r$	$4r$	$4r$
	$A"_0 = 1L'_{4r}$	$2r+1$	$2r$	$4r-1$	$4r+1$
$i = 1$	$A_1 = 01L_{4r}$	$2r+1$	$2r+1$	$4r+1$	$4r+1$
	$A'_1 = 11L_{4r}$	$2r$	$2r+2$	$4r+2$	$4r$
$i = 2$	$A_2 = 011L_{4r}$	$2r+1$	$2r+2$	$4r+2$	$4r+2$
	$A'_2 = 001L_{4r}$	$2r+2$	$2r+1$	$4r+1$	$4r+3$
	$A"_2 = 110L_{4r}$	$2r+1$	$2r+2$	$4r+3$	$4r+1$
$i = 3$	$A_3 = 0011L_{4r}$	$2r+2$	$2r+2$	$4r+2$	$4r+4$
	$A'_3 = 1100L_{4r}$	$2r+2$	$2r+2$	$4r+4$	$4r+2$
	$A"_3 = 1011L_{4r}$	$2r+1$	$2r+3$	$4r+3$	$4r+3$

$m=4s+j$, $j=0,1,2,3$	Labeling of P_m	y_0	y_1	b_0	b_1
$j=0$	$B_0=L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
$j=1$	$B_1=L_{4s}0$	$2s+1$	$2s$	$2s$	$2s$
	$B_1'=1_30_3S_{4s-8}110, s\geq 2$	$2s$	$2s+1$	$2s+1$	$2s-1$
$j=2$	$B_2=L_{4s}10$	$2s+1$	$2s+1$	$2s+1$	$2s$
$j=3$	$B_3=L_{4s}001$	$2s+2$	$2s+1$	$2s+1$	$2s+1$
	$B_3'=S_{4s}011$,	$2s+1$	$2s+2$	$2s+2$	$2s$

Table 4.1. Labelings of Wheels W_n and paths P_m.

$n=4r+i$, $i=0,1,2,3$	$m=4s+j$, $j=0,1,2,3$	W_n	P_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	1
0	1	A_0	B_1	0	-1
0	2	A_0	B_2	-1	1
0	3	A_0	B_3	0	-1
1	0	A_1	B_0	0	1
1	1	A_1	B_1	1	0
1	2	A_1	B_2	0	1
1	3	A_1	B_3	1	0
2	0	A_2	B_0	-1	1
2	1	A_2	B_1	0	-1
2	2	A_2	B_2	-1	1
2	3	A_2	B_3	0	-1
3	0	A_3	B_0	0	-1
3	1	A_3	B_1'	-1	0
3	2	A_3	B_2	0	-1
3	3	A_3	B_3'	-1	0

Tabel 4.2. Combinations of labelings.

Lemma 4.2. The join $W_3+P_m=K_4+P_m$ of wheels W_3 and paths P_m is cordial for $m>3$.

Proof. Appropriate labelings are the following:
W_3+P_{4s}: [0011;L_{4s}], W_3+P_{4s+1}:[0011;$L_{4s}1$], W_3+P_{4s+2}:[0011;$L_{4s}10$] and W_3+W_{4s+3}:[0011;$S_{4s}001$], the lemma follows.

Example 4.1. The graphs W_3+P_1, W_3+P_2 and W_3+P_3 are not cordial.

Solution. It is easy to see that $W_3+P_1=K_4+K_1\equiv K_5$ and $W_3+P_2=K_4+K_2\equiv K_6$ are not cordial from the fact that the complete graph K_n is cordial if and only if $n\leq 3$ [1]. By investigating all possible labelings, we see that $W_3+P_3=K_4+P_3$ does not have a cordial labeling.

Lemma 4.3. The join W_n+P_m is cordial for all $n>3$ and all $m\leq 3$ except for $n\equiv 3(\text{mod }4)$ and m = 1.

