SHORT PROOFS OF COMBINATORIAL IDENTITIES FOR $n!$

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In this note, and using elementary tools from complex analysis, a concise, unifying insight into combinatorial identities for $n!$ is given. These identities were separately treated by the authors of references [1] and [2]. In [2], some generalizations of identity (0.2) were given, showing the connection with Stirling numbers. The first identity that appeared in *Integers* [1], is the following:

Identity 1:

$$n! = \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}i^n. \tag{0.1}$$

Proof. By the Cauchy's Integral Formula for Derivatives:

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{(e^z-1)^n}{z^{n+1}}dz = \frac{f^{(n)}(0)}{n!},$$

where $f(z) = (e^z-1)^n$. Note that, since $f(z) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}e^{iz}$, then $f^{(n)}(0) = \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}i^n$.

On the other hand, the same integral may be considered as follows:

$$I = \frac{1}{2\pi i}\oint_{|z|=1}\frac{(e^z-1)^n}{z^{n+1}}dz = \frac{1}{2\pi i}\oint_{|z|=1}\frac{\left(\frac{e^z-1}{z}\right)^n}{z}dz.$$

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But function $\dfrac{e^z - 1}{z} \in \mathcal{H}(\mathbb{C} - \{O\})$ and shows a removable singularity at the origin, and therefore, by the Cauchy's Integral formula, $I = \lim_{z \to 0} \left(\dfrac{e^z - 1}{z} \right)^n = 1$

And therefore, $\dfrac{f^{(n)}(0)}{n!} = \dfrac{\sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^n}{n!} = 1$, and the proof is done. □

Identity 2: Let n be a positive integer. Then for any complex number k,

$$(0.2) \qquad n! = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+k)^n.$$

It should be noted that in [2] identity (0.2) was proven for all nonnegative integer values of k, applying a method which goes back to Euler, and is based on the higher order differences of polynomial sequences. Here, this identity is proven again by using complex analysis arguments.

Proof. Consider

$$\frac{1}{2\pi i} \oint_{|z|=1} e^{kz} \frac{(e^z - 1)^n}{z^{n+1}} dz = \frac{g^{(n)}(0)}{n!}$$

by the Cauchy's Integral Formula for Derivatives, where $g(z) = e^{kz}(e^z - 1)^n$. Note that, since $g(z) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} e^{(i+k)z}$, then $g^{(n)}(0) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+k)^n$.

On the other hand, the same integral may be considered as follows:

$$I = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{kz}(e^z - 1)^n}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{kz} \left(\frac{e^z - 1}{z}\right)^n}{z} dz = 1,$$

by the Cauchy's Integral formula, as before.

□

References

[1] Roberto Anglani and Margherita Barile, Two very short proofs of a combinatorial identity. *Integers*, paper A18, 3 pp. (electronic), 2005.

[2] Roberto Anglani and Margherita Barile, Factorials as sums. arXiv:math.Ho/0702010v1, 1 Feb 2007.

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