# On Cayley Graphs of Symmetric Inverse Semigroups*

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**Abstract**

Let $\mathcal{I}_X$ be the symmetric inverse semigroup on a finite nonempty set $X$, and let $A$ be a subset of $\mathcal{I}_X^* = \mathcal{I}_X \setminus \{0\}$. Let $\mathrm{Cay}(\mathcal{I}_X^*, A)$ be the graph obtained by deleting vertex 0 from the Cayley graph $\mathrm{Cay}(\mathcal{I}_X, A)$. We obtain conditions on $\mathrm{Cay}(\mathcal{I}_X^*, A)$ for it to be $\mathrm{ColAut}_A(\mathcal{I}_X^*)$-vertex-transitive and $\mathrm{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive. The basic structure of vertex transitive $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is characterized. We also investigate the undirected Cayley graphs of symmetric inverse semigroups, and prove that the generalized Petersen graph can be constructed as a connected component of a Cayley graph of a symmetric inverse semigroup, by choosing an appropriate connecting set.

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# 1 Introduction and Preliminaries

The definition of a Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. Cayley graphs of groups have received serious attention, and many algebraic and combinatorial properties have been actively investigated (see, in particular, [3, 14]). Let $S$ be a semigroup, and $A$ a subset of $S$. The *Cayley graph* $\mathrm{Cay}(S, A)$ of $S$ with respect to $A$ is defined as the graph with vertex set $S$ and edge set $E(Cay(S, A))$ consisting of those ordered pairs $(x, y)$, where $sx = y$ for some $s \in A$. The set $A$ is called the *connecting set* of $\mathrm{Cay}(S, A)$. Cayley graphs of semigroups are generalizations of Cayley graphs of groups. One of the earliest

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references on this subject is [1]; see also [17] for another example of early work. The whole Section 2.4 of the book [9] is devoted to the Cayley graphs of semigroups. All vertex-transitive Cayley graphs produced by periodic semigroups are characterized in [11]. A combinatorial property of infinite semigroups defined in terms of Cayley graphs has been investigated (see [4, 12, 16]). In particular, the Cayley graphs of certain classes of semigroups have been studied, and the combinatorial properties of these Cayley graphs have been described. In [10], Kelarev studied the Cayley graphs of inverse semigroups. In [2], a complete description of all vertex transitive Cayley graphs of bands was obtained. The undirected Cayley graphs of right groups are investigated in [5]. We also refer to [13] for a survey of recent results on the Cayley graphs of semigroups.

In this paper, we obtain the conditions on graph $\mathrm{Cay}(\mathcal{I}_X^*, A)$, which is obtained by deleting vertex 0 from the Cayley graph of symmetric inverse semigroup $\mathcal{I}_X$ with respect to $A$, to be $\mathrm{ColAut}_A(\mathcal{I}_X^*)$ -vertex-transitive and $\mathrm{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive, respectively. The basic structure of vertex-transitive $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is also characterized. We investigate the undirected Cayley graphs of symmetric inverse semigroups, and give a method to construct the generalized Petersen graph, which can not be obtained from Cayley graphs of groups, as a component of the Cayley graphs of symmetric inverse semigroups.

Graphs considered in this paper are finite directed graphs without multiple edges but possibly with loops. For a graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ the vertex set and edge set of $\Gamma$, respectively. Let $D(V, E)$ be a graph with vertex set $V$ and edge set $E \subseteq V \times V$. A bijection $\phi : V \longrightarrow V$ is called an *automorphism* of the graph $D$ if $(\phi(u), \phi(v)) \in E$ for all $(u, v) \in E$. A graph $D(V, E)$ is said to be *vertex-transitive* if, for any two vertices $x$, $y \in V$, there exists an automorphism $\Phi \in Aut(D)$ such that $x\Phi = y$. All Cayley graphs of groups are vertex-transitive, since the group on which the Cayley graph is defined acts by right multiplication as a vertex-transitive group of automorphisms. If $S$ is a semigroup and $A \subseteq S$, then the automorphism group of $\mathrm{Cay}(S, A)$ is denoted by $\mathrm{Aut}_A(S)$. $\mathrm{Cay}(S, A)$ is said to be *$Aut_A(S)$-vertex-transitive* if, for any two vertices $x$, $y \in Cay(S, A)$, there exists $\Phi \in \mathrm{Aut}_A(S)$ such that $x\Phi = y$. An element $\Phi \in \mathrm{Aut}_A(S)$ will be called a *color preserving automorphism* if $sx = y$ implies $s(x\Phi) = y\Phi$, for every $x$, $y \in S$ and $s \in A$. If we regard an edge $(x, sx)$, for $s \in A$, as having color $s$, so that the elements of $A$ are thought of as colors associated with the edges of the Cayley graph, then every color-preserving automorphism maps each edge to an edge of the same color. Denote by $\mathrm{ColAut}_A(S)$ the sets of all color-preserving automorphisms of $\mathrm{Cay}(S, A)$. $\mathrm{Cay}(S, A)$ is said to be *$ColAut_A(S)$-vertex-transitive* if, for any two vertices $x$, $y \in S$, there exists an $\Phi \in \mathrm{ColAut}_A(S)$ such that $x\Phi = y$.

The following theorem due to A.V. Kelarev and C.E. Praeger makes a natural assumption of a finiteness condition on a semigroup $S$ and gives a description of all semigroups $S$ of this sort with properties of the Cayley graphs enjoying a complete analogy with the known properties of the Cayley graphs of groups.

**Theorem 1.1** ([11]) *Let $S$ be a semigroup, and let $A$ be a subset of $S$ which generates a subsemigroup $\langle A\rangle$ such that all principal left ideals of $\langle A\rangle$ are finite.*

*Then, the Cayley graph Cay(S, A) is ColAut$_A$(S)-vertex-transitive if and only if the following conditions hold:*

(a) $aS = S$, *for all* $a \in A$;

(b) $\langle A \rangle$ *is isomorphic to a direct product of a right zero band and a group;*

(c) $|\langle A \rangle s|$ *is independent of the choice of* $s \in S$. □

The next theorem reduces the problem of describing all automorphism vertex-transitive Cayley graphs of semigroups to the special case of completely simple semigroups.

