On Diameter Stability of the Johnson Graph*

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Abstract

In this paper, we study the edge deletion preserving the diameter of the Johnson graph $J(n,k)$. Let $un^-(G)$ be the maximum number of edges of a graph G whose removal maintains its diameter. For Johnson graph $J(n,k)$, we give upper and lower bounds to the number $un^-(J(n,k))$, namely: $\binom{k}{2}\binom{n}{k+1} \leq un^-(J(n,k)) \leq \binom{k+1}{2}\binom{n}{k+1} - \lceil(1+\frac{1}{2k})(\binom{n}{k}-1)\rceil$, for $n \geq 2k \geq 2$.

Keywords: Johnson graph; diameter; edge deletion

1 Introduction

Let n and k be fixed positive integers with $n \geq k$. The vertices of *Johnson graph* $J(n,k)$ are the k-subsets of $\Omega = \{1,2,\cdots,n\} \triangleq [n]$, and two such subsets are adjacent if and only if their intersection has size $k-1$. Then $J(n,k)$ is a $k(n-k)$-regular graph with $\binom{n}{k}$ vertices. In particular, $J(n,1)$ is a complete graph K_n with n vertices, and $J(n,n)$ is only one vertex. Further $J(n,k) \cong J(n,n-k)$ [6]. So we may suppose that $n \geq 2k$. For convenience, we always denote the smallest element of a k-subset a of $[n]$ by a_1. For the notation and terminology not defined here, we refer to [1].

For a graph G, we use $d(x,y)$ to denote the distance between the vertices x and y. Then the diameter of G is denoted by $d(G) = \max\{d(x,y) \mid x,y \in V(G)\}$. Pizaña [9] showed that the diameter of $J(n,k)$ is $\min\{k,n-k\}$. For given integers n and d, what is the minimum number of edges of a graph on n vertices with the property that after deleting any edge, the remaining graph has diameter no more than d? This problem was first proposed by

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Vijayan and Murty [12]. About diameter vulnerability of graphs after edge deletion and edge addition, see refs. [3]-[5] and [7]-[15].

Graham and Harary [7] used $un^-(G)$ to denote the maximum number of edges of the graph G whose removal maintains the diameter. Bounds on $un^-(Q_n)$ were given [2]. In this paper we use a similar way to consider $J(n,k)$ by constructing a spanning subgraph G of small size and diameter k. In the next section, we prove that $un^-(J(n,k)) \geq \frac{k-1}{k+1}|E(J(n,k))| = \binom{k}{2}\binom{n}{k+1}$. In Section 3, we give an upper bound to $un^-(J(n,k))$.

2 A spanning subgraph of *J(n,k)* of diameter *k* and small size

In this section, we construct a spanning subgraph G of $J(n,k)$ with diameter k and size at most $\frac{2}{k+1}|E(J(n,k))|$ for $k \geq 2$.

We partition the vertex-set $V(J(n,k))$ into $n-k+1$ disjoint sets $L_1, L_2, \cdots, L_{n-k+1}$ by the different choice of the smallest element a_1: $L_i = \{k\text{-subset } a \text{ of } [n] : a_1 = i\}$, where $i = 1, 2, ..., n-k+1$. If we denote the graph induced by the vertices of L_i by $J[L_i]$, then $J[L_i] \cong J(n-i, k-1)$.

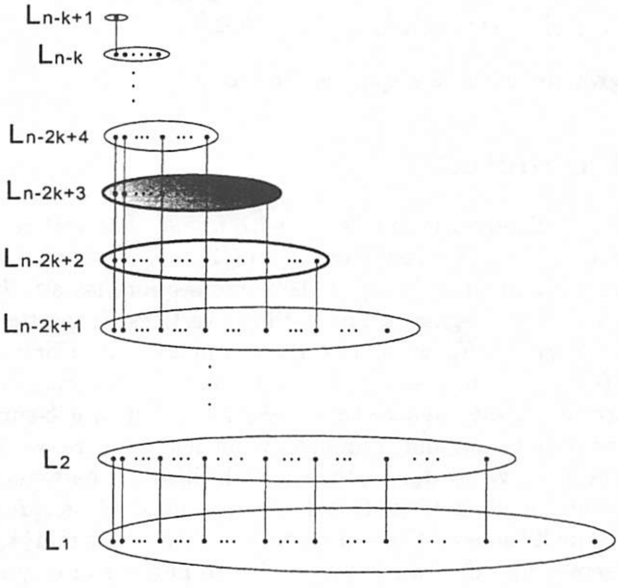

Fig. 1. The vertex partition of $J(n,k)$ and the matching given by Lemma 2.1.

