On edge cover coloring of join graphs *

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Abstract A join graph is the complete union of two arbitrary graphs. An edge cover coloring is a coloring of edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at least one time. The maximum number of colors needed to edge cover color G is called the edge cover chromatic index of G and denoted by $\chi'_c(G)$. It is well known that any simple graph G has the edge cover chromatic index equal to $\delta(G)$ or $\delta(G) - 1$, where $\delta(G)$ is the minimum degree of G. If $\chi'_c(G) = \delta(G)$, then G is of $C1$-class, otherwise G is of $C2$-class. In this paper we give some sufficient conditions for a join graph to be of $C1$-class.

Keywords: edge coloring; edge cover coloring; classification of graphs; join graphs.
AMS subject classification (2000): 05C15, 05C25

1 Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. The graphs in this paper are simple, that is they have no loops or multiple edges. Let $G = (V, E)$ be a graph; the *degree* of a vertex v, denoted by $d_G(v)$, is the number of edges incident to v; the *minimum degree* of G, denoted by $\delta(G)$, is the minimum vertex degree in G; G is *regular* if the degree of every vertex is the same.

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An *edge coloring* Φ of a graph $G = (V, E)$ is an assignment of colors to its edges. The coloring Φ is *proper* if no two adjacent edges have the same color. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a proper edge coloring with k colors. In [5] it was show that every graph G with m edges and $\chi'(G) \leq k$ has an *equalized* k edge coloring Φ: each color a_i in Φ appears on exactly either $\lfloor m/k \rfloor$ or $\lceil m/k \rceil$ edges. Unless otherwise stated, the edge coloring of graphs in this paper are not necessarily proper.

An *edge cover* of G is a subset S of E that saturates every vertex of G, i.e., every vertex of G is an end vertex of an edge in S. An *edge cover coloring* of G is an edge coloring such that the edges assigned the same color formed an edge cover of G. Clearly, the edge cover coloring may not be proper.

The *edge cover chromatic index* $\chi'_c(G)$ of G is the maximum size of a partition of E into edge covers of G. G is said to be *k-edge cover colorable* if there is an edge cover coloring of G with k colors. On the edge cover chromatic index of a graph, Gupta [2] gave the following result.

Theorem 1.1. *Let G be a graph. Then*

$$\delta - 1 \leq \chi'_c(G) \leq \delta.$$

From the above result, we can see that the edge cover chromatic index of any graph must be equal to δ or $\delta - 1$, this immediately gives us a simple way of classifying graphs into two types according to $\chi'_c(G)$. More precisely, we say that G is of $C1$-class if $\chi'_c(G) = \delta$, and that G is of $C2$-class if $\chi'_c(G) = \delta - 1$. The graphs that are $C1$-class are also known as $(1, d)$-factorizable graphs. In general, the problem of determining the edge cover chromatic index of graphs is NP-hard because deciding whether a 3-connected 3-regular graph G is proper 3-edge colorable is NP-complete [4], which is the special case of our general problem.

Hilton generalized the edge cover coloring and obtained many interesting results, the following theorem can be found in [3].

Theorem 1.2. *If G is a bipartite multigraph, then G has a k-edge coloring such that the number of distinct colors represented at v is min $\{k, d(v)\}$ for each $v \in V(G)$.*

By theorem 1.2, for any bipartite graph G with minimum degree δ, it must have a δ-edge cover coloring. So, we can see that all bipartite graphs are of $C1$-class.

Miao [6] considered the classification on edge cover coloring of graphs and gave some sufficient conditions for a graph G to be of $C2$-class. Song

[8] considered the properties of $C2$-class graphs and studied the relation of edge cover chromatic index between the graph G and its complement graph G^c. Wang [9] discuss the classification problem of nearly bipartite graphs and gave some sufficient conditions for a nearly graph G to be of $C1$-class.

The goal of this paper is to find sufficient conditions for a join graph to be $C1$-class.

2 The join graph

Let $G = (V, E)$ be a graph with n vertices. We say that G is a join graph if G is the complete union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In other words, $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2 \bigcup \{uv : u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2, we shall write $G = G_1 + G_2$. Simone and de Mello [7] considered the edge coloring of join graphs and gave some sufficient conditions for a join graph to be Class 1. We adopt some techniques similar to Simone and de Mello and gave some results on edge cover coloring of join graphs.