**Theorem 1.2** ([11]) *Let S be a semigroup, and let A be a subset of S such that all principal left ideals of the subsemigroup* $\langle A \rangle$ *are finite. Then, the Cayley graph Cay(S, A) is* Aut$_A$(S)*-vertex-transitive if and only if the following conditions hold:*

(a) $AS = S$;

(b) $\langle A \rangle$ *is a completely simple semigroup;*

(c) *the Cayley graph* $Cay(\langle A \rangle, A)$ *is* $Aut_A(\langle A \rangle)$*-vertex-transitive;*

(d) $|\langle A \rangle s|$ *is independent of the choice of* $s \in S$. □

Recall that the *symmetric inverse semigroup* $\mathcal{I}_X$ is the set consisting of all partial one-one maps of $X$ with respect to the standard operation $\circ$ of composition of relations: if $\alpha$, $\beta \in \mathcal{I}_X$, then $(x, y) \in \alpha \circ \beta$ if and only if there exists $z$ in $X$ such that $(x, z) \in \alpha$ and $(z, y) \in \beta$. Thus $z = x\alpha$ and $y = z\beta$, and so $y = (x\alpha)\beta$. Notice also that $x \in dom(\alpha\beta)$ if and only if there exist $z$ and $y$ such that $(x, z) \in \alpha$ and $(z, y) \in \beta$. Thus $z \in im\alpha \cap dom\beta$, and so

$$dom(\alpha\beta) = (im\alpha \cap dom\beta)\alpha^{-1}, \; im(\alpha\beta) = (im\alpha \cap dom\beta)\beta.$$

If $im\alpha \cap dom\beta = \emptyset$, then it is said to be $\alpha\beta = 0$, the empty map, which we denote by 0. Hence $\alpha 0 = 0\alpha = 0$, for any $\alpha \in \mathcal{I}_X$.

Then we have the following theorem:

**Theorem 1.3** ([6], Theorem 5.1.5) $\mathcal{I}_X$ *is an inverse semigroup.* □

Now we show the analogue of Cayley's Theorem, a result due to Vagner and (independently) to Preston.

**Theorem 1.4** ([6]) *Let S be an inverse semigroup. Then there exists a symmetric inverse semigroup* $\mathcal{I}_X$ *and a monomorphism* $\phi$ *from S to* $\mathcal{I}_X$. □

Let $S$ be a semigroup which contains 0, and $A$ be a subset of $S$. It is obvious that if Cay$(S, A)$ is ColAut$_A(S)$-vertex-transitive (resp., Aut$_A(S)$-vertex-transitive), then

$$E(Cay(S, A)) = \{(x, x) \mid x \in S\}$$

by Theorem 1.1 (resp., Theorem 1.2). We denote by $\mathcal{I}_X^*$ the set $\mathcal{I}_X \setminus \{0\}$ and let $A$ be a subset of $\mathcal{I}_X$. In this paper, we consider the vertex-transitive Cayley graph Cay($\mathcal{I}_X^*, A$), which is the graph produced by deleting the vertex 0 and the edges adjacent to 0 from Cay($\mathcal{I}_X, A$).

Recall that the *permutation group* $S_n$ is the group of all bijections $X_n \to X_n$, where $X_n = \{1, 2, \ldots, n\}$. The elements of $S_n$ are called *permutations*. Obviously, $S_n$ is a subgroup of symmetric inverse semigroup $\mathcal{I}_{X_n}$. Let $i_1, i_2, \ldots, i_r$, $(r < n)$ be distinct elements of $X_n = \{1, 2, \ldots, n\}$. Then $(i_1 i_2 \cdots i_r)$ denotes the permutation that maps $i_1 \mapsto i_2$, $i_2 \mapsto i_3, \ldots, i_{r-1} \mapsto i_r$, and $i_r \mapsto i_1$, and maps every other element of $X_n$ onto itself. The permutation $(i_1 i_2 \cdots i_r)$ is called a *cycle* of length $r$ or an $r$-cycle; a 2-cycle is called a *transposition*. The permutations $\sigma_1, \sigma_2, \ldots, \sigma_r$ of $S_n$ are said to be *disjoint* provided that for each $1 \le i \le r$, and every $k \in X_n$, $(k)\sigma_i \ne k$ implies $(k)\sigma_j = k$ for all $j \ne i$. Denote by $e$ the identity of $S_n$. A nontrival subgroup $G$ of permutation group $S_n$ is said to be *semiregular* on $X_n$ if no nonidentity element of $G$ fixes an element of $X_n$.

**Theorem 1.5** ([7]) *Every nonidentity permutation in $S_n$ is a product of disjoint cycles, each of which has length at least 2. This product is unique up to the order of factors.* □

For terminology and notation not defined in this paper, we refer the reader to [6], [7] and [15].

# 2 Vertex-transitive Cayley Graph Cay($\mathcal{I}_X^*, A$)

Now we give the main result of this section.

**Theorem 2.1** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and $A$ be a subset of $\mathcal{I}_X^*$. Then the following statements are equivalent,*

(i) *Cay($\mathcal{I}_X^*, A$) is $ColAut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(ii) *Cay($\mathcal{I}_X^*, A$) is $Aut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(iii) *$A \subseteq S_n \subset \mathcal{I}_X$, and $\langle A \rangle$ is semiregular on $X$.*

Firstly, some lemmas are given which will be used in the proof of the main result. We define that $\mathcal{I}_X^* = I_1 \cup I_2 \cup \cdots \cup I_n$, where $I_k = \{\alpha \in \mathcal{I}_X \mid |dom\alpha| = |im\alpha| = k\}$, $k = 1, 2, \ldots, n$. It is obvious that $I_n = S_n$, the permutation group on $X$.

