On the spectra of tricyclic graphs

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Abstract

Let $\mathscr{T}_n$ be the set of tricyclic graphs of order n. In this paper, we use a new proof to determine the unique graph with maximal spectral radius among all graphs in $\mathscr{T}_n$ for each $n \geq 4$. Also, we determine the unique graph with minimal least eigenvalue among all graphs in this class for each $n \geq 52$. We can observe that the graph with maximal spectral radius is not the same as the one with minimal least eigenvalue in $\mathscr{T}_n$, which is different from those on the unicyclic and bicyclic graphs.

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1 Introduction

All graphs considered here are simple and undirected. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, let $G[S]$ be the subgraph induced by S. The degree of a vertex v, written by $d_G(v)$ or $d(v)$, is the number of edges incident with v. A *pendant vertex* is a vertex of degree 1. k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have *almost equal lengths* if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. The set of the neighbors of a vertex v is denoted by $N_G(v)$ or $N(v)$. The *girth* $g(G)$ of a graph G is the length of the shortest cycle in G, with the

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girth of an acyclic graph being infinite. Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. The *complete product* $G_1 \nabla G_2$ of two vertex-disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

Let $A(G)$ or A be the adjacency matrix of a graph G. Since A is symmetric and real, the eigenvalue of A, i.e., the zeros of the characteristic polynomial $P(G, \lambda) = det(\lambda I - A)$, can be arranged as follows:

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G).$$

Since G is connected, then A is irreducible non-negative and by Perron-Frobenius Theorem, the spectral radius $\rho(G) = \lambda_1(G)$ is simple and has a unique positive eigenvector. We will refer to such an eigenvector as Perron vector of G. It is known [5] that $\lambda_n(G) = -\rho(G)$ for a bipartite graph G.

Brualdi and Solheid [3] proposed the following general problem, which became one of the classical problems of spectral graph theory:

Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius in this set and characterize the graphs in which the maximal spectral radius is attained.

A lot of researchers have showed dense interest to the above problem. At the same time, they have turned their attention to the similar problems on the Laplacian spectral radius and the least eigenvalue, which became more popular in spectral graph theory:

Given a set $\mathcal{G}$ of graphs, find an upper bound for the Laplacian spectral radius or a lower bound for the least eigenvalue in this set and characterize the graphs in which the maximal Laplacian spectral radius or the minimal least eigenvalue is attained.

This paper mainly focuses on the lower bound of the least eigenvalue in the set of tricyclic graphs.

For the above classical problems, unicyclic and bicyclic graphs have become two popular sets of graphs. There are many results in the literature on the spectral radius and least eigenvalue of unicyclic and bicyclic graphs with n vertices(see [4, 8, 13, 16, 19, 24]). A tricyclic graph is a connected graph in which the number of edges equals the number of vertices plus two. The set of tricyclic graphs is also a very important class of graphs in spectral graph theory. Recently, tricyclic graphs have aroused extensive attention of many researchers. Let $\mathcal{T}_{n,k}$ be the set of tricyclic graphs with n vertices and k pendant vertices. In [9], Geng et al. characterized the tricyclic graphs with maximal spectral radius in $\mathcal{T}_{n,k}$. Guo [12] determined the tricyclic graphs with maximal Laplacian spectral radius in $\mathcal{T}_{n,k}$. Let $\mathcal{T}(2k)$ be the set of all tricyclic graphs on $2k(k \geq 2)$ vertices with perfect matchings.

Geng et al. [10] characterized the tricyclic graphs with maximal spectral radius in $\mathscr{T}(2k)$. In [18], Li et al. determined the tricyclic graphs of a given diameter with minimal energy. This paper focuses on the spectral radius and least eigenvalue of tricyclic graphs.

Let $\mathscr{T}_n$ be the set of tricyclic graphs of order n. In this paper, we use a new proof to determine the unique graph with maximal spectral radius among all graphs in $\mathscr{T}_n$ for each $n \geq 4$. Also, we determine the unique graph with minimal least eigenvalue among all graphs in this class for each $n \geq 52$. We can observe that the graph with maximal spectral radius is not the same as the one with minimal least eigenvalue in $\mathscr{T}_n$, which is different from those on the unicyclic and bicyclic graphs.

2 Preliminaries

In this section, we list some known results which will be used in this paper.

