On (l, m)-Walk-regular Digraphs *

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Abstract

In this paper, we introduce a class of digraphs called (l, m)-walk-regular digraphs, a common generalization of both weakly distance-regular digraphs[1] and k-walk-regular digraphs[3] and give several characterization of them about their regularity properties that are related to distance and about the number of walks of given length between vertices at a given distance.

1 Introduction

Let $\Gamma = (V, E)$ be a digraph with the vertex set V and the arc set E. If $(u, v) \in E$, we say that u dominates v. The set of vertices of Γ dominated by u is said to be the *out-neighbors* of u, denoted by $\Gamma_1^+(u)$. The set of vertices of Γ dominating u is said to be the *in-neighbors* of u, denoted by $\Gamma_1^-(u)$. A digraph is said to be *out(in)-regular* if the number of out(in)-neighbors of u, $|\Gamma_1^+(u)|(|\Gamma_1^-(u)|)$, is independent of u for any $u \in V$ and the digraph is said to be *regular* if $|\Gamma_1^+(u)| = |\Gamma_1^-(u)|$ for every $u \in V$. A *walk* of length t in Γ is a sequence $(u_0, u_1, ..., u_t)$ of vertices such that $(u_{i-1}, u_i) \in E$, $i = 1, 2, ..., t$. The number of arcs traversed in a shortest walk from u to v is called the *distance* from u to v in Γ, denoted by $\partial(u, v)$. The maximum value of the distance function in Γ is called the *diameter* of Γ. A digraph Γ is said to be *strongly connected* if, for any two distinct vertices u and v, there is a walk from u to v. The distance$-k$ digraph Γ_k is the digraph with vertex set V and where there is an arc from u to v if and only if $\partial(u, v) = k$ in Γ for any vertices u and v.

Let Γ be a digraph with diameter D. For $0 \le k \le D$, the *distance-k matrix* A_k, is defined by

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$$(A_k)_{uv} := \begin{cases} 1 & \text{if } \partial(u,v) = k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $A_0 = I$, $A_1 = A$ is called the adjacency matrix of Γ and A_k is the Γ_k's adjacency matrix.

The spectrum of the digraph Γ, consisting of the eigenvalues of A together with their multiplicities, is denoted by spΓ: sp$\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \cdots, \lambda_d^{m_d}\}$. Thus $m(x)$, the minimal polynomial of Γ can be described as $m(x) = (x-\lambda_0)^{r_0}(x-\lambda_1)^{r_1}\cdots(x-\lambda_d)^{r_d}$, where $r_i \leq m_i$, $i = 0, 1, \cdots, d$, with degree $\sum_{i=0}^{d} r_i \triangleq t+1 \geq d+1$ since A may not be a symmetric matrix .

Let $\mathcal{A}(\Gamma)$ be the adjacency algebra of Γ, that is, $\mathcal{A}(\Gamma)$ is the algebra spanned by A in $\mathbb{C}$. It is known that $\{I, A, \cdots, A^t\}$ is a basis of $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma) \simeq \mathbb{C}_t[A] = span\{I, A, A^2, \cdots, A^t\}$. It is immediate that $t \geq D$ if we notice that the powers $I, A, A^2, \cdots, A^D$ are linearly independent.

If $AA^* = A^*A$, where A^* is the transpose of A's conjugate, then A is said to be normal and the digraph with A as its adjacency matrix is said to be a normal digraph. About normal matrices, there are the following properties:

Proposition 1.1 *([1]) Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. Then A is normal if and only if any of the following assertions holds:*

(a) *$U^*AU = D$ for some matrix U such that $UU^* = I$, and $D = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$.*

(b) *$A^* = p(A)$ for some polynomial $p \in \mathbb{C}[x]$.*

(c) *$\mathrm{tr}(AA^*) = \sum_{i=0}^{n-1} |\lambda_i|^2$.*

If Γ is normal, then $r_i = 1$, $(i = 0, 1, \cdots, d)$, $t = d$ and from Proposition 1.1(a), we know that the eigenvectors of a normal $n \times n$ square matrix constitute an orthogonal basis of the vector space $\mathbb{C}^n$. Notice that If Γ is a normal digraph, then for any vertex u, we have $|\Gamma_1^+(u)| = (AA^*)_{uu} = (A^*A)_{uu} = |\Gamma_1^-(u)|$ and then