We can obtain a bipartite graph $G_i = (L_i, L_{i+1})$ for each $1 \le i \le n-k$: Its vertices are ones of $L_i \cup L_{i+1}$ and its edges are ones of $J(n,k)$ joining vertices of L_i to vertices of L_{i+1}. For any vertex $a \in L_{i+1}$, $a_1 = i+1$. We can obtain a new vertex b from a by replacing a_1 by i. Then $b \in L_i$ and $ab \in E(G_i)$. By this way, we give a matching M_i in G_i which saturates all the vertices of L_{i+1} for any $1 \le i \le n-k$. So we have the following.

Lemma 2.1 For any $1 \le i \le n-k$, M_i is a matching of G_i which saturates all the vertices of L_{i+1}. □

Now we give two remarks about the lemma:

Remark 1. Let us denote the set of vertices in L_i which are saturated by M_i in Lemma 2.1 by N_i for $1 \le i \le n-k$. Then N_i consists of vertices in L_i that do not contain element $i+1$ and $|N_i| = |L_{i+1}|$. Let $R_i = L_i \setminus N_i$. So $|R_i| = |L_i| - |N_i| = \binom{n-i}{k-1} - \binom{n-i-1}{k-1} = \binom{n-i-1}{k-2}$. The three following assertions hold for $1 \le i \le n-k$: Each vertex in L_{i+1} has k neighbors in L_i, namely $k-1$ neighbors in R_i and one neighbor in N_i; Each vertex in N_i has one neighbor in L_{i+1}; and each vertex in R_i has $n-k-i+1$ neighbors in L_{i+1}.

Remark 2. The matchings $M_1, M_2, \cdots, M_{n-k}$ form $|L_2|$ disjoint paths P_i, $1 \le i \le |L_2|$. Note that the vertices on each P_i are adjacent to each other in $J(n,k)$, that is, the graph induced by the vertices on each P_i is a clique C_i. If we denote $C = \{C_i | 1 \le i \le |L_2|\}$, then there are $\binom{n-j-1}{k-2}$ elements in C with size j, where $2 \le j \le n-k+1$.

Lemma 2.2 If a semi-regular bipartite graph $G = (X, Y)$ has $|X| \le |Y|$ and edges, then G contains a matching which saturates X.

Proof. Let d_X (respectively d_Y) be the degree of each vertex in X (respectively Y). Since $|X| \le |Y|$, we have $d_X \ge d_Y > 0$. For any subset $S \subseteq X$, let m be the number of edges from S to $N(S)$. Then these m edges are incident to vertices in $N(S)$. Since G is semi-regular, $m = d_X \cdot |S| \le d_Y \cdot |N(S)|$. This implies that $|S| \le \frac{d_Y}{d_X} \cdot |N(S)| \le |N(S)|$. By Hall's Theorem, we know that G has a matching which saturates all the vertices of X. □

Lemma 2.3 For $k \ge 2$, the bipartite graph $G_i' = (L_{i+1}, R_i)$ is semi-regular and contains a matching which saturates all the vertices of R_i for $1 \le i \le n-2k+2$, a matching which saturates all the vertices of L_{i+1} for $n-2k+2 \le i \le n-k$, and a perfect matching for $i = n-2k+2$.

Proof. By Remark 1, we know that G_i' is semi-regular and $|L_{i+1}| = \binom{n-i-1}{k-1}$, $|R_i| = \binom{n-i-1}{k-2}$ for $1 \le i \le n-k$. By the unimodality of binomial coefficients we have $|L_{i+1}| \geq |R_i|$ for $1 \leq i \leq n-2k+2$, $|L_{i+1}| \leq |R_i|$ for $n-2k+2 \leq i \leq n-k$, and $|L_{i+1}| = |R_i|$ for $i = n-2k+2$. Hence the remaining of the lemma follows by Lemma 2.2. □

In the following, let $I_1 = [n-2k+3]$, $I_2 = [n-k+1] \setminus [n-2k+3]$, $I_3 = I_1 \setminus \{n-2k+2\}$, $I_4 = [n-2k+1]$ and $L_{I_t} = \bigcup_{i \in I_t} L_i$ for $1 \leq t \leq 4$. Now we construct a spanning subgraph of $J(n,k)$ by choosing the following four types of edges:

- $E_1 = \bigcup_{i \in I_3} E(J'[L_i])$, where $J'[L_i]$ is a spanning subgraph of $J[L_i]$ with the same diameter of $J[L_i]$ having as minimal number of edges as possible for $i \in I_3$.
- E_2 is the set of edges in each clique in C with one endvertex in L_{I_2} and the other one in L_{I_3}.
- E_3 is the set of edges in each clique in C with endvertices in L_{I_1}.
- E_4 is the perfect matching in G'_{n-2k+2} which we obtained in Lemma 2.3.