Write $n_1 = |V_1|$, $n_2 = |V_2|$, $\delta_1 = \delta(G_1)$ and $\delta_2 = \delta(G_2)$. Clearly, $n = n_1 + n_2$ and $\delta(G) = min\{n_1 + \delta_2, n_2 + \delta_1\}$. Without loss of generality we shall assume that $n_1 \leq n_2$.

To every join graph $G = G_1 + G_2$ we shall associate the complete bipartite graph B_G obtained from G by removing all edges of G_1 and G_2. For every minimum edge cover L in B_G, let G_L denote the subgraph of G obtained by removing all edges of B_G but the edges in L. Figure 2 shows two G_L for the graph G in Figure 1.

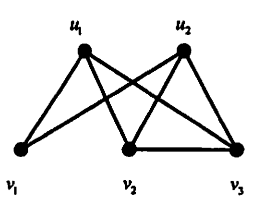

Figure 1: G.

We have the following key lemma.

Lemma 2.1. *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$ such that $\delta_1 \geq \delta_2$, or such that $\delta_1 < \delta_2$ and $n_1 = n_2$. If there exists a minimum edge*

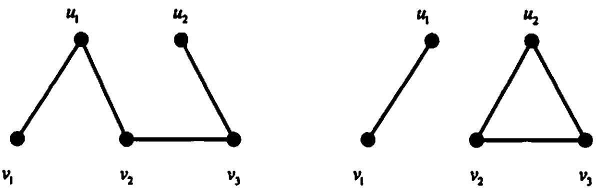

Figure 2: Two G_L for the graph G.

cover L in B_G such that the corresponding graph G_L is $C1$-class, then G is $C1$-class.

Proof. Let L be a minimum edge cover of B_G such that $\chi_c'(G_L) = \delta(G_L)$ and let B' be the bipartite graph obtained from B_G by removing all edges in L. Note that $\chi_c'(G) \geq \chi_c'(G_L) + \chi_c'(B')$, and that $\chi_c'(B') = \delta(B') = n_1 - 1$ (because $n_1 \leq n_2$). If $\delta_1 \geq \delta_2$, then $\delta(G) = \delta_2 + n_1$ and $\delta(G_L) = \delta_2 + 1$ and so $\chi_c'(G) \geq \delta(G)$. If $\delta_1 < \delta_2$ and $n_1 = n_2$, then $\delta(G) = \delta_1 + n_1$ and $\delta(G_L) = \delta_1 + 1$, and so $\chi_c'(G) \geq \delta(G)$. ∎

If some assumption in Lemma 2.1 does not hold, then G could be $C2$-class even though G_L is $C1$-class for every minimum edge cover L. For instance, the case of graph $G = G_1 + G_2$ when $G_1 = K_2$ and $G_2 = K_3$ (here $n_1 < n_2$ and $\delta_1 < \delta_2$).

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$. In view of Lemma 2.1, it is natural to ask when there exists a minimum edge cover L of B_G such that the corresponding graph G_L is $C1$-class. We have the following result.

Theorem 2.2. *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\delta_1 \geq \delta_2$ and G_2 is $C1$-class, then for every minimum edge cover L of B_G, the corresponding graph G_L is $C1$-class.*

Proof. Since $\delta_1 \geq \delta_2$ and G_2 is $C1$-class, G_2 is δ_2-edge cover colorable with colors a_0, a_1, $\cdots$, $a_{\delta_2 - 1}$. Suppose L is a minimum edge cover of B_G, we can give all the edges in L the color a_{δ_2}. This is an edge cover coloring of G_L with $\delta_2 + 1$ colors. By assumption, $\delta_{G_L} = \delta_2 + 1$, so the corresponding graph G_L is $C1$-class. ∎

An instant corollary of Theorem 2.2 and Lemma 2.1 is as follows.