**Lemma 2.2** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and $A$ be a subset of $\mathcal{I}_X^*$. If Cay($\mathcal{I}_X^*, A$) is $Aut_A(\mathcal{I}_X)$-vertex-transitive, then $A \subseteq S_n \subset \mathcal{I}_X$.*

**Proof.** Suppose by contradiction that there exists $\alpha \in A$, such that $im\alpha = \{a_1, a_2, \ldots, a_k\}$, where $k < n$. We may choose $\gamma \in \mathcal{I}_X^*$ with $dom\gamma \cap im\alpha = \emptyset$. It follows that $\alpha\gamma = 0$. On the other hand, for any $\delta \in S_n \subset \mathcal{I}_X^*$, $\alpha\delta \ne 0$ for all

$\alpha \in \mathcal{I}_X^*$, thus $0 \notin A\delta$. It implies that the out-degree of $\alpha$ is less than the out-degree of $\delta$ in $\text{Cay}(\mathcal{I}_X^*, A)$. Then there is no $\Phi \in \text{Aut}_A(\mathcal{I}_X^*)$, such that $(\alpha)\Phi = \delta$. It contradicts the fact that the $\text{Cay}(\mathcal{I}_X^*, A)$ is $\text{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive. □

**Remark.** Let $X$ be a finite nonempty set with $|X| = n$, and let $\mathcal{I}_X$ be the symmetric inverse semigroup on $X$ with a subset $A$. From the definition of Cayley graph and the above discussion, there is no edge from $v$ to 0 if $A \subseteq S_n \subset \mathcal{I}_X$, for all $v \in \mathcal{I}_X^*$.

**Lemma 2.3** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A \subseteq S_n \subset \mathcal{I}_X$, where $S_n$ is the permutation group on $X$. Then the following statements are equivalent,*

(i) *$|\langle A\rangle\alpha|$ is independent of the choice of $\alpha \in \mathcal{I}_X^*$;*

(ii) *$\langle A\rangle$ is semiregular on $X$.*

**Proof.** (i)⇒(ii). Take any $\alpha \in S_n \subset \mathcal{I}_X$, then $|\langle A\rangle\alpha| = |\langle A\rangle|$. Hence we conclude that for any $\alpha \in \mathcal{I}_X^*$, $|\langle A\rangle\alpha| = |\langle A\rangle|$. Suppose by contradiction that there exists $i \in X$ and $\delta \in \langle A\rangle \setminus \{e\}$, such that $(i)\delta = (i)e = i$. We take $\binom{i}{i'} \in \mathcal{I}_X$. Then $\delta\binom{i}{i'} = e\binom{i}{i'} = \binom{i}{i'}$. Since $\delta \neq e$, then $|\langle A\rangle\binom{i}{i'}| < |\langle A\rangle|$, a contradiction. Hence $\langle A\rangle$ is semiregular on $X$.

(ii)⇒(i). Take any $\alpha = \binom{i_1\ i_2\ \cdots\ i_k}{i_1'\ i_2'\ \cdots\ i_k'} \in \mathcal{I}_X^*$. If $k = n$, then $\alpha \in S_n$, and it is obvious that $|\langle A\rangle\alpha| = |\langle A\rangle|$. If $1 \leq k < n$, then we suppose by contradiction that $|\langle A\rangle\alpha| < |\langle A\rangle|$ ($\mathcal{I}_X$ is finite). It follows that there exist $\beta, \gamma \in \langle A\rangle$ with $\beta \neq \gamma$, such that $\beta\alpha = \gamma\alpha$. It implies that $(i_j)\beta^{-1} = (i_j)\gamma^{-1}$, $j = 1, 2, \ldots, k$, then $(i_j)\beta^{-1}\gamma = i_j$. Since $\beta^{-1}\gamma \neq e$, it contradicts the fact that $\langle A\rangle$ is semiregular. Hence $|\langle A\rangle\alpha|$ is independent of the choice of $\alpha$ in $\mathcal{I}_X^*$. □

**Lemma 2.4** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A \subseteq S_n \subset \mathcal{I}_X^*$, where $S_n$ is the permutation group on $X$. Then $\langle A\rangle\alpha$ is equal to the strongly connected component of $\text{Cay}(\mathcal{I}_X^*, A)$ containing $\alpha$, for each $\alpha \in \mathcal{I}_X^*$.*

**Proof.** For any edge $(\alpha, \beta)$ of $\text{Cay}(\mathcal{I}_X^*, A)$, there exists $\gamma \in A \subseteq \langle A\rangle \subseteq S_n$ such that $\beta = \gamma\alpha$. It implies that $\alpha = \gamma^{-1}\beta$, where $\gamma^{-1} \in \langle A\rangle \subseteq S_n$. Hence there exist $\gamma_1, \gamma_2, \ldots, \gamma_k \in A$, such that $\gamma^{-1} = \gamma_1\gamma_2\cdots\gamma_k \in \langle A\rangle$, thus $\alpha = \gamma_1\gamma_2\cdots\gamma_k\beta$. It follows that $(\beta, \gamma_k\beta), (\gamma_k\beta, \gamma_{k-1}\gamma_k\beta), \ldots, (\gamma_2\cdots\gamma_k\beta, \alpha)$ are edges of $\text{Cay}(\mathcal{I}_X^*, A)$. Therefore, there exists a path from $\beta$ to $\alpha$. Since $(\alpha, \beta)$ was chosen arbitrarily, then every connected component of $\text{Cay}(\mathcal{I}_X^*, A)$ is strongly connected.

Let $C$ be the set of vertices of a connected component of $\text{Cay}(\mathcal{I}_X^*, A)$, and let $\alpha \in C$. Since $e\alpha = \alpha$ for any $\alpha \in \mathcal{I}_X^*$ and identity $e \in S_n$, and $\langle A\rangle$ is a subgroup of $\mathcal{I}_X^*$, then $\alpha = e\alpha \in \langle A\rangle\alpha$. Since $C$ is strongly connected and $\langle A\rangle\alpha$ is the set of all vertices $\delta$ of the Cayley graph $\text{Cay}(\mathcal{I}_X^*, A)$ such that there exists a path from $\alpha$ to $\delta$, then $\langle A\rangle\alpha \subseteq C$, thus $\langle A\rangle\alpha = C$, which completes the proof. □

Now we are in a position to give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** (i)⇒(ii) is obvious.