Lemma 2.1 ([17]) *Let v be a vertex in a connected graph G and suppose that two new paths $P : vv_1v_2\cdots v_k$ and $Q : vu_1u_2\cdots u_m$ of length k, m ($k \geq m \geq 1$) are attached to G at v, respectively, to form a new graph $G_{k,m}$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_m$ are distinct new vertices. Then for any $\lambda \geq \rho(G_{k,m})$, we have*

$$P(G_{k+1,m-1}, \lambda) > P(G_{k,m}, \lambda).$$

In particular,

$$\rho(G_{k,m}) > \rho(G_{k+1,m-1}).$$

Lemma 2.2 ([20]) *Let v be a vertex of G and $\mathscr{C}(v)$ be the set of all cycles containing v. Then*

$$P(G, \lambda) = \lambda P(G-v, \lambda) - \sum_{u \in N(v)} P(G-v-u, \lambda) - 2 \sum_{Z \in \mathscr{C}(v)} P(G-V(Z), \lambda),$$

where $G - V(Z)$ is the graph obtained by removing from G the vertices belonging to Z.

Lemma 2.3 ([17]) *Let G and H be two connected graphs such that $P(G, \lambda) > P(H, \lambda)$ for $\lambda \geq \rho(H)$. Then $\rho(G) < \rho(H)$.*

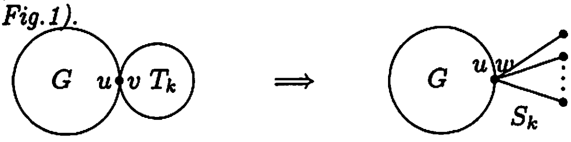
Lemma 2.4 ([5]) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ eigenvalues of an induced subgraph H. Then*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad (i = 1, \ldots, m).$$

Lemma 2.5 ([23]) *Let u, v be two distinct vertices of a connected graph G, $\{v_i | i = 1, 2, \ldots, s\} \subseteq N_G(v) \backslash N_G(u)$, and $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G. Let $G^* = G - \sum_{i=1}^{s} v_i v + \sum_{i=1}^{s} v_i u$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Let G, H be two disjoint connected graphs with $u \in V(G)$ and $w \in V(H)$, we denote by $GuwH$ the graph obtained from G and H by identifying u with w.

Corollary 2.6 ([23]) *Let G be a nontrivial connected graph, T_k be a tree of order k and S_k be a star with center w. Then $\rho(GuvT_k) \leq \rho(GuwS_k)$ for any $u \in V(G)$ and $v \in V(T_k)$. The equality holds if and only if $GuvT_k \cong GuwS_k$ (See Fig.1).*



$GuvT_k$ $GuwS_k$

Fig.1. Operations on a general graph and a tree.

Lemma 2.7 ([11]) *Let G, G', G'' be three connected graphs disjoint in pairs. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $\rho(G_1) < \rho(G_2)$ or $\rho(G_1) < \rho(G_3)$.*

Let G be a connected graph with $uv \in E(G)$. We denote by G_{uv} the graph obtained from G by subdividing the edge uv, that is, introducing a new vertex on the edge uv. A walk $v_1 v_2 \cdots v_k$ ($k \geq 2$) in a graph G is called an *internal path*, if these k vertices are distinct (except possibly $v_1 = v_k$), $d_G(v_1) > 2$, $d_G(v_k) > 2$ and $d_G(v_2) = \cdots = d_G(v_{k-1}) = 2$ (unless $k = 2$). Let W_n ($n \geq 6$) be the graph obtained from a path $v_1 v_2 \cdots v_{n-4}$ by attaching two pendant vertices to v_1 and another two to v_{n-4}. Hoffman and Smith showed the following result.

Lemma 2.8 ([14]) *Let G be a connected graph with $uv \in E(G)$. If uv belongs to an internal path of G and $G \ncong W_n$, then $\rho(G_{uv}) < \rho(G)$.*

Lemma 2.9 *Let G and H be two connected graphs on n vertices.*

(i) When n is even, if $P(G, \lambda) - P(H, \lambda) < 0$ for $\lambda = \lambda_n(H)$, then $\lambda_n(G) < \lambda_n(H)$.

(ii) When n is odd, if $P(G, \lambda) - P(H, \lambda) > 0$ for $\lambda = \lambda_n(H)$, then $\lambda_n(G) < \lambda_n(H)$.

Proof. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the zeros of $P(G, \lambda)$. Then we have

$$P(G, \lambda) = (\lambda - \lambda_1(G))(\lambda - \lambda_2(G)) \cdots (\lambda - \lambda_n(G)).$$

For $\lambda = \lambda_n(H)$, we have

$$P(G, \lambda) - P(H, \lambda) = (\lambda_n(H) - \lambda_1(G))(\lambda_n(H) - \lambda_2(G)) \cdots (\lambda_n(H) - \lambda_n(G)).$$

(i)When n is even, if $\lambda_n(G) \geq \lambda_n(H)$, then $P(G, \lambda) - P(H, \lambda) \geq 0$, a contradiction.