Proof. Appropriate labelings are the following:
$W_{4r}+P_3$:[1L_{4r};001],$W_{4r+1}+P_3$:[01L_{4r};001], $W_{4r+2}+P_3$:[011L_{4r};001], $W_{4r+3}+P_3$:[1100L_{4r};010],$W_{4r}+P_2$:[1L_{4r};01],$W_{4r+1}+P_2$:[01L_{4r};01], $W_{4r+2}+P_2$:[011L_{4r};01], $W_{4r+3}+P_2$:[1100L_{4r};01], $W_{4r}+P_1$:[1L_{4r};0],$W_{4r+1}+P_1$:[01L_{4r};0] and $W_{4r+2}+P_1$:[011L_{4r};0], the lemma follows.

Lemma 4.4. If $n \equiv 3 (\text{mod } 4)$ and m = 1, then $W_n + P_m$ is not cordial.

Proof. It is easy to verify that the graph $W_{4r+3} + P_1$ is Eulerian which has a size congruent 2(mod4) and from Cahit's theorem [1], which states that " An Eulerian graph is not cordial if its size is congruent 2(mod4)", so we obtain that $W_{4r+3} + P_1$ is not cordial, the lemma follows.

Theorem 3.1. The join W_n+P_m of wheels W_n and paths P_m is cordial for all n and all m if and only if one of the following conditions is not satisfied:
(i) (n, m) = (3,3),(3,2) and (3,3),
(ii) $n \equiv 3 (\text{mod } 4)$ and m = 1.

Proof. The proof follows directly from lemma 4.1, lemma 4.2, lemma 4.3, lemma 4.4 and example 4.1, the theorem follows.

Lemma 4.5. The union $W_n \cup P_m$ of wheels W_n and paths P_m is cordial for all $n > 3$ and all $m > 3$.

Proof. For given values of i and j with $0 \le i \le 3$ and $0 \le j \le 3$, we use the labeling A_i or A'_i or $A"_i$ for the wheel W_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the path P_m , where $m > 3$ as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 4.3. Since these are all 0,1, or -1. For $n \equiv 3 (\text{mod } 4)$ and m = 3, the following labeling is suffice $W_{4r+3} \cup P_3$: [0011L_{4r};00111], the lemma follows.

$n=4r+i$, $i=0,1,2,3$	$m=4s+j$, $j=0,1,2,3$	W_n	P_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	1
0	1	A_0	B_1	0	0
0	2	A_0	B_2	-1	1
0	3	A_0	B_3	0	0
1	0	A_1	B_0	0	1
1	1	A_1	B_1	1	0
1	2	A_1	B_2	0	1
1	3	A_1	B_3	1	0
2	0	A_2	B_0	-1	1
2	1	A_2	B_1	0	0
2	2	A_2	B_2	-1	1
2	3	A_2	B_3	0	0
3	0	A_3	B_0	0	-1
3	1	A_3	B_1'	-1	0
3	2	A_3	B_2	0	-1
3	3	A_3	B_3'	-1	0

Tabel 4.3. Combinations of labelings.

Lemma 4.6. The union $W_3 \cup P_m = K_4 \cup P_m$ of wheels W_3 and paths P_m is cordial for $m > 3$.
Proof. Appropriate labelings are the following:
$W_3 \cup P_{4s}$: [0011;L_{4s}], $W_3 \cup P_{4s+1}$:[0011;L_{4s}1], $W_3 \cup P_{4s+2}$:[0011;L_{4s}10] and $W_3 \cup P_{4s+3}$:[0011;S_{4s}001], the lemma follows.
Example 4.1. The graphs $W_3 \cup P_1$, $W_3 \cup P_2$ and $W_3 \cup P_3$ are cordial.
Solution. Appropriate labelings are the following:
$W_3 \cup P_1$:[1000;1], $W_3 \cup P_2$: [1000;11] and $W_3 \cup P_3$: [1000;110].
Lemma 4.7. The union $W_n \cup P_m$ of wheels W_n and paths P_m is cordial for all $n > 3$ and all $m \leq 3$.
Proof. Appropriate labelings are the following:
$W_{4r} \cup P_3$:[1L_{4r};001], $W_{4r+1} \cup P_3$:[01L_{4r};001], $W_{4r+2} \cup P_3$:[011L_{4r};001], $W_{4r+3} \cup P_3$:[1100L_{4r};010], $W_{4r} \cup P_2$:[1L_{4r};01], $W_{4r+1} \cup P_2$:[01L_{4r};01], $W_{4r+2} \cup P_2$:[011L_{4r};01], $W_{4r+3} \cup P_2$:[1100L_{4r};01], $W_{4r} \cup P_1$:[1L_{4r};0], $W_{4r+1} \cup P_1$:[01L_{4r};0], $W_{4r+2} \cup P_1$:[011L_{4r};0] and $W_{4r+3} \cup P_1$: [$0_4 1_{r+1} M_{2r} 0_{r-1}$;0], the lemma follows.
Theorem 4.2. The union $W_n \cup P_m$ of wheels W_n and paths P_m is cordial for all n and all m.
Proof. The proof follows directly from lemma 4.5, lemma 4.6, lemma 4.7, and example 4.2, the theorem follows.