(ii)⇒(iii). Suppose that $\text{Cay}(\mathcal{I}_X^*, A)$ is $\text{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive, then $A \subseteq S_n \subset \mathcal{I}_X$ by Lemma 2.2. Since the vertex sets of all connected components of a vertex-transitive graph have the same cardinality, then $|\langle A\rangle\alpha|$ is independent of the choice of $\alpha \in \mathcal{I}_X^*$ by Lemma 2.4. Therefore $\langle A\rangle$ is semiregular on $X$ by Lemma 2.3.

(iii)⇒(i). For any $\alpha$, $\beta \in \mathcal{I}_X^*$, we define a mapping $\Phi$: $\mathcal{I}_X^* \longrightarrow \mathcal{I}_X^*$ such that $(\alpha)\Phi = \beta$, and show that $\Phi$ is a color preserving automorphism of $\text{Cay}(\mathcal{I}_X^*, A)$. It follows from Lemma 2.4 that $\alpha$, $\beta$ belong to the connected component $\langle A\rangle\alpha$ and $\langle A\rangle\beta$, respectively. If $\langle A\rangle\alpha \cap \langle A\rangle\beta \neq \emptyset$, then $\gamma\alpha = \gamma'\beta$, for some $\gamma, \gamma' \in \langle A\rangle$. For any $\theta \in \langle A\rangle\alpha$, $\theta = \gamma''\alpha = \gamma''\gamma^{-1}\gamma'\beta \in \langle A\rangle\beta$, for some $\gamma'' \in \langle A\rangle$. Similarly, $\langle A\rangle\beta \subseteq \langle A\rangle\alpha$. Therefore $\langle A\rangle\alpha = \langle A\rangle\beta$. It follows that $\mathcal{I}_X^*$ is a disjoint union of $\langle A\rangle\alpha$ for all $\alpha \in \mathcal{I}_X^*$. Consider two cases.

**Case 1.** $\langle A\rangle\alpha \neq \langle A\rangle\beta$. Then, for $\gamma \in \mathcal{I}_X^*$, we define

$$(\gamma)\Phi = \begin{cases} \theta\beta & \text{if } \gamma = \theta\alpha \text{ for some } \theta \in \langle A\rangle; \\ \theta\alpha & \text{if } \gamma = \theta\beta \text{ for some } \theta \in \langle A\rangle; \\ \gamma & \text{if } \gamma \notin \langle A\rangle\alpha \cup \langle A\rangle\beta. \end{cases}$$

Then $\Phi$ is well defined and is a bijection, since $|\langle A\rangle\alpha| = |\langle A\rangle\beta|$. Take any $\gamma \in \mathcal{I}_X^*$, $\delta \in A$. If $\gamma = \theta\alpha \in \langle A\rangle\alpha$, then $\delta\gamma \in \langle A\rangle\alpha$. Therefore the edge $(\gamma, \delta\gamma)$ is mapped by $\Phi$ to $((\gamma)\Phi, (\delta\gamma)\Phi) = (\theta\beta, \delta\theta\beta) = ((\gamma)\Phi, \delta((\gamma)\Phi))$. An analogous property holds if $\gamma \in \langle A\rangle\beta$. Also, $\Phi$ leaves invariant all edges involving vertices of $\mathcal{I}_X^* \setminus (\langle A\rangle\alpha \cup \langle A\rangle\beta)$. Thus $(\delta\gamma)\Phi = \delta((\gamma)\Phi)$, i.e. $\Phi$ is a color-preserving automorphism of $\text{Cay}(\mathcal{I}_X^*, A)$.

**Case 2.** $\langle A\rangle\alpha = \langle A\rangle\beta$. Then, for $\gamma \in \mathcal{I}_X^*$, we define

$$(\gamma)\Phi = \begin{cases} \theta\beta & \text{if } \gamma = \theta\alpha \in \langle A\rangle\alpha; \\ \gamma & \text{if } \gamma \notin \langle A\rangle\alpha. \end{cases}$$

Then $\Phi$ is well defined and is a bijection. Take any $\gamma \in \mathcal{I}_X^*$, $\delta \in A$. If $\gamma = \theta\alpha \in \langle A\rangle\alpha$, then $\delta\theta \in \langle A\rangle$ and $\delta\gamma \in \langle A\rangle\alpha$. Therefore $(\delta\gamma)\Phi = \delta\theta\beta$, and so $(\delta\gamma)\Phi = \delta((\gamma)\Phi)$. On the other hand, if $\gamma \notin \langle A\rangle\alpha$, then $\delta\gamma \notin \langle A\rangle\alpha$. Therefore $(\delta\gamma)\Phi = \delta\gamma = \delta((\gamma)\Phi)$. It follows that $\Phi$ is a color-preserving automorphism of $\text{Cay}(\mathcal{I}_X^*, A)$.

Thus we have verified that $\text{Cay}(\mathcal{I}_X^*, A)$ is $\text{ColAut}_A(\mathcal{I}_X^*)$-vertex-transitive. □

**Corollary 2.5** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A$ be a subset of $\mathcal{I}_X^*$. If Cayley graph $Cay(\mathcal{I}_X^*, A)$ is $ColAut_A(\mathcal{I}_X^*)$-vertex-transitive, then for any $\alpha \in A \subseteq S_n$, one of the following cases holds,*

*(i) $\alpha$ is a $n$-cycle, or*

*(ii) $\alpha$ is a product of $r$ disjoint $s$-cycles, where $rs = n$.*

*Moreover, if $A$ contains a $n$-cycle, then $A$ is a subset of a cyclic group of order $n$.*

**Proof.** Since the Cayley graph $\text{Cay}(\mathcal{I}_X^*, A)$ is $ColAut_A(\mathcal{I}_X^*)$-vertex-transitive, then $A \subseteq S_n \subset \mathcal{I}_X^*$ by Theorem 2.1. For any $\alpha \in A$ and $\alpha \neq e$, we suppose that $\alpha$ is not a $n$-cycle. Since $(i)\alpha \neq i$ for all $i \in X$ by Lemma 2.3, then $\alpha$ is a product of disjoint cycles whose total length is $n$ by Theorem 1.5. We may suppose by contradiction that $\alpha = (i_1 i_2 \cdots i_r)(i_{r+1} i_{i+2} \cdots i_{i+s}) \cdots (\cdots i_n)$ where $r < s$. It implies that $(i_k)\alpha^r = i_k$ for $k = 1, 2, \ldots, r$, and $(i_k)\alpha^r \neq i_k$ for $k = r+1, r+2, \ldots, r+s$. Since $\alpha^r \in \langle A \rangle \setminus \{e\}$, then $\langle A \rangle$ is not semiregular on $X$. It contradicts the fact that $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is $\mathrm{ColAut}_A(\mathcal{I}_X^*)$-vertex-transitive by Theorem 2.1. Hence $\alpha$ is a product of $r$ disjoint cycles with equal length $s$, where $rs = n$.