(ii)When n is odd, if $\lambda_n(G) \geq \lambda_n(H)$, then $P(G, \lambda) - P(H, \lambda) \leq 0$, a contradiction. $\square$

3 Tricyclic graphs with maximal spectral radius

By [9], we know that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, and there do not exist 5 cycles in G. Let $\mathscr{T}_{n,k}$ $(k \geq 1)$ be the set of tricyclic graphs on n vertices and k pendent vertices. Then $\mathscr{T}_{n,k} = \mathscr{T}^3_{n,k} \cup \mathscr{T}^4_{n,k} \cup \mathscr{T}^6_{n,k} \cup \mathscr{T}^7_{n,k}$, where $\mathscr{T}^i_{n,k}$ denotes the set of tricyclic graphs in $\mathscr{T}_{n,k}$ with exact i cycles for $i = 3, 4, 6, 7$. Correspondingly, $\mathscr{T}_n = \mathscr{T}^3_n \cup \mathscr{T}^4_n \cup \mathscr{T}^6_n \cup \mathscr{T}^7_n$.

Denote by G_1, G_2, G_3 and G_4 the connected tricyclic graphs of order n presented in Fig.2. Let $G_{i,k}$ $(k \geq 1)$ be the graph obtained from G_i by substituting all pendant edges with k paths with almost equal lengths at vertex v, where $i = 1, 2, 3, 4$. Clearly, $G_1 = G_{1,n-7}, G_2 = G_{2,n-6}, G_3 = G_{3,n-5}$ and $G_4 = G_{4,n-4}$.

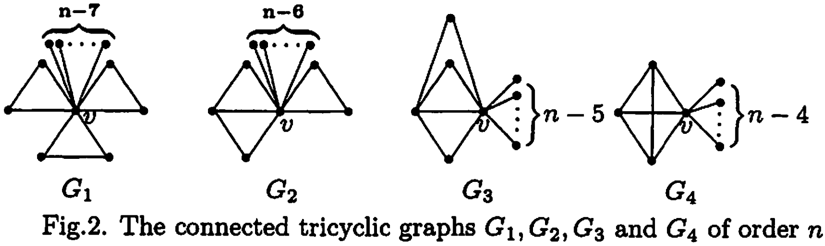


Fig.2. The connected tricyclic graphs G_1, G_2, G_3 and G_4 of order n.

Lemma 3.1 ([9]) *Let G have maximal spectral radius in $\mathscr{T}_{n,k}$, where $k \geq 1$.*

(i)If $G \in \mathscr{T}^3_{n,k}$ $(1 \leq k \leq n-7)$, then $G \cong G_{1,k}$.
(ii)If $G \in \mathscr{T}^4_{n,k}$ $(1 \leq k \leq n-6)$, then $G \cong G_{2,k}$.
(iii)If $G \in \mathscr{T}^6_{n,k}$ $(1 \leq k \leq n-5)$, then $G \cong G_{3,k}$.
(iv)If $G \in \mathscr{T}^7_{n,k}$ $(1 \leq k \leq n-4)$, then $G \cong G_{4,k}$.

In [4], R.A. Brualdi and E.S. Solheid characterized the unique graph with maximal spectral radius among all graphs in $\mathscr{T}_n$ for each $n \geq 4$. We will give a new proof of the result.

Theorem 3.2 ([4]) *Let G have maximal spectral radius in $\mathscr{T}_n$, where $n \geq 4$. Then $G \cong G_4$.*

Proof. It is easy to see from the tables of eigenvalues of connected graphs on 4, 5, 6 and 7 vertices [5, 6, 7] that theorem holds for $4 \leq n \leq 7$. Now suppose that $n \geq 8$. Note that $G \in \mathscr{T}_n = \mathscr{T}_n^3 \cup \mathscr{T}_n^4 \cup \mathscr{T}_n^6 \cup \mathscr{T}_n^7$.

