QUALITATIVE BEHAVIOR OF A RATIONAL DIFFERENCE EQUATION

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ABSTRACT

In this paper we study the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_{n-k}^{r} x_{n-(2k+1)}^{s}}, \quad n = 0, 1, \ldots$$

where a, b, c are nonnegative parameters, initial conditions are nonnegative real numbers, k is a nonnegative integer and $r, s \geq 1$.
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1. INTRODUCTION

Difference equations play an important role in the analysis of the mathematical models of biology, physics and engineering. Qualitative analysis of rational difference equations is a fertile research area. Recently there has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations. For example see Refs. [1-18].

Hamza et al. [1] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}.$$

Also Hamza [2] investigated the global stability, periodic nature, oscilation and the boundedness of the difference equation

$$x_{n+1} = \frac{A \prod_{i=l}^{k} x_{n-2i-1}}{B + C \prod_{i=l}^{k-1} x_{n-2i}}.$$

Elabbasy et al. [6] investigated some qualitative behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

El-Owaidy et al. [9] studied the dynamics of the recurcive sequence

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}.$$

Karatas [16] studied the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{A x_{n-2l}}{B + C \prod_{i=0}^{2k} x_{n-i}}.$$

Cinar [4] investigated the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Yalcinkaya [10] studied the global behaviour of the rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Elsayed [7] investigated the qualitative behavior of the solution of the difference equation

$$x_{n+1} = a x_n + \frac{b x_n x_{n-1}}{c x_n + d x_{n-1}}.$$

In this paper we study the behavior of the positive solutions of the following difference equation

$$x_{n+1} = \frac{a x_{n-k}}{b + c x_{n-k}^{r} x_{n-(2k+1)}^{s}}, \quad n = 0, 1, \ldots \tag{1.1}$$

where a, b, c are nonnegative real numbers, initial conditions are nonnegative, k is a nonnegative integer and $r, s \geq 1$.

2. Preliminaries

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \ n = 0, 1, ...$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. *An equilibrium point for Eq.(2.1) is a point* $\overline{x} \in I$ *such that* $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$.

Definition 2. *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period p if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

Definition 3. *(i) An equilibrium point* $\overline{x}$ *for Eq.(2.1) is called locally stable if for every* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that every solution* $\{x_n\}$ *with initial conditions* $x_{-k}, x_{-k+1}, ..., x_0 \in]\overline{x} - \delta, \overline{x} + \delta[$ *is such that* $x_n \in]\overline{x} - \epsilon, \overline{x} + \epsilon[$ *for all* $n \in \mathbb{N}$. *Otherwise* $\overline{x}$ *is said to be unstable.*

(ii) The equilibrium point $\overline{x}$ *of Eq.(2.1) is called locally asymptotically stable if it is locally stable and there exists* $\gamma > 0$ *such that for any initial conditions* $x_{-k}, x_{-k+1}, ..., x_0 \in]\overline{x} - \gamma, \overline{x} + \gamma[$, *the corresponding solution* $\{x_n\}$ *tends to* $\overline{x}$.

(iii) An equilibrium point $\overline{x}$ *for Eq.(2.1) is called a global attractor if every solution* $\{x_n\}$ *converges to* $\overline{x}$ *as* $n \to \infty$.

(iv) The equilibrium point $\overline{x}$ *of Eq.(2.1) is called globally asymptotically stable if it is locally asymptotically stable and is a global attractor.*

Let

$c_i = \frac{\partial f}{\partial x_{n-i}}(\overline{x}, ..., \overline{x}), \quad$ *for* $i = 0, 1, ..., k$

denote the partial derivatives of $f(x_n, x_{n-1}, ..., x_{n-k})$ *with respect to* x_{n-i} *evaluated at the equilibrium point* $\overline{x}$*of Eq.(2.1). Then the equation*

$$y_{n+1} = \sum_{i=0}^{k} c_i y_{n-i}, \qquad n = 0, 1, ... \tag{2.2}$$

is called the linearized equation associated with Eq.(2.1) about the equilibrium point $\overline{x}$.

The characteristic equation associated with Eq.(2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} c_i \lambda^{k-i} = 0. \tag{2.3}$$

Theorem 1. [17] *Assume that f is a* C^1 *function and let* $\overline{x}$ *be an equilibrium point of Eq.(2.1). Then the following statements are true.*

(i) If all roots of Eq.(2.3) lie in open disk $|\lambda| < 1$, *then* $\overline{x}$ *is locally asymptotically stable.*

(ii) If at least one root of Eq.(2.3) has absolute value greater than one, then $\overline{x}$ is unstable.

3. Dynamics of Eq.(1.1)

In this section, we study the dynamics of Eq.(1.1).

The change of variables $x_n = \left(\frac{b}{c}\right)^{\frac{1}{r+s}} y_n$ reduces *Eq.*(1.1) to the difference equation

$$y_{n+1} = \frac{py_{n-k}}{1 + y_{n-k}^r y_{n-(2k+1)}^s}, \quad n = 0, 1, ... \tag{3.1}$$

where $p = \frac{a}{b}$. We can see that *Eq.*(3.1) has two nonnegative equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = (p-1)^{\frac{1}{r+s}}$ when $p > 1$ and the zero equilibrium point only when $p \leq 1$.