Proposition 1.2 *A normal digraph Γ is out-regular if and only if Γ is in-regular, further a normal digraph Γ is regular if and only if Γ is out(in)-regular.*

In the next section, we will introduce the definition of (l, m)-walk-regular digraphs, which is a generalization of weakly distance-regular digraphs[1], a class of digraphs different with those defined in [2] by K.Wang and H.Suzuki, and also a generalization of m-walk-regular digraphs[3]. The following are the definitions about weakly distance-regular digraphs and m-walk-regular digraphs.

Definition 1.1 *([1]) A digraph Γ of diameter D is weakly distance-regular if, for each nonnegative integer $l \leq D$, the number a_{uv}^l of walks of length l from vertex u to vertex v only depends on their distance $\partial(u,v) = k$, for any $l = 0, 1, \cdots, D$. In this case we write $a_{uv}^l = a_k^l, 0 \leq k, l \leq D$.*

Definition 1.2 *([3]) $\Gamma = (V, E)$ is said to be a k-walk-regular digraph if, for a given integer k, $(0 \leq k \leq D)$, the number of walks of length l, $a_{uv}^l = (A^l)_{uv}$, from vertex u to vertex v only depends on the distance from u to v, provided that this distance does not exceed k. In this case we just denote the number by a_k^l.*

2 Main Results

Definition 2.1 *Let Γ be a strongly connected digraph with diameter D and d+1 distinct eigenvalues. Γ is said to be (l,m)-walk-regular if the number of walks of length $i \leq l$ from u to v with $\partial(u,v) = k \leq m$, $a_{uv}^i = (A^i)_{uv}$ does not depend on such vertices u and v, but depends only on i and k, where $l \leq t$ and $m \leq D$ satisfying $l \geq m$ are two given integers. In this case we just denote the number by a_k^i.*

It is easy to see that if Γ is (l,m)-walk-regular and " $\circ$ " is the Hadamard-entrywise-product of matrices, then for each $0 \leq i \leq l$ and $0 \leq j \leq m$, we have $A^i \circ A_j = \begin{cases} a_j^i A_j & if\ \ i \geq j \\ 0 & otherwise \end{cases}$. Also the (l,m)-walk-regular digraphs must be (l',m')-walk-regular for any positive integers l', m' with $l' \leq l$ and $m' \leq m$. And this concept does generalize the concepts of weakly distance-regular digraphs and m-walk-regular digraphs. In fact, (t,m)-walk-regular digraphs coincide with the m-walk-regular digraphs, and (D,D)-walk-regular digraphs are exactly the weakly distance-regular digraphs.

Theorem 2.1 *Let Γ be a strongly connected digraph with diameter D and distance matrices $\{A_k\}_{k=0}^D$. Let m be an integer with $0 \leq m \leq D$. Then the following is equivalent.*

(i) Γ is (m,m)-walk-regular;

(ii)The distance matrices A_k, $k = 0, 1, \cdots, m$, is a polynomial of degree k in the adjacency matrix A; that is, $A_k = p_k(A)$, for each $k = 0, 1, \cdots, m$, where $p_k \in \mathbb{C}[x]$;

(iii) For each pair of i, k, $0 \leq k \leq m$, $0 \leq i \leq m-1$, the number

$$p_{i,1}^k(u,v) = |P_{i,1}^k(u,v)| = |\Gamma_i^+(u) \cap \Gamma_1^-(v)|$$

is a constant for all u, v such that $\partial(u,v) = k$.

(iv) For each triple of i, j, k, $0 \le i, j, k \le m$ and $i+j \le m$, the number, $p^k_{i,j}(u,v) := |P^k_{i,j}(u,v)| := |\Gamma^+_i(u) \cap \Gamma^-_j(v)|$, is a constant for all u, v such that $\partial(u,v) = k$.