For $G \in \mathscr{T}_n^3 = \bigcup_{k=0}^{n-7} \mathscr{T}_{n,k}^3$. If $G \in \mathscr{T}_{n,0}^3$, let G_0 be the graph obtained from G by contracting a vertex on an internal path of G and adding a pendant edge to a vertex of G, by Lemma 2.8, $\rho(G_0) > \rho(G)$ and $G_0 \in \mathscr{T}_{n,1}^3$, a contradiction. Hence $G \in \bigcup_{k=1}^{n-7} \mathscr{T}_{n,k}^3$. For each fixed k $(1 \leq k \leq n-7)$, by Lemma 3.1(i), $G \cong G_{1,k}$. Consider graph $G_{1,k} \in \mathscr{T}_{n,k}^3$. Let $1 \leq k < n-7$, it follows that there exists a path $P_{l+1} = vv_1 \cdots v_l$ attached to the vertex v of $G_{1,k}$ such that $l \geq 2$. Let $G' = G_{1,k} - \{v_l v_{l-1}\} + \{v_l v\}$. Then $G' \in \mathscr{T}_{n,k+1}^3$. By Lemma 2.1, we have $\rho(G_{1,k}) < \rho(G')$. By Lemma 3.1(i), we have $\rho(G') < \rho(G_{1,k+1})$. Hence $\rho(G_{1,k}) < \rho(G_{1,k+1})$. Thus $G \cong G_{1,n-7} = G_1$.

Similarly, for $G \in \mathscr{T}_n^4$, then $G \cong G_2$. For $G \in \mathscr{T}_n^6$, then $G \cong G_3$. For $G \in \mathscr{T}_n^7$, then $G \cong G_4$. Hence $\rho(G) = \max\{\rho(G_i) | 1 \leq i \leq 4\}$.

By applying Lemma 2.2 to the vertex v of G_1 we obtain

$$P(G_1, \lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 - 6\lambda^5 + (3n-6)\lambda^4 + 12\lambda^3 - (3n-14)\lambda^2 - 6\lambda + (n-7)].$$

In the analogous manner we have

$$P(G_2, \lambda) = \lambda^{n-6}[\lambda^6 - (n+2)\lambda^4 - 6\lambda^3 + (3n-9)\lambda^2 + 8\lambda - (2n-12)].$$

$$P(G_3, \lambda) = \lambda^{n-4}[\lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5)].$$

$$P(G_4, \lambda) = \lambda^{n-5}[\lambda^5 - (n+2)\lambda^3 - 8\lambda^2 + 3(n-5)\lambda + 2(n-4)].$$

For $\lambda \geq \rho(G_2) > \rho(K_{1,n-1}) = \sqrt{n-1}$, we have

$$\begin{aligned} P(G_1, \lambda) - P(G_2, \lambda) &= \lambda^{n-8}[3\lambda^4 + 4\lambda^3 - (n-2)\lambda^2 - 6\lambda + (n-7)] \\ &= \lambda^{n-8}[\lambda^2(3\lambda^2 - n + 2) + 2\lambda(2\lambda^2 - 3) + (n-7)] > 0. \end{aligned}$$

By Lemma 2.3, $\rho(G_1) < \rho(G_2)$.
For $\lambda \geq \rho(G_3) > \rho(K_{1,n-1}) = \sqrt{n-1}$, we have

$$P(G_2, \lambda) - P(G_3, \lambda) = \lambda^{n-6}[6\lambda^2 + 8\lambda - (2n-12)] > 0.$$

By Lemma 2.3, $\rho(G_2) < \rho(G_3)$.
For $\lambda \geq \rho(G_4) > \rho(K_{1,n-1}) = \sqrt{n-1}$, we have

$$P(G_3, \lambda) - P(G_4, \lambda) = \lambda^{n-5}[2\lambda^2 - 2(n-4)] > 0.$$

By Lemma 2.3, $\rho(G_3) < \rho(G_4)$.
Hence $G \cong G_4$, this completes the proof of Theorem 3.2. □

4 Tricyclic graphs with minimal least eigenvalue

In [1], Bell et al. study connected graphs whose least eigenvalue is minimal among graphs of prescribed order and size. They state the following structural result.

Theorem 4.1 ([1]) *Let G be a connected graph whose least eigenvalue $\lambda_n(G)$ is minimal among the connected graphs of order n and size m $(0 < m < \binom{n}{2})$. Then G is either*

(i)a bipartite graph, or

(ii)a complete product of two nested split graphs (not both totally disconnected).

Here a graph G is called a *nested split graph* (*or threshold graph*) if its vertices can be ordered so that $jq \in E(G)$ implies $ip \in E(G)$ whenever $i \leq j$ and $p \leq q$. The nested split graphs are the graphs without $2K_2, P_4$ or C_4 as an induced subgraph. They are precisely the graphs with a stepwise adjacency matrix. There are many spectral results on these graphs in the literature (see for example, [2, 15, 21, 22]).

G_1^* G_2^*

Fig.3. The connected unicyclic graph G_1^* and bicyclic graph G_2^* of order n.