5 Joins and Unions of Wheels and Cycles

In this section, we show that the join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all n and all m if and only if $(m,n) \neq$ (3,3), (3,2) and (3,3). Also, we prove that the union $W_n \cup C_m$ of wheels W_n and cycles C_m is cordial for all n and all m.

Lemma 5.1. The join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all $n > 3$ and all $m > 3$ except for $n \equiv 3 (\text{mod } 4)$.

Proof. For given values of i and j with $0 \leq i \leq 2$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i or $A"_i$ for the wheel W_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the cycle C_m, where $m > 3$ as given in Table 5.1. Using Table 5.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 5.2. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0,1,2,3$	Labeling of W_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r+1$	$4r$	$4r$
	$A'_0 = 0L_{4r}$	$2r+1$	$2r$	$4r$	$4r$
	$A"_0 = 1L'_{4r}$	$2r+1$	$2r$	$4r-1$	$4r+1$
$i = 1$	$A_1 = 01L_{4r}$	$2r+1$	$2r+1$	$4r+1$	$4r+1$
	$A'_1 = 11L_{4r}$	$2r$	$2r+2$	$4r+2$	$4r$
$i = 2$	$A_2 = 011L_{4r}$	$2r+1$	$2r+2$	$4r+2$	$4r+2$
	$A'_2 = 001L_{4r}$	$2r+2$	$2r+1$	$4r+1$	$4r+3$

$m = 4s + j$, $j = 0,1,2,3$	Labeling of C_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s$
$j = 1$	$B_1 = 1L_{4s}$	$2s$	$2s+1$	$2s+1$	$2s$
	$B'_1 = L_{4s}0$	$2s+1$	$2s$	$2s+1$	$2s$
$j = 2$	$B_2 = 0L_{4s}1$	$2s+1$	$2s+1$	$2s+2$	$2s$
$j = 2$	$B'_2 = 0L_{4s}0$	$2s+2$	$2s$	$2s+2$	$2s$
$j = 3$	$B_3 = L_{4s}110$	$2s+1$	$2s+2$	$2s+3$	$2s$
	$B'_3 = L_{4s}001,$	$2s+2$	$2s+1$	$2s+1$	$2s+2$

Table 5.1. Labelings of Wheels W_n and cycles C_m.

$n=4r+i$, $i=0,1,2$	$m=4s+j$, $j=0,1,2,3$	W_n	C_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	0
0	1	A'_0	B_1	0	0
0	2	$A"_0$	B_2	1	0
0	3	$A"_0$	B_3	0	0
1	0	A_1	B_0	0	0
1	1	A_1	B_1	-1	1
1	2	A'_1	B'_2	0	0
1	3	A'_1	B'_3	-1	-1
2	0	A_2	B_0	-1	0
2	1	A_2	B'_1	0	0
2	2	A'_2	B_2	1	-1
2	3	A'_2	B_3	0	0

Tabel 5.2. Combinations of labelings.

Lemma 5.2. If $n \equiv 3 (\text{mod } 4)$, then the join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all $n > 3$ and all $m > 3$.