Proof. (i)$\Longrightarrow$(ii): Suppose for any nonnegative integer $i \le m$, the number a^i_{uv} only depends on the distance from u to v. Then

$$A^i = a^i_0 I + a^i_1 A + a^i_2 A_2 + \dots + a^i_i A_i, (0 \le i \le m), \tag{1}$$

where, necessarily, $a^i_i \neq 0$ and $a^i_k = 0$ for any $k > l$. In matrix form

$$\begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^m \end{pmatrix} = \begin{pmatrix} a^0_0 & 0 & 0 & 0 & \dots & 0 \\ a^1_0 & a^1_1 & 0 & 0 & \dots & 0 \\ a^2_0 & a^2_1 & a^2_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^m_0 & a^m_1 & a^m_2 & a^m_3 & \dots & a^t_t \end{pmatrix} \begin{pmatrix} I \\ A \\ A_2 \\ \vdots \\ A_m \end{pmatrix}, \tag{2}$$

where $C := (a^i_k)$ is a lower triangular matrix. Since $a^i_i > 0$ for any $0 \le i \le m$, the matrix C is non-singular and its inverse C^{-1} is also a lower triangular matrix. Hence A_k is a polynomial of degree of k in A for any $0 \le k \le m$:

$$A_k = p_k(A) = \alpha^k_0 I + \alpha^k_1 A + \alpha^k_2 A^2 + \dots + \alpha^k_k A^k, (0 \le k \le m), \tag{3}$$

where $\alpha^k_k \neq 0$, as desired.

(ii)$\Longrightarrow$(i): Let us assume that there are constants α^k_i, with $\alpha^k_k \neq 0$, $(0 \le k, i \le m)$ satisfying (3). This implies that Equations (2) and (1) hold with $C = (a^i_k)$ being the inverse matrix of $C^{-1} = (\alpha^i_k)$, and, hence the number of walks of length i $(0 \le i \le m)$ from one vertex to another vertex only depends on their distance.

(ii)$\Longrightarrow$(iv) Suppose $A_k = p_k(A)$, for $k = 0, 1, \cdots, m$, is a polynomial of degree k in A. Set $\mathcal{M} = \{f(A) | f(A) \in \mathcal{A}(\Gamma), deg(f) \le m\}$. Then $\mathcal{M}$ is a subspace of $\mathcal{A}(\Gamma)$ and $\{I, A, \cdots, A_m\}$ is a basis of $\mathcal{M}$. Thus for any i, j, k such that $0 \le i, j, k \le m$ and $0 \le i + j \le m$,

$$A_i A_j = p_i(A) p_j(A) = \sum_{k=0}^{m} \gamma^k_{i,j} A_k.$$

Let u and v be two vertices of digraph Γ such that $\partial(u,v) = k$ $(0 \le k \le m)$. Notice that the number $p^k_{i,j}(u,v)$ representing the number of vertices at distance i from u and at distance j to v, coincides with the (u,v)-th entry of the matrix $A_i A_j$, we have $p^k_{i,j}(u,v) = \gamma^k_{i,j} = p^k_{i,j}$, $0 \le i, j, k \le m, 0 \le i + j \le m$, for any two vertices u, v at distance k. (iv)$\Longrightarrow$(iii) is obvious so it suffices to prove (iii)$\Longrightarrow$(ii).

If for $0 \leq i \leq m-1$,

$$A_i A_1 = \sum_{k=0}^{i+1} p_{i,1}^k A_k.$$

We can use an inductive argument starting from $A_0 = I$, $A_1 = A$ to deduce that the distance matrix A_k $(0 \leq k \leq t)$ is indeed a polynomial of degree k in the adjacency matrix A. □

For a given digraph Γ with adjacency matrix A, we consider the following scalar product in $\mathbb{C}[x]$:(see [1])

$$\langle p, q \rangle = \frac{1}{n} tr(p(A)q(A)^*)$$

It is obvious that the product is well defined in the quotient ring $\mathbb{C}[x]/(m(x))$. Notice that $1, x, x^2, \cdots, x^t$ are linear independent in $\mathbb{C}_t[x]$, then by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existance and the uniqueness of the orthogonal system of polynomials $\{p_k\}_{0 \leq k \leq t}$ called predistance polynomials, which, for any $0 \leq h, k \leq t$, satisfy:

(1) $deg(p_k) = k$;

(2) $\langle p_h, p_k \rangle = 0$, if $h \neq k$;

(3) $\| p_k \|^2 = p_k(\lambda_0)$.