We can see that $E_i \bigcap E_j = \emptyset$ for $i \neq j$ and $i,j \in \{1,2,3,4\}$. It follows that

$$|E_2| = |L_{I_2}|(n-2k+2) = \binom{2k-3}{k}(n-2k+2),$$
$$|E_3| = \sum_{i=2}^{n-2k+2} \binom{n-i-1}{k-2}\binom{i}{2} + \binom{2k-3}{k-1}\binom{n-2k+3}{2}, \text{ and}$$
$$|E_4| = \binom{2k-3}{k-1}.$$

Let G be the spanning subgraph of $J(n,k)$ with edge-set $\bigcup_{i=1}^{4} E_i$. For example, the spanning subgraph G of $J(4,2)$ is shown in Figure 2 and has diameter 2, where $E_1 = \{\{14,13\},\{14,12\},\{13,12\}\}, E_2 = \emptyset, E_3 = \{\{14,24\},\{14,34\},\{24,34\},\{13,23\}\}$, $E_4 = \{\{23,34\}\}$.

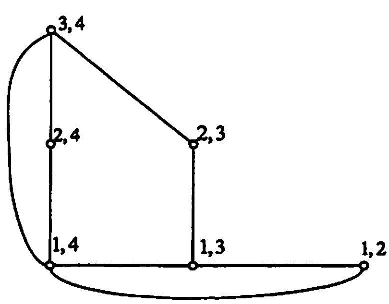

Fig. 2. A spanning subgraph G of $J(4,2)$ with diameter 2.

Next we shall verify that G has diameter k and $|E(G)| \le \frac{2}{k+1}|E(J(n,k))|$, where

$$|E(J(n,k))| = \frac{k(n-k)}{2}\binom{n}{k} = \frac{k(n-k)}{2}\frac{k+1}{n-k}\binom{n}{k+1} = \binom{k+1}{2}\binom{n}{k+1}. \tag{1}$$

Lemma 2.4 For $n \ge 2k \ge 2$, the spanning subgraph G of $J(n,k)$ has diameter k.

Proof. For $k = 1$, we have $I_1 = [n]$, $E_1 = E_2 = E_4 = \emptyset$ and $E_3 = E(J(n,1))$. Hence $G = J(n,1) = K_n$ and the lemma is true. For $1 \le i \le n-k+1$, $J[L_i] \cong J(n-i,k-1)$, and $d(J[L_i]) = d(J'[L_i]) = \min\{k-1, n-k-i+1\} \le k-1$. For $k = 2$, we have $I_1 = [n-1]$ and $I_2 = \emptyset$. Let $v_1 \in L_i, v_2 \in L_j$, where $1 \le i \le j \le n-k+1$. We will check $d_G(v_1,v_2) \le k$ for any $v_1, v_2 \in V(G)$ by considering the three following cases.

Case 1 : $i,j \in I_2$. We can find two edges in E_2 which join v_1 and v_2 to vertices in L_{n-2k+3} respectively. Since $d(J'[L_{n-2k+3}]) = k-2$, we have $d_G(v_1,v_2) \le k$.

Case 2: $i \in I_1, j \in I_2$. If $i = n-2k+2$, then we can find an edge in E_2 joining v_2 and a vertex in L_{n-2k+3}, and an edge in E_3 or E_4 joining v_1 and a vertex in L_{n-2k+3} since M_{n-2k+2} together with a perfect matching of G'_{n-2k+2} saturate all vertices in L_{n-2k+2}. Hence $d_G(v_1,v_2) \le k$; If $i \in I_3$, we can find an edge in E_2 joining v_2 to a vertex in L_i. Since $d(G'[L_i]) \le k-1$ for $i \in I_3$, we have $d_G(v_1,v_2) \le k$.