The linearized equation associated with Eq.(3.1) about $\overline{y}$ is

$$z_{n+1} = \frac{p + (1-r)p\overline{y}^{r+s}}{\left(1+\overline{y}^{r+s}\right)^2} z_{n-k} - \frac{ps\overline{y}^{r+s}}{\left(1+\overline{y}^{r+s}\right)^2} z_{n-(2k+1)}, \quad n = 0, 1, ...$$

The characteristic equation associated with this equation is

$$\lambda^{2k+2} - \frac{p + (1-r)p\overline{y}^{r+s}}{\left(1+\overline{y}^{r+s}\right)^2}\lambda^{k+1} + \frac{ps\overline{y}^{r+s}}{\left(1+\overline{y}^{r+s}\right)^2} = 0.$$

Theorem 2. *The following statements are true:*

(i) If $p < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is locally asymptotically stable,

(ii) If $p > 1$, then the equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = (p-1)^{\frac{1}{r+s}}$ of Eq.(3.1) are unstable.

Proof. The linearized equation of *Eq.*(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} - pz_{n-k} = 0, \quad n = 0, 1, ...$$

The characteristic equation of *Eq.*(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{k+1}\left(\lambda^{k+1} - p\right) = 0.$$

So

$\lambda = 0$ and $\lambda = p^{\frac{1}{k+1}}$. In view of Theorem 1:

If $p < 1$, then $|\lambda| < 1$ for all roots and the equilibrium point $\overline{y}_1 = 0$ is locally asymptotically stable.

If $p > 1$, it follows that the equilibrium point $\overline{y}_1 = 0$ is unstable.

The linearized equation of $Eq.(3.1)$ about the equilibrium point $\overline{y}_2 = (p-1)^{\frac{1}{r+s}}$ is

$$z_{n+1} = \frac{p+r-pr}{p} z_{n-k} - \frac{sp-s}{p} z_{n-(2k+1)}, \qquad n = 0, 1, \ldots.$$

The associated characteristic equation is

$$\lambda^{2k+2} - \frac{p+r-rp}{p}\lambda^{k+1} + \frac{sp-s}{p} = 0.$$

In view of Theorem 1 it follows that the equilibrium point $\overline{y}_2 = (p-1)^{\frac{1}{r+s}}$ is unstable.

Theorem 3. *Assume that $p < 1$, then the equilibrium point $\overline{y}_1 = 0$ of $Eq.(3.1)$ is globally asymptotically stable.*

□

Proof. Let $\{y_n\}_{n=-(2k+1)}^{\infty}$ be a solution of $Eq.(3.1)$. From Theorem 2 we know that the equilibrium point $\overline{y}_1 = 0$ of $Eq.(3.1)$ is locally asymptotically stable. So it suffices to show that

$$\lim_{n\to\infty} y_n = 0.$$

We get

$$y_{n+1} \le p y_{n-k}$$

from $Eq.(3.1)$. Then it can be written for $t = 0, 1, \ldots$

(3.2) $$y_{t(k+1)+i} \le p^{t+1} y_{-(k+1-i)}, \quad i = 1, 2, \ldots, k+1$$

If $p < 1$, then $\lim_{t\to\infty} p^{t+1} = 0$

and

$$\lim_{n\to\infty} y_n = 0.$$

The proof is complete. □

Theorem 4. *The following statements are true:*

(i) If $p = 1$, then every solution of $Eq.(3.1)$ is bounded.

(ii) If $p > 1$, then every solution of $Eq.(3.1)$ is unbounded.

Proof. (i) Let $\{y_n\}_{n=-(2k+1)}^{\infty}$ be a solution of $Eq.(3.1)$. It follows from $Eq.(3.1)$ that

$$y_{n+1} = \frac{y_{n-k}}{1 + y_{n-k}^r y_{n-(2k+1)}^s} \le y_{n-k}$$

Then from inequalitiy (3.2) we have for $t = 0, 1, \ldots$

$$y_{t(k+1)+1} \le y_{-k},$$

$$y_{t(k+1)+2} \le y_{-k+1},$$

$$\ldots$$

$$y_{t(k+1)+k+1} \leq y_0.$$

So every solution of $Eq.(3.1)$ is bounded from above by $A = \max\{y_{-k}, y_{-k+1}, ..., y_0\}$.

(ii) Similarly from inequlitiy (3.2) we have for $t = 0, 1, ...$

$$y_{t(k+1)+1} \leq p^{t+1} y_{-k},\ y_{t(k+1)+2} \leq p^{t+1} y_{-k+1}, ...,\ y_{t(k+1)+k+1} \leq p^{t+1} y_0.$$

So every solution of $Eq.(3.1)$ is unbounded when $p > 1$. □

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