If $A$ contains a $n$-cycle $\alpha$, then $\langle \alpha \rangle \cong \mathbb{Z}_n$. It is easy to see that $\mathbb{Z}_n$ is isomorphic to a maximal semiregular subgroup of $S_n$. Also since $\langle A \rangle$ is semiregular on $X$, then $A \subseteq \langle \alpha \rangle \cong \mathbb{Z}_n$. □

**Lemma 2.6** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A \subseteq S_n \subset \mathcal{I}_X$, where $S_n$ is the permutation group on $X$. For any $\alpha$, $\beta \in \mathcal{I}_X$, where $\alpha = \binom{a_1 a_2 \ldots a_l}{a'_1 a'_2 \ldots a'_l}$ and $\beta = \binom{b_1 b_2 \ldots b_m}{b'_1 b'_2 \ldots b'_m}$, $(\alpha, \beta) \in E(Cay(\mathcal{I}_X^*, A))$ if and only if $l = m = k$, $a_i = (b_i)\gamma$ for some $\gamma \in A$ and $a'_i = b'_i$, for $i = 1, 2, \ldots, k$.*

**Proof.** For any $\alpha$, $\beta \in \mathcal{I}_X$, we get

$$\begin{aligned}(\alpha, \beta) \in E(Cay(\mathcal{I}_X^*, A)) &\Leftrightarrow \text{there exists } \gamma \in A, \text{ such that } \gamma\alpha = \beta \\ &\Leftrightarrow \begin{pmatrix} (a_1)\gamma^{-1} \ldots (a_l)\gamma^{-1} \\ a'_1 \quad \ldots \quad a'_l \end{pmatrix} = \begin{pmatrix} b_1 \ldots b_m \\ b'_1 \ldots b'_m \end{pmatrix} \\ &\Leftrightarrow l = m = k, a_i = (b_i)\gamma, \text{ for some } \gamma \in A \text{ and} \\ &\quad a'_i = b'_i, \text{ for } i = 1, 2, \ldots, k,\end{aligned}$$

as required. □

**Lemma 2.7** ([6]) *Let $X$ be a finite nonempty set with $|X| = n$, and let $\mathcal{I}_X$ be the symmetric inverse semigroup on $X$. Then*

$$|\mathcal{I}_X^*| = \sum_{k=1}^{n} \binom{n}{k}^2 k!.$$

□

Let $X$ be a finite nonempty set, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A$ be a subset of $\mathcal{I}_X$. We define $\Gamma(X, A)$ to be a graph whose vertex set is $X$ and edge set $E(\Gamma(X, A)) = \{(i, j) \mid (j)\alpha = i \text{ for some } \alpha \in A\}$. Now we are in a position to give the structure of $\mathrm{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive Cayley graph $\mathrm{Cay}(\mathcal{I}_X^*, A)$.

**Proposition 2.8** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A$ be a subset of $\mathcal{I}_X^*$. If $Cay(\mathcal{I}_X^*, A)$ is $Aut_A(\mathcal{I}_X^*)$-vertex-transitive, then $Cay(\mathcal{I}_X^*, A) \cong N\Gamma(X, A)$, where $N = \frac{\sum_{k=1}^{n} \binom{n}{k}^2 k!}{n}$, that is, $Cay(\mathcal{I}_X^*, A)$ is isomorphic to a disjoint union of $N$ copies of $\Gamma(X, A)$.*

**Proof.** For any $I_k \subseteq \mathcal{I}_X$, given any $k$-subset of $X$, denoted by $R = \{r_1, r_2, \ldots, r_k\}$, we define

$$S_R = \{\alpha \in I_k \mid (a_i)\alpha = r_i \text{ for all } (a_1, a_2, \ldots, a_k) \in X^k,\ i = 1, 2, \ldots, k\}.$$

It implies that $|S_R| = \binom{n}{k}k!$ by Lemma 2.7. We can denote the elements of $S_R$ by the $k$-ordered arrays $(a_1, a_2, \ldots, a_k)$. For any $(a_1, a_2, \ldots, a_k)$, $(b_1, b_2, \ldots, b_k) \in S_R$, it follows that

$$((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k)) \in E(Cay(\mathcal{I}_X^*, A))$$

if and only if $(a_i)\alpha = b_i$ for some $\alpha \in A$, $i = 1, 2, \ldots, k$, by Lemma 2.6.

We define a relation $\beta$ on $S_R$ as

$$(a_1, a_2, \ldots, a_k)\beta(b_1, b_2, \ldots b_k) \text{ if } (a_1, a_2, \ldots, a_k) \in (b_1, b_2, \ldots, b_k)\langle A\rangle.$$

It follows that $\beta$ is an equivalence relation on $S_R$. Since $\langle A\rangle$ is semiregular on $X$, then $|((a_1, a_2, \ldots, a_k))\langle A\rangle| = |\langle A\rangle|$, for any $(a_1, a_2, \ldots, a_k) \in S_R$. Furthermore, $\beta_{(a_1, a_2, \ldots, a_k)}$ is equal to the connected component of $\text{Cay}(\mathcal{I}_X^*, A)$ containing $(a_1, a_2, \ldots, a_k)$ by Lemma 2.4.

We also define a relation $\gamma$ on $X$ as $i\gamma j$ if $i \in (j)\langle A\rangle$. It implies that $\gamma$ is an equivalence relation on $X$. On the other hand, since $\langle A\rangle$ is semiregular on $X$, then $|(i)\langle A\rangle| = |\langle A\rangle|$, for any $i \in X$. Here $\gamma_i$ is equal to the connected component of $\Gamma(X, A)$ containing $i$ by the definition of $\Gamma(X, A)$.