Case 3 : $i,j \in I_1$. If $i = j \ne n-2k+2$, we have $d_G(v_1,v_2) \le d(J'[L_i]) \le k-1$; If $i = j = n-2k+2$, similar to the reason in Case 2 we can find two edges in E_3 or E_4 joining v_1 and v_2 to vertices in L_{n-2k+3} respectively. Hence $d_G(v_1,v_2) \le k$; If $n-2k+2 = i < j$, we can find an edge in E_3 or E_4 joining v_1 to a vertex in L_{n-2k+3}. That implies that $d_G(v_1,v_2) \le k-1$; For the remaining case, i.e. $n-2k+2 \ne i < j \le n-2k+3$, we can find an edge in E_3 joining v_2 to a vertex in L_i. Hence $d_G(v_1,v_2) \le k$.

Hence $d(G) \le k$. On the other hand, $d(G) \ge d(J(n,k)) = k$. That is, $d(G) = k$. □

Lemma 2.5 The spanning subgraph G of $J(n,k)$ has the following bound on the number of edges: $|E(G)| \le \frac{2}{k+1}|E(J(n,k))| = k\binom{n}{k+1}$ for $n \ge 2k$.

Proof. We proceed by induction on the diameter k of $J(n,k)$. For $k = 1$, we have $G = J(n,1)$. For $k = 2$, we have $I_1 = [n-1]$, $I_2 = \emptyset$, $I_3 = [n-1] \setminus \{n-2\}$ and $I_4 = [n-3]$. $J[L_i]$ is a complete graph for $i \in I_3$. The diameter of $J[L_i]$ will be changed if we delete any one edge. Hence $J'(L_i) = J(L_i)$, and $|E_1| = \sum_{i \in I_3} |E(J[L_i])| = \sum_{i \in I_3} |E(J(n-i,1))| = \sum_{i \in I_3} \binom{n-i}{2} = \binom{n}{3} - 1$, $|E_2| = 0$, $|E_3| = \sum_{i=1}^{n-2} \binom{n-i}{2} = \binom{n}{3}$ and $|E_4| = 1$.

So $|E(G)| = \sum_{i=1}^{4}|E_i| = 2\binom{n}{3} = \frac{2}{3}|E(J(n,2))|$. Hence in the two trivial cases, the lemma is true.

In the following let $k \geq 3$. The diameter of $J[L_i]$ is $k-1$ for $i \in I_4$ and $k-2$ for $i = n-2k+3$. By induction hypothesis, we have

$$\begin{aligned}|E_1| &= \textstyle\sum_{i\in I_3}|E(J'[L_i])|\\ &\leq \textstyle\sum_{i\in I_4}\frac{2}{k}|E(J[L_i])| + \frac{2}{k-1}|E(J[L_{n-2k+3}])|\\ &= \textstyle\sum_{i\in I_4}\frac{2}{k}|E(J(n-i,k-1))| + \frac{2}{k-1}|E(J(2k-3,k-2))|\\ &= \textstyle\sum_{i\in I_4}\frac{2}{k}\frac{k(k-1)}{2}\binom{n-i}{k} + \frac{2}{k-1}\frac{(k-1)(k-2)}{2}\binom{2k-3}{k-1} \text{ (by Eq. (1))}\\ &= \textstyle\sum_{i\in I_4}(k-1)\binom{n-i}{k} + (k-2)\binom{2k-3}{k-1}.\end{aligned}$$

Since $(k-1)\binom{2k-3}{k-1} - (k-1)\binom{2k-1}{k+1} \leq 0$, we have

$$\begin{aligned}|E_1| + |E_4| &\leq \textstyle\sum_{i\in I_4}(k-1)\binom{n-i}{k} + (k-2)\binom{2k-3}{k-1} + \binom{2k-3}{k-1}\\ &= (k-1)[\binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{2k-1}{k}] + (k-1)\binom{2k-3}{k-1}\\ &= (k-1)[\binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{2k-1}{k} + \binom{2k-1}{k+1}]\\ &\quad - (k-1)\binom{2k-1}{k+1} + (k-1)\binom{2k-3}{k-1}\\ &= (k-1)\binom{n}{k+1} - (k-1)\binom{2k-1}{k+1} + (k-1)\binom{2k-3}{k-1}\\ &\leq (k-1)\binom{n}{k+1}.\end{aligned}$$

We now estimate $|E_2| + |E_3|$ in an another way. For any vertex $a \in L_i$, $a_1 = i$. We can obtain exactly k vertices of L_j by replacing each a_l by j for $1 \leq l \leq k$ and $1 \leq j \leq i-1$. That is, each vertex of L_i has exactly k neighbors in each L_j for $1 \leq j \leq i-1$. Those edges and the edges of $J[L_i]$ for $1 \leq i \leq n-k+1$ constitute the edges of $J(n,k)$. In E_2 and E_3, we just remain at most one edge incident with a for any $a \in L_i$ and the other endpoint in L_j for $2 \leq i \leq n-k+1$ and $1 \leq j \leq i-1$.