Corollary 2.3. *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\delta_1 \geq \delta_2$ and G_2 is $C1$-class, then $\chi_c'(G) = \delta(G)$.*

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$. Assume that $\delta_1 = \delta_2$, it is natural to ask when there exists a minimum edge cover L of B_G such that the corresponding graph G_L is $C1$-class. It follows that we can still make use of Lemma 2.1 by finding sufficient conditions for the existence of edge cover L.

Theorem 2.4. *Let $G = G_1 + G_2$ be a join graph with $\delta_1 = \delta_2$. If one of the following two conditions hold,*

(a) Both G_1 and G_2 are $C1$-class;

(b) Both G_1 and G_2 are disjoint unions of cliques;

then there exists a minimum edge cover L of B_G such that the corresponding graph G_L is $C1$-class.

Proof. Without loss of generality, we can assume that $n_1 \leq n_2$.

First, assume that (a) holds. Let L be an arbitrary minimum edge cover of B_G. Since G_1 and G_2 are $C1$-class, and since $\delta_1 = \delta_2$, it follows that we can give an edge cover color G_1 and G_2 with δ_1 colors. If we use an extra color to color all the edges in L, then $\chi_c'(G_L) \geq \delta_1 + 1 = \delta(G_L)$, and so G_L is $C1$-class.

Secondly, assume that (b) holds. Order the vertices of G_1, $u_1, \cdots, u_{n_1}$, so that all the vertices in a same connected component of G_1 are consecutive, and such that if u_i belongs to a clique K_t and u_j belongs to a clique K_s with $t > s$ then $i > j$. Similarly, we can order the vertices of G_2, $v_1, \cdots, v_{n_2}$, so that all the vertices in a same connected component of G_2 are consecutive, and such that if v_i belongs to a clique K_t and v_j belongs to a clique K_s with $t > s$, then $i > j$.

Let $\Phi = \{a_0, \cdots, a_{\delta_1}\}$ be the $\delta_1 + 1$ edge coloring of G_1 obtained in the following way: to every edge $u_i u_j$ assign color a_h with $h = (i+j) \bmod (\delta_1 + 1)$. To show that each vertex misses at most one color, we only need verify that at least δ_1 colors presented at each vertex.

For this purpose, assume that the edges $u_i u_j$ and $u_j u_k$ (with $i \neq k$) have been assigned the same color a_h. Then, by construction, $h = (i + j) \bmod (\delta_1 + 1)$ and $h = (j+k) \bmod (\delta_1 + 1)$. It follows that $| k - i | = t(\delta_1 + 1)$ for some integer t, which implies that $| k - i | \geq \delta_1 + 1$. On the other hand, the chosen ordering of the vertices of G_1 implies that $u_i u_j$ and $u_j u_k$ (with $i \neq k$) will be assigned different colors if $| k - i | \leq \delta_1$. So there are at least δ_1 colors presented at each vertex of G_1.

Note that, by construction, for every $i = 1, \cdots, n_1$, vertex u_i misses color $a_{(2i)\ mod\ (\delta_1+1)}$. Since $\delta_1 = \delta_2$, we can color the edges of G_2 in a similar way using the same colors in Φ: to every edge v_iv_j of G_2, assign color a_h with $h = (i+j)\ mod\ (\delta_1 + 1)$. By construction, for every $i = 1, \cdots, n_2$, vertex v_i misses color $a_{(2i)\ mod\ (\delta_1+1)}$.

Now we are ready to choose the desired minimum edge cover L of B_G: $L = \{u_iv_i, i = 1, \cdots, n_1\}$ when $n_1 = n_2$ or $L = \{u_iv_i, i = 1, \cdots, n_1\} \bigcup \{u_{(i\ mod\ n_1)+1}v_i, i = n_1+1, \cdots, n_2\}$ when $n_1 < n_2$. Indeed, for every $i = 1, \cdots, n_2$, we can assign to edge incident to v_i in L the color $a_{(2i)\ mod\ (\delta_1+1)}$. Thus, G_L is $C1$-class and the theorem follows. ■

By Theorem 2.4, it is easy to verify the following results.