Recall that, in a weakly distance-regular digraph, we have $D = d = t$([1],Theorem 2.2) and such polynomials satisfy $p_k(A) = A_k$, $0 \leq k \leq d$. If the $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ satisfy the condition that $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_d|$, then by the Perron-Frobenius theorem, λ_0 is simple and has a positive eigenvector $\mathbf{v}$. Further if Γ is k-regular, then we may pick $\mathbf{v} = \mathbf{j}$, where $\mathbf{j}$ denotes the all 1- vector, and $\lambda_0 = k$. Consequently, if Γ is regular and (m, m)-walk-regular, $A_k\mathbf{j} = p_k(A)\mathbf{j} = p_k(\lambda_0)\mathbf{j}$. So the number n_k of vertices at distance k from any given vertex is equal to $p_k(\lambda_0)$, for each $k = 0, 1, \cdots, m$. In the following we will give some characterization of (l, m)-walk-regular digraphs,

Lemma 2.2 *Let Γ be a (l, m)-walk-regular digraph, then for any $p \in \mathcal{A}_l$ with $deg(p) = h \leq l$ and any integer j with $0 \leq j \leq m$, $p(A) \circ A_j = r_j(p)A_j$, where $r_j(p)$ is a real number depending only on the polynomial p and the integer j.*

Proof. Let $p(x) = a_0 + a_1 x + \cdots + a_h x^h$. Then

$$p(A) \circ A_j = (a_0 I + a_1 A + \cdots + a_h A^h) \circ A_j$$

$$= (a_0 I) \circ A_j + (a_1 A) \circ A_j + \cdots + (a_h A^h) \circ A_j = \sum_{i=j}^{h} a_i a_j^i A_j = (\sum_{i=j}^{h} a_i a_j^i) A_j \triangleq r_j(p) A_j,$$

where $r_j(p) = \sum_{i=j}^{h} a_i a_j^i$, a real number depending only on p and j. $\square$

Theorem 2.3 *Let Γ be a regular digraph with predistance polynomials $p_0, p_1, \cdots, p_t$. Then the following statements are equivalent.*
(i) Γ is (l,m)-walk-regular;
(ii) $p_i(A) \circ A_j = \delta_{ij} A_i$, $0 \le i \le l$, $0 \le j \le m$.

Proof. (i)⇒(ii). Suppose that Γ is (l,m)-walk-regular, we have that $p_i(A) = A_i$, $0 \le i \le m$ by Theorem 2.1. It is obvious that $p_i(A) \circ A_j = \delta_{ij} A_i$ for $0 \le i,j \le m$.

For $m+1 \le i \le l$, $0 \le j \le m$, $p_i(A) \circ A_j = r_j(p_i) A_j$ from the Lemma above. Notice that Γ is regular and (l,m)-walk-regular we have $p_j(\lambda_0) = \frac{1}{n} Sum(A_j) = \frac{1}{n} Sum(p_j(A))$. So

$$r_j(p_i) \cdot p_j(\lambda_0) = r_j(p_i) \cdot \frac{1}{n} Sum(p_j(A)) = \frac{1}{n} Sum(r_j(p_i) p_j(A))$$

$$= \frac{1}{n} Sum(r_j(p_i) A_j) = \frac{1}{n} Sum(p_i(A) \circ A_j) = \frac{1}{n} Sum(p_i(A) \circ p_j(A))$$

$$= \frac{1}{n} Sum(p_i(A) \circ \overline{p_j(A)}) = \frac{1}{n} tr(p_i(A) p_j(A)^*) = \langle p_i, p_j \rangle = 0.$$

Thus $r_j(p_i) = 0$ for $p_j(\lambda_0) \neq 0$. Therefore $p_i(A) \circ A_j = 0$, as desired.