Denote by $D_\beta$ the connected component of $\text{Cay}(\mathcal{I}_X^*, A)$, which is induced by $\beta_{(a_1, a_2, \ldots, a_k)}$. For any $a_i \in \{a_1, a_2, \ldots, a_k\}$, denote by $D_\gamma$ the connected component of $\Gamma(X, A)$, which is induced by $\gamma_{a_i}$. Define $\Phi$: $D_\beta \longrightarrow D_\gamma$ by $(a_1, a_2, \ldots, a_k) \mapsto a_i$. Since $|\beta_{(a_1, a_2, \ldots, a_k)}| = |\gamma_{a_i}| = |\langle A\rangle|$, and $b_i \neq c_i$ for any distinct two $(b_1, b_2, \ldots, b_k)$, $(c_1, c_2, \ldots, c_k) \in \beta_{(a_1, a_2, \ldots, a_k)}$, then $\Phi$ is a bijection. Let $(a_1, a_2, \ldots, a_k)$, $(b_1, b_2, \ldots, b_k) \in \beta_{(a_1, a_2, \ldots, a_k)}$. Then

$$\begin{aligned}
&((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k)) \in E(Cay(\mathcal{I}_X^*, A))\\
\Leftrightarrow\ & a_i = (b_i)\alpha, i = 1, 2, \ldots, k, \text{ for some } \alpha \in A\\
\Leftrightarrow\ & (a_i, b_i) \in E(\Gamma(X, A))\\
\Leftrightarrow\ & (\Phi((a_1, a_2, \ldots, a_k)), \Phi((b_1, b_2, \ldots, b_k))) \in E(\Gamma(X, A)).
\end{aligned}$$

Therefore $\Phi$ is an isomorphism from $D_\beta$ to $D_\gamma$.

If $\gamma_i \neq \gamma_j$ for some $i, j \in X$. We show that $D_{\gamma_i} \cong D_{\gamma_j}$, where $D_{\gamma_i}$ (resp. $D_{\gamma_j}$) is the subgraph of $\Gamma(X, A)$ induced by $\gamma_i$ (resp. $\gamma_j$). Define $\Lambda$: $D_{\gamma_i} \longrightarrow D_{\gamma_j}$ by $((i)\alpha)\Lambda = (j)\alpha$, for some $\alpha \in \langle A\rangle$. Since $|\gamma_i| = |\gamma_j| = |\langle A\rangle|$, then $\Lambda$ is a bijection, and we have that

$$\begin{aligned}
((i)\alpha, (i)\beta) \in E(D_{\gamma_i}), \text{ for some } \alpha, \beta \in \langle A\rangle &\Leftrightarrow (i)\alpha = (i)\beta\beta', \text{ for some } \beta' \in A\\
&\Leftrightarrow \beta\beta'\alpha^{-1} = e, \text{ since } \langle A\rangle \text{ is semiregular}\\
&\Leftrightarrow (j)\beta\beta'\alpha^{-1} = j \Leftrightarrow (j)\alpha = (j)\beta\beta'\\
&\Leftrightarrow ((j)\alpha, (j)\beta) \in E(D_{\gamma_j})\\
&\Leftrightarrow (((i)\alpha)\Lambda, ((i)\beta)\Lambda) \in E(D_{\gamma_j}).
\end{aligned}$$

Therefore, $\Lambda$ is a isomorphism from $D_{\gamma_i}$ to $D_{\gamma_j}$, thus there are $\frac{n}{|\langle A\rangle|}$ isomorphic connected components in $\Gamma(X, A)$, and $\frac{\binom{n}{k}k!}{|\langle A\rangle|}$ isomorphic connected components in the subgraph induced by $S_R$ by Lemma 2.4 and Theorem 2.1. Then the subgraph of $\mathrm{Cay}(\mathcal{I}_X^*, A)$ induced by $S_R$ is isomorphic to

$$\frac{\binom{n}{k}k!/|\langle A\rangle|}{n/|\langle A\rangle|} = \frac{\binom{n}{k}k!}{n}$$

copies of $\Gamma(X, A)$. Moreover, the subgraph induced by $I_k$ is isomorphic to $\frac{\binom{n}{k}^2 k!}{n}$ copies of $\Gamma(X, A)$ by Lemma 2.7 and the fact that $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is $\mathrm{ColAut}_A(\mathcal{I}_X^*)$-vertex-transitive. Therefore, $\mathrm{Cay}(\mathcal{I}_X^*, A) \cong N\Gamma(X, A)$, where $N = \frac{\sum_{k=1}^{n}\binom{n}{k}^2 k!}{n}$.□

We consider the strongly connected component of $\mathrm{Cay}(\mathcal{I}_X^*, A)$ induced by $\langle A\rangle e = \langle A\rangle$, where $e$ is the identity of $S_n \subset \mathcal{I}_X$. It is obvious that this connected component is a Cayley graph of group $\langle A\rangle$ with respect to $A$. If $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is $\mathrm{Aut}_A(\mathcal{I}_X^*)$-vertex-transitive, then its connected components are isomorphic to each other. Therefore,

$$\mathrm{Cay}(\mathcal{I}_X^*, A) \cong \frac{\sum_{k=1}^{n}\binom{n}{k}^2 k!}{|\langle A\rangle|} Cay(\langle A\rangle, A)$$

by Lemma 2.7. On the other hand, it is well known that the Cayley graphs of groups are vertex-transitive, so are the disjoint union of Cayley graphs of groups. Hence we have the second main result of this paper which is a stronger version of Theorem 2.1. It shows that any $Aut_A(\mathcal{I}_X^*)$-vertex-transitive graph $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is a disjoint union of Cayley graphs of some group.