Let G have minimal least eigenvalue among all connected unicyclic graphs of order n $(n \geq 12)$. Xu et al. [24] and Fan et al. [8] independently by the different methods show that $G \cong G_1^*$. In fact, it is much easier by Theorem 4.1 to obtain the same conclusion. If G is a bipartite graph, then by Lemmas 2.7, 2.8, and Corollary 2.6, $G \cong G_0$, where G_0 is the graph obtained from C_4 by attaching $n-4$ pendant edges at one vertex. If G is a complete product of two nested split graphs, then $G \cong G_1^*$. By [24], for $n \geq 12$, we have $\lambda_n(G_0) > \lambda_n(G_1^*)$, hence $G \cong G_1^*$.

Let G have minimal least eigenvalue among all connected bicyclic graphs of order n $(n \geq 28)$. In [19], by Theorem 4.1, M. Petrović et al. show that $G \cong G_2^*$.

Let G have minimal least eigenvalue among all connected tricyclic graphs of order n $(n \geq 52)$. In this section, by Theorem 4.1, we will prove $G \cong G_3$.

Bipartite graph

Let $\mathscr{BT}_n$ denote the set of all connected bipartite tricyclic graphs. Note that $\lambda_n(G) = -\rho(G)$ for a bipartite graph G, hence the minimal least eigenvalue problem in $\mathscr{BT}_n$ is equivalent to the maximal spectral radius problem in this class.

In the following, let $G_5, G_6, \ldots, G_{12}$ be the connected bipartite tricyclic graphs as shown in Fig.4.

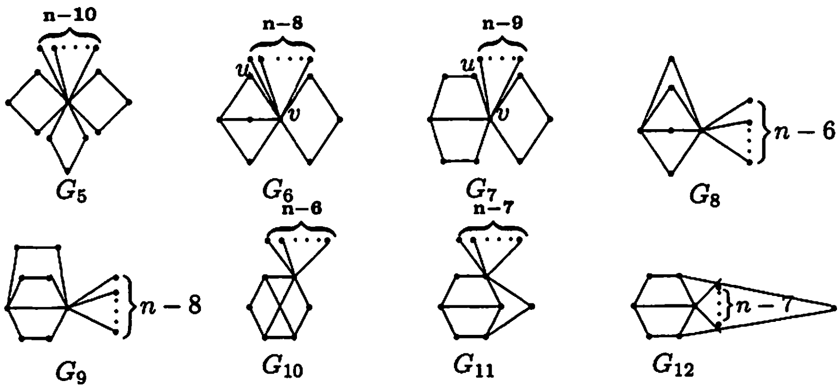


Fig.4. Eight connected bipartite tricyclic graphs G_i $(5 \leq i \leq 12)$.

Lemma 4.2 *Let G have maximal spectral radius among all connected bipartite tricyclic graphs of order n $(n \geq 10)$. Then $G \cong G_8$(see Fig.4).*

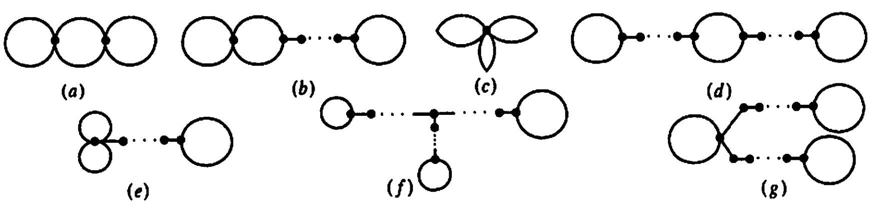


Fig.5. Seven possible cases for the arrangement of three cycles in G.

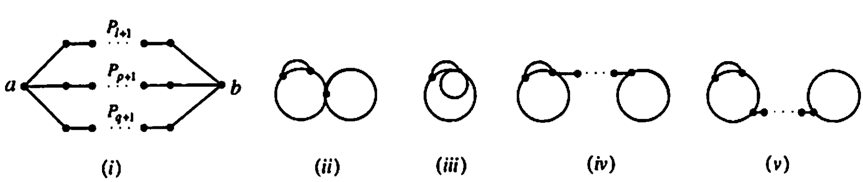


Fig.6. $P(l, p, q)$ and four possible cases for the arrangement of four cycles in G.

Proof. Let G have maximal spectral radius among all connected bipartite tricyclic graphs of order n ($n \geq 10$), then $G \in \mathscr{BT}_n = \mathscr{BT}_n^3 \cup \mathscr{BT}_n^4 \cup \mathscr{BT}_n^6 \cup \mathscr{BT}_n^7$. Let X be the Perron vector of G corresponding to $\rho(G)$.