Proof. We have two cases for the labeling of the vertices of wheels W_n as Case 1. Let n = 4r+3 and $r \geq 1$, then we label the vertices of the wheel W_n as $A_3 = 1110L_{4r}$, i.e. x_0 = 2r+1, x_1 = 2r+3, a_0 = 4r+5 and a_1 = 4r+1. Then we have two subcases for the labeling of the vertices of the cycle C_m, where $m > 3$ as

(i) If m = 4s, the we label the vertices of the cycle C_{4s} as $B_0 = L'_{4s}$, i.e. y_0 =2s+1, y_1 = 2s-1, b_0 = 2s and b_1 = 2s. Hence from the fact that $v_0 - v_1 = (x_0-x_1)+(y_0-y_1) = 0$ and $e_0-e_1 = (a_0-a_1)+(b_0-b_1)+(x_0-x_1)(y_0-y_1) = 0$.

(ii) If m = 4s+1, the we label the vertices of the cycle C_{4s+1} as $B_1 = L'_{4s}0$, i.e. y_0 =2s+2, y_1 = 2s-1, b_0 = 2s+1 and b_1 = 2s. Hence from the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$.

Case 2. Let n = 4r+3 and $r \geq 1$, then we label the vertices of the wheel W_n as $A'_3 = 0011L_{4r}$,i.e. x_0 = 2r+2, x_1 = 2r+2, a_0 = 4r+2 and a_1 = 4r+4. Then we have two subcases for the labeling of the vertices of the cycle C_m , where $m > 3$ as

(i) If m = 4s+2, the we label the vertices of the cycle C_{4s+2} as $B_2 = 0L_{4s}1$, i.e. y_0 =2s+1, y_1 = 2s+1, b_0 = 2s+2 and b_1 = 2s. Hence from the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$.

(ii)If m = 4s+3, the we label the vertices of the cycle C_{4s+3} as $B_3 = L_{4s}110$, i.e. y_0 =2s+1, y_1 = 2s+2, b_0 = 2s+3 and b_1 = 2s. Hence from the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) =$ -1 and $e_0 - e_1 = (a_0 - a_1) + (b_0 -$

$b_1) + (x_0 - x_1)(y_0 - y_1) = 1$. Therefore from the above cases we conclude that the join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all $n > 3$ and all $m > 3$ if $n \equiv 3 (\text{mod } 4)$, the lemma follows.

Example 5.1. The graph $W_3 + C_3$ is not cordial.

Solution. Similar to lemma 4.4, we see that $W_3 + C_3 = K_4 + K_3 \equiv K_7$ is not cordial.

Example 5.2. The graph $W_3 + C_4$ is not cordial.

Solution. By investigating all possible labelings, we see that $W_3 + C_4 = K_4 + C_4$ does not have a cordial labeling.

Lemma 5.3. The join $W_3 + C_m$ of the wheel W_3 and cycles C_m is cordial for all $m > 4$.

Proof. Appropriate labelings are the following:
$W_3 + C_{4s}$:[0001;$0_3 1_5 M'_{4s-8}$], where $s > 1$, $W_3 + C_{4s+1}$:[0001;$L_{4s}1$], $W_3 + C_{4s+2}$:[0011;$0L_{4s}1$] and $W_3 + C_{4s+3}$:[0011;$L_{4s}110$], the lemma follows.

Theorem 5.1. The join $W_n + C_m$ of wheels W_n and cycles C_m is cordial for all n and all m if and only if $(m, n) \neq$ (3,3), (3,2) and (3,3).

Proof. The proof follows directly from lemma 5.1, lemma 5.2, lemma 5.3, example 5.1 and example 5.2, the theorem follows.

Lemma 5.4. The union $W_n \cup C_m$ of wheels W_n and cycles C_m is cordial for all $n > 3$ and $m > 3$.