**Corollary 2.9** *Let $X$ be a finite nonempty set with $|X| = n$, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A$ be a subset of $\mathcal{I}_X^*$. Then the following statements are equivalent,*

(i) *$Cay(\mathcal{I}_X^*, A)$ is $ColAut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(ii) *$Cay(\mathcal{I}_X^*, A)$ is $Aut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(iii) *$A \subseteq S_n \subset \mathcal{I}_X$, and $\langle A\rangle$ is semiregular on $X$;*

(iv) *$Cay(\mathcal{I}_X^*, A) \cong \frac{\sum_{k=1}^{n}\binom{n}{k}^2 k!}{|\langle A\rangle|} Cay(\langle A\rangle, A)$.* □

**Example 2.10** Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and let $\mathcal{I}_X$ be the symmetric inverse semigroup on $X$. Put $A = \{(123)(456)(789), (147)(258)(369)\} \subset S_9$. It is routine to verify that $\langle A\rangle$ is semiregular on $X$. Then $\mathrm{Cay}(\mathcal{I}_X^*, A)$ is a disjoint union of $\frac{\sum_{k=1}^{9}\binom{9}{k}^2 k!}{9}$ copies of $\Gamma(X, A)$ (Illustrated by Figure 1). □

The following result is a special case of Corrollary 2.9. It gives a complete description of the $Aut_A(\mathcal{I}_X^*)$-vertex-transitive graph $\mathrm{Cay}(\mathcal{I}_X^*, A)$ when $|X|$ is a prime.

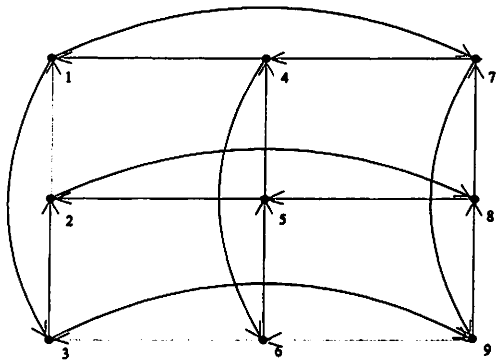


Figure 1: $\Gamma(X, A)$

**Corollary 2.11** *Let $X$ be a finite nonempty set with $|X| = p$, where $p$ is a prime, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$, and let $A$ be a subset of $\mathcal{I}_X^*$. Then the following statements are equivalent,*

(i) *$Cay(\mathcal{I}_X^*, A)$ is $ColAut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(ii) *$Cay(\mathcal{I}_X^*, A)$ is $Aut_A(\mathcal{I}_X^*)$-vertex-transitive;*

(iii) *$Cay(\mathcal{I}_X^*, A) \cong \frac{\sum_{k=1}^{p} \binom{p}{k}^2 k!}{p} Cay(\mathbb{Z}_p, \Phi(A))$, where $\Phi$ is an isomorphism from $\langle A \rangle$ to $\mathbb{Z}_p$.* □

**Proof.** (i)⇒ (ii) and (iii)⇒(i) are obvious. We only need to show that (ii) implies (iii). Let Cay$(\mathcal{I}_X^*, A)$ be $Aut_A(\mathcal{I}_X^*)$-vertex-transitive, we have $A \subseteq S_p \subset \mathcal{I}_X^*$ by Theorem 2.1. Since $p$ is a prime, then each nonidentity element of $A$ is a $p$-cycle. Also since that $\langle A \rangle$ is semiregular on $X$, then $\langle A \rangle \cong \mathbb{Z}_p$. We may suppose that $X = \{0, 1, \ldots, p-1\}$, and $\alpha = (01 \cdots p-1) \in A$, thus $\langle A \rangle = \langle \alpha \rangle$, Define a mapping from $\langle A \rangle$ to $\mathbb{Z}_p$ by

$$\Phi : \langle A \rangle \longrightarrow \mathbb{Z}_p;$$
$$(i_0 i_1 \cdots i_{p-1}) \mapsto \overline{i_1 - i_0}, \; e \mapsto \overline{0}.$$

It is easy to verify that $\Phi$ is an isomorphism. We define a bijection as follows,

$$\Lambda : X \longrightarrow \mathbb{Z}_p; \; i \mapsto \overline{i}.$$

It follows that

$$\begin{aligned}(i, j) \in E(\Gamma) &\Leftrightarrow i = (j)\alpha, \text{ for some } \alpha \in A \\ &\Leftrightarrow \overline{j - i} \in \Phi(A) \\ &\Leftrightarrow (i, j) \in E(\text{Cay}(\mathbb{Z}_p, \Phi(A)).\end{aligned}$$

We conclude that $\Lambda$ is an isomorphism from $\Gamma(X.A)$ to Cay$(\mathbb{Z}_p, \Phi(A))$. Hence (iii) holds by Proposition 2.8 and Corollary 2.9. □

## 3 Undirected Cayley Graphs and Generalized Petersen Graph

In this section, we investigate the undirected Cayley graphs of symmetric inverse semigroups, and construct the generalized Petersen graph as a connected component of the Cayley graph of the symmetric inverse semigroup. The following lemma was produced by A.V. Kelarev which characterizes all undirected Cayley graphs of semigroups.

**Lemma 3.1** ([8]) *Let $S$ be a finite semigroup, $A$ a subset of $S$. Then Cayley graph $Cay(S, A)$ is undirected if and only if the following conditions hold,*

(i) $AS = S$;

(ii) $\langle A\rangle = \mathcal{M}[G; I, \Lambda; P]$ *is a completely simple semigroup;*

(iii) *for any* $(i, g, \lambda) \in A$, $j \in I$, *there exists* $\mu \in \Lambda$ *such that* $(j, P_{\lambda j}^{-1} g p_{\mu i}^{-1}, \mu)$ *belongs to* $A$. □

Here we give the complete description of all undirected Cayley graphs of symmetric inverse semigroup.

**Theorem 3.2** *Let $X$ be a finite nonempty set, $\mathcal{I}_X$ the symmetric inverse semigroup on $X$ , and let $A$ be a subset of $\mathcal{I}_X$. Then Cayley graph $Cay(\mathcal{I}_X, A)$ is undirected if and only if $A \subseteq S_n \subset \mathcal{I}_X$, and $A^{-1} = A$.*

**Proof.** Necessity. We suppose that there exists $\beta \in A$, such that $|im\beta| = |dom\beta| = k < n$. We may choose $\alpha \in \mathcal{I}_X$, such that $im\beta \cap dom\alpha = \emptyset$, thus $\beta\alpha = 0$, that is $(\alpha, 0) \in E(\text{Cay}(\mathcal{I}_X, A))$. On the other hand, there is no directed edge from 0 to any non-zero vertex. It contradicts the fact that $\text{Cay}(\mathcal{I}_X, A)$ is undirected. It shows that for any $\beta \in A$, $|im\beta| = |dom\beta| = n$. Therefore $A \subseteq S_n \subset \mathcal{I}_X$.