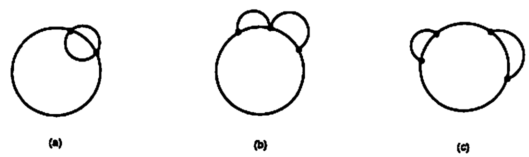


Fig.7. Three possible cases for the arrangement of six cycles in G.

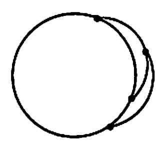


Fig.8. One possible case for the arrangement of seven cycles in G.

For $G \in \mathscr{BT}_n^3$. The arrangement of three cycles C_p, C_q, C_r in G has seven possible cases(see Fig.5). Suppose that the arrangement of the three cycles contained in G is just (b). Denote by $v_1v_2\cdots v_l$ ($l \geq 2$) the path connecting two cycles C_q, C_r. Suppose that $v_1 \in C_q, v_l \in C_r$. Without loss of generality, we may assume that $x_1 \geq x_l$. Denote $N_{C_r}(v_l) = \{w_1, w_2\}$. Let $G_0 = G - w_1v_l - w_2v_l + w_1v_1 + w_2v_1$. Then $G_0 \in \mathscr{BT}_n^3$, and by Lemma 2.5, we have $\rho(G_0) > \rho(G)$, a contradiction. Hence $l = 1$. Similarly, we can also show that G cannot contain three cycles whose arrangement is as (d), (e), (f) or (g). Hence the arrangement of three cycles in G is (a) or (c). By Lemma 2.7, we know that the arrangement of three cycles in G is (c). From a repeated use of Lemma 2.7, we have G has exactly one tree T attaching to the common vertex v of these three cycles. We claim that $p = q = r = 4$. Otherwise, note that G is bipartite, without loss of generality, we may assume that $p \geq 6$. Let G' be the graph obtained from G by replacing C_p with C_{p-2} and adding two pendant edges to vertex v. Then $G' \in \mathscr{BT}_n^3$, and by Lemma 2.8, $\rho(G') > \rho(G)$, a contradiction. Furthermore, by Corollary 2.6, then $G \cong G_5$. By Lemma 2.2, we have

$$P(G_5, \lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 + 6(n-4)\lambda^4 - 4(3n-22)\lambda^2 + 8(n-10)].$$

$$P(G_8, \lambda) = \lambda^{n-4}[\lambda^4 - (n+2)\lambda^2 + 4(n-6)].$$

For $\lambda \geq \rho(G_8) > \rho(K_{1,n-2}) = \sqrt{n-2}$, we have

$$P(G_5, \lambda) - P(G_8, \lambda) = \lambda^{n-8}[2n\lambda^4 - 4(3n-22)\lambda^2 + 8(n-10)] > 0.$$

By Lemma 2.3, $\rho(G_5) < \rho(G_8)$. That is to say, for any $\widetilde{G} \in \mathscr{B}\mathscr{T}_n^3$, we always have $\rho(\widetilde{G}) < \rho(G_8)$.

For $G \in \mathscr{B}\mathscr{T}_n^4$. Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex-disjoint paths, where $l, p, q \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph(see Fig.6), denoted by $P(l,p,q)$, is called a θ-graph. Furthermore, let C_r be a cycle. Join $P(l,p,q)$ and C_r by a path P_s and denote the resulting graph by G_0, where $s \geq 1$ and G_0 has four cases(see Fig.6). We will prove that G_0 is either (ii) or (iii). Assume, on the contrary, that it is not true. Then there exists a path P_s joining $P(l,p,q)$ and C_r, where $s \geq 2$. Suppose that $v_1 \in P(l,p,q), v_s \in C_r$. Without loss of generality, we may assume that $x_1 \geq x_s$. Denote $N_{C_r}(v_s) = \{w_1, w_2\}$. Let $G' = G - w_1v_s - w_2v_s + w_1v_1 + w_2v_1$. Then $G' \in \mathscr{B}\mathscr{T}_n^4$, and by Lemma 2.5, $\rho(G') > \rho(G)$, a contradiction. Hence $s = 1$. From a repeated use of Lemma 2.7, we have G has exactly one tree T attaching to the common vertex v of $P(l,p,q)$ and C_r. By Lemma 2.8, we have $r = 4$, one of p,q,l is 1 and the other two are 3, or p,q,l are all 2. Furthermore, by Corollary 2.6, then $G \cong G_6, G_6', G_7$ or G_7', where G_6' and G_7' are the graphs obtained from G_6 and G_7 by moving C_4 and all the pendant edges from v to u, respectively(see Fig.4). By applying Lemma 2.2 to the vertex u of G_6' and G_7' we have

$$P(G_6',\lambda) = \lambda^{n-6}[\lambda^6 - (n+2)\lambda^4 + (6n-28)\lambda^2 - 8(n-8)].$$

$$P(G_7',\lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 + (7n-32)\lambda^4 - (12n-89)\lambda^2 + (4n-34)].$$