By using the known combinatorial formula: $\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$, we have

$$\begin{aligned}\textstyle\sum_{i=1}^{n-k+1}|E(J[L_i])| &= \textstyle\sum_{i=1}^{n-k+1}|E(J(n-i,k-1))|\\ &= \textstyle\sum_{i=1}^{n-k}\frac{k(k-1)}{2}\binom{n-i}{k} = \frac{k(k-1)}{2}\binom{n}{k+1}.\end{aligned} \tag{2}$$

Hence we obtain that

$$\begin{aligned}|E_2| + |E_3| &\leq \textstyle\frac{1}{k}[|E(J(n,k))| - \sum_{i=1}^{n-k+1}|E(J(n-i,k-1))|]\\ &\leq \textstyle\frac{1}{k}[\frac{k(k+1)}{2}\binom{n}{k+1} - \frac{k(k-1)}{2}\binom{n}{k+1}] = \binom{n}{k+1}.\end{aligned}$$

Hence $|E(G)| = \sum_{i=1}^{4}|E_i| \leq k\binom{n}{k+1} = \frac{2}{k+1}|E(J(n,k)|$. □

Theorem 2.6 The maximum number of edges of $J(n,k)$ whose removal leaves the diameter unchanged is bounded by

$un^-(J(n,k)) \geq \frac{k-1}{k+1}|E(J(n,k))| = \binom{k}{2}\binom{n}{k+1}$, for $n \geq 2k \geq 2$.

Proof. By Lemmas 2.4 and 2.5, we have $un^-(J(n,k)) \geq |E(J(n,k))| - |E(G)| \geq \frac{k-1}{k+1}|E(J(n,k))|$. □

The following immediate consequence is a particular case of the well-known Vandermonde convolution.

Corollary 2.7 $\sum_{i=2}^{n-k+1}(i-1)\binom{n-i}{k-1} = \binom{n}{k+1}$.

Proof. From the proof of Lemma 2.5, we know that each vertex of L_i has exactly k neighbors in each L_j for $1 \leq j \leq i-1$ and $2 \leq i \leq n-k+1$. Those edges and the edges of each $J[L_i]$ for $1 \leq i \leq n-k+1$ constitute the edges of $J(n,k)$. So we obtain a formula as following: $\sum_{i=2}^{n-k+1} k(i-1)\binom{n-i}{k-1} + \sum_{i=1}^{n-k+1} |E(J[L_i])| = \frac{k(n-k)}{2}\binom{n}{k}$. By inserting Eq.(2) into the above equation and by a simple computation, we obtain the lemma. □

3 An upper bound to $un^-(J(n,k))$

In this section, we give an upper bound to $un^-(J(n,k))$.

Lemma 3.1 Let G be a spanning subgraph of $J(n,k)$ of diameter k for $n \geq 2k \geq 2$. Then every edge e of G is in a cycle of length at most $2k+1$.

Proof. Let $e = ab$ be an edge of G. Let us partition $V(G)$ according to their distances to a and b. Let $M_i(a)$, for $0 \leq i \leq k$, be the set of vertices at distance i from a; We define similarly $M_i(b)$, for $0 \leq i \leq k$. Since $d_G(a,b) = 1$, we have that $M_i(b) \subseteq M_{i-1}(a) \bigcup M_i(a) \bigcup M_{i+1}(a)$ for each $0 < i < k$. Put $L_i(a) := M_i(a) \bigcap M_{i+1}(b)$ and $L_i(b) := M_i(b) \bigcap M_{i+1}(a)$ for $0 \leq i \leq k-1$, and $L_i(a,b) := M_i(a) \bigcap M_i(b)$ for $1 \leq i \leq k$. Then $V(G) = \bigcup_{i=0}^{k-1} L_i(a) \cup \bigcup_{i=0}^{k-1} L_i(b) \cup \bigcup_{i=1}^{k} L_i(a,b)$, where $L_0(a) = \{a\}$ and $L_0(b) = \{b\}$. We proceed by distinguishing the following two cases.