(ii)⇒(i). Let $x^h = \sum_{i=0}^{h} a_i^h p_i$ for $h \le l$. Then for each pair of vertices u, v with $\partial(u,v) = j \le m$ and $h \le l$ we have

$$(A^h)_{uv} = (A^h \circ A_j)_{uv} = \sum_{i=0}^{h} a_i^h (p_i(A) \circ A_j)_{uv} = a_j^h.$$

Consequently, Γ is (l,m)-walk-regular. $\square$

We define the preintersection number in the same way as in [4]:

$$\xi_{ij}^k = \frac{\langle p_i p_j, p_k \rangle}{\|p_k\|^2} = \frac{1}{n\|p_k\|^2} tr(p_i(A) p_j(A) p_k(A)^*).$$

Proposition 2.4 *Let $t \ge l \ge m \le D$, Γ be a (l,m)-walk-regular digraph, and let $i,j,k \le m$. If $i+j \le l$, then the preintersection numbers ξ_{ij}^k equals the well defined intersection numbers p_{ij}^k. If $i+j \ge l+1$, then the preintersection numbers equal the average $\overline{p}_{ij}^k$ of the values $p_{ij}^k(u,v) = |\Gamma_i^+(u) \cap \Gamma_j^-(v)|$ over all vertices u, v with $\partial(u,v) = k$.*

Proof. For $i + j \le l$,

$$\xi_{ij}^k = \frac{1}{n\|p_k\|^2} tr(p_i(A) p_j(A) p_k(A)^*) = \frac{1}{n p_k(\lambda_0)} Sum(p_i(A) p_j(A) \circ \overline{p_k(A)})$$

$$= \frac{1}{np_k(\lambda_0)} Sum(p_i(A)p_j(A) \circ p_k(A)) = \frac{1}{np_k(\lambda_0)} Sum(A_iA_j \circ A_k)$$

$$= \frac{1}{np_k(\lambda_0)} \sum_{\partial(u,v)=k} (A_iA_j)_{uv} = \frac{1}{nn_k} \sum_{\partial(u,v)=k} \sum_w (A_i)_{uw}(A_j)_{wv}$$

$$= \frac{1}{nn_k} \sum_{\partial(u,v)=k} |\Gamma_i^+(u) \cap \Gamma_j^-(v)| = |\Gamma_i^+(u) \cap \Gamma_j^-(v)|$$

with $\partial(u,v) = k$ and the number is p_{ij}^k.

For $i + j \geq l + 1$,

$$\xi_{ij}^k = \frac{1}{np_k(\lambda_0)} Sum(A_iA_j \circ A_k) = \frac{1}{np_k(\lambda_0)} \sum_{\partial(u,v)=k} (A_iA_j)_{uv}$$

$$= \frac{1}{nn_k} \sum_{\partial(u,v)=k} \sum_w (A_i)_{uw}(A_j)_{wv} = \frac{1}{nn_k} \sum_{\partial(u,v)=k} |\Gamma_i^+(u) \cap \Gamma_j^-(v)| = \overline{p}_{ij}^k.$$

□

References

[1] F.Comellas, M.A.Fiol, J.Gimbert and M.Mitjana, Weakly distance-regular digraphs, J. Combin. Theory Ser. B 90(2004), 233-255.

[2] K.S.Wang and H.Suzuki, Weakly distance-regular digraphs, Discrete Mathematics 264(2003)225-236.

[3] Wen Liu, k-walk-regular Digraphs, to appear in Journal of Mathematical Research and Exposition.

[4] C.Dalfoa, E.R.Van Damb, M.A. Fiol, E.Garriga, and B.L.Gorissenb, On almost distance-regular graphs, Journal of Combinatorial Theory, Series A(2010), doi:10.1016/j.jcta.2010.10.005.

[5] C.Dalfoa,M.A. Fiol and E.Garriga, Characterizing (l,m)-walk-regular graphs, Linear Algebra and Its Applications, 433(2010), 1821-1826.