By applying Lemma 2.2 to the vertex v of G_6 and G_7 we have

$$P(G_6,\lambda) = \lambda^{n-6}[\lambda^6 - (n+2)\lambda^4 + (5n-22)\lambda^2 - 6(n-8)].$$

$$P(G_7,\lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 + (6n-25)\lambda^4 - (11n-78)\lambda^2 + 6n - 52)].$$

For $\lambda \geq \rho(G_6) > \rho(K_{1,n-3}) = \sqrt{n-3}$, we have

$$P(G_6',\lambda) - P(G_6,\lambda) = \lambda^{n-6}[(n-6)\lambda^2 - 2(n-8)] > 0.$$

By Lemma 2.3, $\rho(G_6') < \rho(G_6)$.
For $\lambda \geq \rho(G_7) > \rho(K_{1,n-4}) = \sqrt{n-4}$, we have

$$P(G_7',\lambda) - P(G_7,\lambda) = \lambda^{n-8}[(n-7)\lambda^4 - (n-11)\lambda^2 - 2(n-9)] > 0.$$

By Lemma 2.3, $\rho(G_7') < \rho(G_7)$.

Hence $G \cong G_6$ or G_7. Similarly, we can prove $\rho(G_6) < \rho(G_8)$ and $\rho(G_7) < \rho(G_8)$. That is to say, for any $\widetilde{G} \in \mathscr{B}\mathscr{T}_n^4$, we always have $\rho(\widetilde{G}) < \rho(G_8)$.

For $G \in \mathscr{B}\mathscr{T}_n^6$, G can be obtained from G_0 by planting some trees at some vertices of G_0, where G_0 consists of six cycles. G_0 can be obtained from $P(l,p,q)$ by adding a new path P_{r+1}, where the two endpoints of P_{r+1} meanwhile belong to one of P_{l+1}, P_{p+1} and P_{q+1}. Hence G_0 has three cases(see Fig.7). From a repeated use of Lemma 2.7, we have G has exactly one big tree T attaching to one vertex v of G_0. Note that G have maximal spectral radius and G_0 is a bipartite graph. By Lemma 2.8, G_0 has exactly eight structures(see Fig.9). By Corollary 2.6, then G must be obtained from G_0 by attaching some pendant edges to exactly one vertex of G_0. Clearly, G has many structures. For example, $G \cong G_8$ or G_9(see Fig.4). Note that

$$P(G_9, \lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 + (6n-27)\lambda^4 - (9n-56)\lambda^2 + 4(n-7)].$$

For $\lambda \geq \rho(G_8) > \rho(K_{1,n-2}) = \sqrt{n-2}$, we have

$$P(G_9, \lambda) - P(G_8, \lambda) = \lambda^{n-8}[(2n-3)\lambda^4 - (9n-56)\lambda^2 + 4(n-7)] > 0.$$

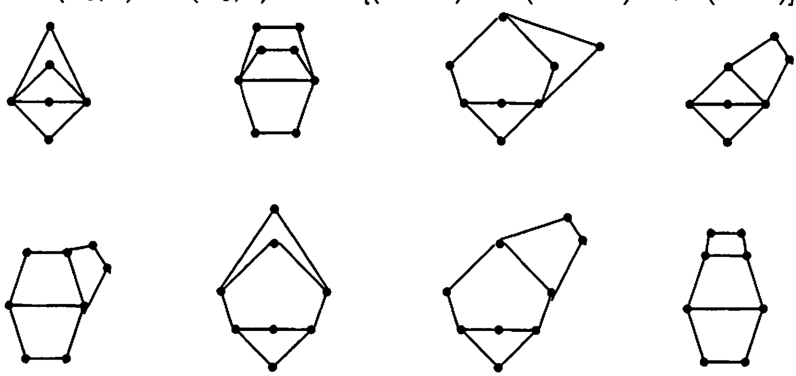


Fig.9. Eight structures of G_0.

By Lemma 2.3, $\rho(G_9) < \rho(G_8)$. Similarly, by computing the characteristic polynomial of graphs and Lemma 2.3, we can show that the spectral radii of other structures of G are all less than that of G_8 or G_9. That is to say, for any $\widetilde{G} \in \mathscr{B}\mathscr{T}_n^6$, we always have $\rho(\widetilde{G}) < \rho(G_8)$ unless $\widetilde{G} \cong G_8$.