Proof. For given values of i and j with $0 \le i \le 3$ and $0 \le j \le 3$, we use the labeling A_i or A'_i or $A"_i$ for the wheel W_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the cycle C_m, where $m > 3$ as given in Table 5.3. Using Table 5.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 5.4. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of W_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r+1$	$4r$	$4r$
	$A'_0 = 0L_{4r}$	$2r+1$	$2r$	$4r$	$4r$
	$A"_0 = 1L'_{4r}$	$2r+1$	$2r$	$4r-1$	$4r+1$
$i = 1$	$A_1 = 01L_{4r}$	$2r+1$	$2r+1$	$4r+1$	$4r+1$
	$A'_1 = 11L_{4r}$	$2r$	$2r+2$	$4r+2$	$4r$
$i = 2$	$A_2 = 011L_{4r}$	$2r+1$	$2r+2$	$4r+2$	$4r+2$
	$A'_2 = 001L_{4r}$	$2r+2$	$2r+1$	$4r+1$	$4r+3$
	$A"_2 = 110L_{4r}$	$2r+1$	$2r+2$	$4r+3$	$4r+1$
$i = 3$	$A_3 = 0011L_{4r}$	$2r+2$	$2r+2$	$4r+2$	$4r+4$
	$A'_3 = 1100L_{4r}$	$2r+2$	$2r+2$	$4r+4$	$4r+2$
	$A"_3 = 1011L_{4r}$	$2r+1$	$2r+3$	$4r+3$	$4r+3$

$m = 4s + j$, $j = 0,1,2,3$	Labeling of C_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s$
	$B'_0 = L'_{4s}$	$2s+1$	$2s-1$	$2s$	$2s$
$j = 1$	$B_1 = L_{4s}0$	$2s+1$	$2s$	$2s+1$	$2s$
$j = 2$	$B_2 = 0L_{4s}1$	$2s+1$	$2s+1$	$2s+2$	$2s$
$j = 2$	$B'_2 = 01L'_{4s}$	$2s+2$	$2s$	$2s$	$2s+2$
$j = 3$	$B_3 = L_{4s}011$	$2s+1$	$2s+2$	$2s+1$	$2s+2$
	$B'_3 = L_{4s}001$,	$2s+2$	$2s+1$	$2s+1$	$2s+2$
	$B"_3 = L_{4s}110$,	$2s+1$	$2s+2$	$2s+3$	$2s$

Table 5.3. Labelings of Wheels W_n and cycles C_m.

$n = 4r + i$, $i = 0,1,2,3$	$m = 4s + j$, $j = 0,1,2,3$	W_n	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	-1	0
0	1	A_0	B_1	0	1
0	2	$A"_0$	B_2	1	0
0	3	A_0	B'_3	0	-1
1	0	A_1	B_0	0	0
1	1	A_1	B_1	1	1
1	2	A'_1	B'_2	0	0
1	3	A_1	B_3	-1	-1
2	0	A_2	B_0	-1	0
2	1	A_2	B_1	0	1
2	2	A'_2	B_2	1	0
2	3	A'_2	$B"_3$	0	1
3	0	$A"_3$	B'_0	0	0
3	1	$A"_3$	B_1	-1	1
3	2	A_3	B_2	0	0
3	3	A_3	$B"_3$	-1	1

Tabel 5.4. Combinations of labelings.

Lemma 5.5. The union $W_3 \cup C_m = K_4 \cup C_m$ of wheel W_3 and cycles C_m is cordial for all m.

Proof. Appropriate labelings are the following:
$W_3 \cup C_{4s}$: [1110;L'_{4s}], where $s > 1$, $W_3 \cup C_{4s+1}$: [0001;1L_{4s}], $W_3 \cup C_{4s+2}$:[0011;0L_{4s}1] and $W_3 \cup C_{4s+3}$:[0011;L_{4s}110]. For n =3 and m = 4, the following labeling is suffice $W_3 \cup C_4$:[0001;1110], and for n =3 and m = 3, the following labeling is suffice $W_3 \cup C_3$:[0111;001], the lemma follows.

Theorem 5.2. The union $W_n \cup C_m$ of wheels W_n and cycles C_m is cordial for all n and all m.

Proof. The proof follows directly from lemma 5.4 and lemma 5.5, the theorem follows.

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