Corollary 2.5. *Let $G = G_1 + G_2$ be a join graph with $\delta_1 = \delta_2$. If both G_1 and G_2 are $C1$-class, or if both G_1 and G_2 are disjoint unions of cliques, then $\chi'_c(G) = \delta(G)$.*

Corollary 2.6. *Let G be a complete graph with an even number of vertices. Then $\chi'_c(G) = \delta(G)$.*

Now we show that, if G is a regular join graph with $\delta_1 = \delta_2$, then for every minimum edge cover L of B_G, the corresponding graph G_L is $C1$-class, and so G is $C1$-class.

Theorem 2.7. *Every regular join graph $G = G_1 + G_2$ with $\delta_1 = \delta_2$ is $C1$-class.*

Proof. Let m_i denote the number of edges of G_i, $i = 1, 2$. Since G is regular and that $\delta_1 = \delta_2$, it follows that $n_1 = n_2$ and $m_1 = m_2$. Let $\Phi_1 = \{a_1, \cdots, a_{\delta_1+1}\}$ be an equalized edge coloring of G_1; and let $\Phi_2 = \{b_1, \cdots, b_{\delta_2+1}\}$ be an equalized edge coloring of G_2. Since Φ_1 is equalized, each color a_i, $(i = 1, \cdots, \delta_1 + 1)$ is missed by exactly $n_1 - 2\lfloor m_1/(\delta_1+1)\rfloor$ or $n_1 - 2\lceil m_1/(\delta_1+1)\rceil$ vertices of G_1; similarly, each color b_i, $(i = 1, \cdots, \delta_2+1)$ is missed by exactly $n_2 - 2\lfloor m_2/(\delta_2 + 1)\rfloor$ or $n_2 - 2\lceil m_2/(\delta_2 + 1)\rceil$ vertices of G_2.

Without loss of generality, we can assume that colors $a_1, \cdots, a_p$ are missed by exactly $n_1 - 2\lfloor m_1/(\delta_1+1)\rfloor$ vertices of G_1, that colors $a_{p+1}, \cdots, a_{\delta_1+1}$ are missed by exactly $n_1 - 2\lceil m_1/(\delta_1 + 1)\rceil$ vertices of G_1, that colors $b_1, \cdots, b_q$ are missed by exactly $n_2 - 2\lfloor m_2/(\delta_2 + 1)\rfloor$ vertices of G_2, that colors $b_{q+1}, \cdots, b_{\delta_2+1}$ are missed by exactly $n_2 - 2\lceil m_2/(\delta_2 + 1)\rceil$ vertices of G_2.

It is easy to verify that G_1 is δ_1-regular and that G_2 is δ_2-regular, and so each vertex u_i of G_1 misses exactly one color a_j and each vertex v_i of G_2 misses exactly one color b_h. So, we have

$$p(n_1 - 2\lfloor m_1/(\delta_1+1)\rfloor) + (\delta_1+1-p)(n_1 - 2\lceil m_1/(\delta_1+1)\rceil) = n_1,$$

$$q(n_2 - 2\lfloor m_2/(\delta_2+1)\rfloor) + (\delta_2+1-q)(n_2 - 2\lceil m_2/(\delta_2+1)\rceil) = n_2.$$

Since $m_1 = m_2$ and $n_1 = n_2$, so we have

$$(p-q)(\lfloor m_1/(\delta_1+1)\rfloor - \lceil m_1/(\delta_1+1)\rceil) = 0.$$

But note that $\lfloor m_1/(\delta_1+1)\rfloor = \lceil m_1/(\delta_1+1)\rceil$ implies that $p = q$. Hence $p = q$, and we can assume that $a_i = b_i$ for every $i = 1, \cdots, \delta_1+1$. Now, let $L = \{u_iv_i : i = 1, \cdots, \delta_1+1\}$. For every $i = 1, \cdots, \delta_1+1$, we can assign to edge u_iv_i the same color missed at both u_i and v_i. Now we get an edge cover coloring of G_L with δ_1+1 colors, and so G is $C1$-class. ∎

From the results above, we probably find some general sufficient conditions for the existence of minimum edge cover L of B_G such that the corresponding graph G_L is $C1$-class, and so the corresponding join graph G is $C1$-class.

Other interesting results on the edge cover coloring can be found in [6], [8] and [9].

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