For any $\alpha \in \mathcal{I}_X$, $\gamma \in A$, we have $(\alpha, \gamma\alpha) \in E(\text{Cay}(\mathcal{I}_X, A))$, then $(\gamma\alpha, \alpha) \in E(\text{Cay}(\mathcal{I}_X, A))$. It implies that $\alpha = \gamma'\gamma\alpha$, for some $\gamma' \in A$. Since $A \subseteq S_n$, then $\gamma'\gamma = e$. Hence $\gamma'$ and $\gamma^{-1} \in A$. It follows that $A \subseteq A^{-1}$. The converse case is obviously. Therefore $A^{-1} = A$.

Sufficiency. Since $A \subseteq S_n \subset \mathcal{I}_X$, then $A\mathcal{I}_X = \mathcal{I}_X$. The condition (ii) and (iii) of Lemma 3.1 hold since $\langle A\rangle$ is a group. Therefore Cayley graph $\text{Cay}(\mathcal{I}_X, A)$ is undirected. □

Let $k$ and $n$ be positive integers with $n > 2k$. The generalized Petersen graph $P_{k,n}$ is the simple graph with vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ and undirected edges $(x_i, x_{i+1})$, $(y_i, y_{i+k})$, $(x_i, y_i)$, $i = 1, 2, \ldots, n$, indices being taken modulo $n$ (Note that $P_{2,5}$ is the Petersen graph). It is known that the generalized Petersen graph can not be obtained as the Cayley graphs of groups (see [3]).

Let $\mathcal{I}_X$ be the symmetric inverse semigroup on a finite nonempty set $X$, and $A$ be a subset of $\mathcal{I}_X$. We put $I_1 = \{\alpha \in \mathcal{I}_X \mid |dom\alpha| = |im\alpha| = 1\}$, and define $\Gamma$ be the subgraph of $\text{Cay}(\mathcal{I}_X^*, A)$ induced by $I_1$. The following result shows that $\Gamma$ is a disjoint union of the generalized Petersen graphs via constructing a appropriate set $A$.

**Theorem 3.3** *Let $X = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ be a finite nonempty set, $\mathcal{I}_X$ be the symmetric inverse semigroup on $X$, take $S = \{(x_1x_2\cdots x_n), (y_1y_2\cdots y_n)^k, (x_1y_1)(x_2y_2)\cdots(x_ny_n)\}$ and $A = S \cup S^{-1}$. Then $\Gamma \cong 2nP_{k,n}$.*

**Proof.** Let $V(P_{k,n}) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$, $(x_i, x_{i+1})$, $(y_i, y_{i+k})$ and $(x_i, y_i)$ are undirected edges, for $i = 1, 2, \ldots, n$, indices being taken modulo $n$. For any $x_i \in X$, we define $I_1^{x_i} = \{\alpha \in I_1 \mid im\alpha = \{x_i\}\}$, and define a map

$$\phi_i : I_1^{x_i} \longrightarrow V(P_{k,n}) \text{ by } \binom{x_j}{x_i} \mapsto x_j.$$

It is obvious that $\phi_i$ is a bijection. Let $u$, $v \in I_1^{x_i}$ If $u = \binom{x_l}{x_i}$, $v = \binom{x_m}{x_i}$, for some $x_l$, $x_m \in X$, where $l$, $m$ are taken modulo $n$. Then

$$\begin{aligned}(u,v) \in E(\Gamma) &\Leftrightarrow l = m \pm 1, \text{ by } (x_1x_2\cdots x_n) \in A \text{ and Lemma 2.6}\\ &\Leftrightarrow (x_l, x_m) \in E(P_{k,n})\\ &\Leftrightarrow (\phi_i(u), \phi_i(v)) \in E(P_{k,n}).\end{aligned}$$

If $u = \binom{y_l}{x_i}$, $v = \binom{y_m}{x_i}$, for some $y_l$, $y_m \in X$, where $l$, $m$ are taken modulo $n$. Then

$$\begin{aligned}(u,v) \in E(\Gamma) &\Leftrightarrow l = m \pm k, \text{ by } (y_1y_2\cdots y_n)^k \in A \text{ and Lemma 2.6}\\ &\Leftrightarrow (y_l, y_m) \in E(P_{k,n})\\ &\Leftrightarrow (\phi_i(u), \phi_i(v)) \in E(P_{k,n}).\end{aligned}$$

If $u = \binom{x_l}{x_i}$, $v = \binom{y_m}{x_i}$, for some $x_l$, $y_m \in X$, where $l$, $m$ are taken modulo $n$. Then

$$\begin{aligned}(u,v) \in E(\Gamma) &\Leftrightarrow l = m, \text{ by } (x_1y_1)(x_2y_2)\cdots(x_ny_n) \in A \text{ and Lemma 2.6}\\ &\Leftrightarrow (x_l, y_m) \in E(P_{k,n})\\ &\Leftrightarrow (\phi_i(u), \phi_i(v)) \in E(P_{k,n}).\end{aligned}$$

Therefore, the subgraph of $\Gamma$ induced by $I_1^{x_i}$ is isomorphic to $P_{k,n}$. On the other hand, for any $i$, $j \in \{1, 2, \ldots, n\}$ and $i \neq j$, it is obvious that the subgraphs of $\Gamma$ induced by $I_1^{x_i}$, $I_1^{x_j}$, $I_1^{y_i}$ and $I_1^{y_j}$ are isomorphic to each other by Lemma 2.6. It follows that the subgraph induced by $I_1$ is isomorphic to $2nP_{k,n}$. Therefore $\Gamma \cong 2nP_{k,n}$, as required. □

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