Case 1. $n \geq 2k+1$. One can choose a k-subset c in $[n] \setminus (a \cup b)$ since $|a \cup b| = k+1$. Then $d_J(a,c) = d_J(b,c) = k$. Since G is a spanning subgraph of $J(n,k)$ with diameter k, we have $k \geq d_G(a,c) \geq d_J(a,c) = k$. So c is an antipodal vertex of both a and b, and $c \in L_k(a,b)$. Hence shortest paths between c and a and between c and b (not passing through ab), and the edge ab constitute a graph that contains a cycle of length at most $2k+1$, passing through e.

Case 2. $n = 2k$. If $L_d(a,b) \neq \emptyset$ for some $d \leq k$, then there is a vertex $z \in L_d(a,b)$ for some $d \leq k$. Similar to Case 1, G has a cycle of length at most $2d+1$ which contains e.

If $L_d(a,b) = \emptyset$ for all $d \leq k$, let $\bar{a} = [n] \setminus a$ and $\bar{b} = [n] \setminus b$. Then $\bar{a}$ and $\bar{b}$ are antipodal vertices of a and b in G respectively by the same reason as Case 1; namely, $\bar{a} \in M_k(a)$ and $\bar{b} \in M_k(b)$. Since $\bar{a} \notin M_k(b)$, $k-1 = d_J(\bar{a},b) \leq d_G(\bar{a},b) < k$. Hence $d_G(b,\bar{a}) = k-1$, that is, $\bar{a} \in L_{k-1}(b)$. Similarly we have that $\bar{b} \in L_{k-1}(a)$. Since $V(G) = \bigcup_{i=0}^{k-1} L_i(a) \cup \bigcup_{i=0}^{k-1} L_i(b)$

is a partition, a shortest path in G between $\bar{a}$ and $\bar{b}$ must pass through an edge between $x \in L_d(a)$ and $y \in L_d(b)$ for some $d \leq k-1$, since there are no edges between $L_i(a)$ and $L_j(b)$ for $j \neq i$. Further $d \geq 1$ since $1 = d_J(\bar{a}, \bar{b}) \leq d_G(\bar{a}, \bar{b}) \leq k$. Then the edge xy, two shortest paths between x and a and between y and b (not passing through ab), and the edge ab form a cycle of length at most $2d+2 \leq 2k$ which contains e. □

Lemma 3.2 [2] Let G be a connected multigraph. If every edge of G is in a cycle of length at most l then G has at least $|V(G)|-1+\lceil(|V(G)|-1)/(l-1)\rceil$ edges.

Theorem 3.3 For $n \geq 2k \geq 2$, the maximum number of edges of $J(n,k)$ whose removal does not alter the diameter is bounded by
$un^-(J(n,k)) \leq \binom{k+1}{2}\binom{n}{k+1} - \lceil(1+\frac{1}{2k})(\binom{n}{k}-1)\rceil$.

Proof. For any spanning subgraph G of $J(n,k)$ of diameter k, by Lemmas 3.1 and 3.2 we have that $|E(G)| \geq |V(J(n,k))|-1+\lceil(|V(J(n,k))|-1)/2k\rceil$. Hence $un^-(J(n,k)) \leq |E(J(n,k))| - |E(G)|$ and the theorem follows. □

4 Conclusion

By Theorems 2.6 and 3.3, we conclude that, for $n \geq 2k \geq 2$

$$\binom{k}{2}\binom{n}{k+1} \leq un^-(J(n,k)) \leq \binom{k+1}{2}\binom{n}{k+1} - \left\lceil\left(1+\frac{1}{2k}\right)\left(\binom{n}{k}-1\right)\right\rceil. \tag{3}$$

Our result about the lower bound is optimal for $k=1$. For $k=2$, we find a spanning subgraph G of $J(4,2)$ with diameter 2 and size 7, see Figure 3. Hence $5 \leq un^-(J(4,2))$. By Eq. (3) we have that $4 \leq un^-(J(4,2)) \leq 5$ and the upper bound can be achieved.

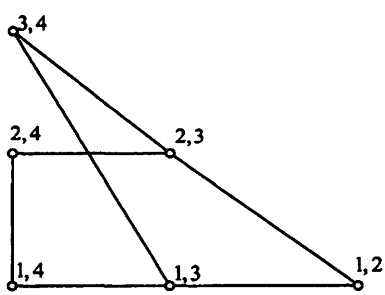

Fig. 3. The spanning subgraph of $J(4,2)$ with diameter 2 and size 7.

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