For $G \in \mathscr{B}\mathscr{T}_n^7$, G can be obtained from G_0 by planting some trees at some vertices of G_0, where G_0 consists of seven cycles. G_0 can be obtained from $P(l,p,q)$ by adding a new path P_{r+1}, where the two endpoints of P_{r+1} are on the different paths P_{l+1}, P_{p+1} or P_{q+1}. Hence G_0 has only one case(see Fig.8). From a repeated use of Lemma 2.7, we have G has exactly one big tree T attaching to vertex v of G_0. By Corollary 2.6, $G \cong G_{10}, G_{11}, G_{12}$ or other graphs obtained from G_{10}, G_{11}, G_{12} by moving all pendant edges to one vertex other than v, respectively. Note that

$$P(G_{10}, \lambda) = \lambda^{n-6}[\lambda^6 - (n+2)\lambda^4 + (5n-26)\lambda^2 - 2(n-6)].$$

$$P(G_{11},\lambda) = \lambda^{n-6}[\lambda^6 - (n+2)\lambda^4 + (6n-27)\lambda^2 - (5n-28)].$$

$$P(G_{12},\lambda) = \lambda^{n-8}[\lambda^8 - (n+2)\lambda^6 + (6n-27)\lambda^4 - (9n-56)\lambda^2 + 4(n-7)].$$

By Lemma 2.3, $\rho(G_i) < \rho(G_8)$ for $i = 10, 11, 12$. Similarly, we can show that the spectral radii of other structures of G are all less than that of G_{10}, G_{11} or G_{12}. That is to say, for any $\widetilde{G} \in \mathcal{BT}_n^7$, we always have $\rho(\widetilde{G}) < \rho(G_8)$.

Hence $G \cong G_8$. This completes the proof of Lemma 4.2. □

Complete products of two nested split graphs

Lemma 4.3 *Let G be a connected non-bipartite tricyclic graph of order n $(n \geq 6)$ with minimal least eigenvalue among all connected tricyclic graphs, and be the complete product of two nested split graphs. Then $G \cong G_3$.*

Proof. Let $G = H_1 \nabla H_2$, where H_1 and H_2 are nested split graphs. Let $|H_1| = k, |H_2| = n-k$ $(1 \leq k \leq n-2)$. Then $k = 1$, because in the opposite case

$$|E(G)| = |E(H_1 \nabla H_2)| > k(n-k) \geq 2(n-2) \geq n+2,$$

when $n \geq 6$ and $2 \leq k \leq n-2$, a contradiction. So $|H_1| = 1$ and $|H_2| = n-1$. By the definition of nested split graph, then $G \cong G_3$ or G_4. Note that

$$P(G_3,\lambda) = \lambda^{n-4}[\lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5)].$$

$$P(G_4,\lambda) = \lambda^{n-5}[\lambda^5 - (n+2)\lambda^3 - 8\lambda^2 + 3(n-5)\lambda + 2(n-4)].$$

By Lemma 2.4, for $\lambda = \lambda_n(G_4) \leq \lambda_n(K_{1,n-3}) = -\sqrt{n-3}$, when n is even, we have

$$P(G_3,\lambda) - P(G_4,\lambda) = \lambda^{n-5}[2\lambda^2 - 2(n-4)] < 0.$$

When n is odd, we have

$$P(G_3,\lambda) - P(G_4,\lambda) = \lambda^{n-5}[2\lambda^2 - 2(n-4)] > 0.$$

According to Lemma 2.9, $\lambda_n(G_3) < \lambda_n(G_4)$. Hence $G \cong G_3$. □

Having in mind Lemmas 4.2 and 4.3, we get the main result.

Theorem 4.4 *Let G have minimal least eigenvalue among all connected tricyclic graphs of order n $(n \geq 52)$. Then $G \cong G_3$.*

Proof. By Lemmas 4.2 and 4.3, we have $G \cong G_8$ or G_3. By Lemma 2.2, we have

$$P(G_8,\lambda) = \lambda^{n-4}[\lambda^4 - (n+2)\lambda^2 + 4(n-6)].$$

$$P(G_3,\lambda) = \lambda^{n-4}[\lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5)].$$

Obviously, $\lambda_n(G_8) = -\sqrt{\frac{n+2+\sqrt{(n-6)^2+64}}{2}}$. Let $f(\lambda) = \lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5)$. Then $f(\lambda)$ has the same nonzero roots as $P(G_3, \lambda)$ and

$$f(\lambda_n(G_8)) = -n + 9 + 6\sqrt{\frac{n+2+\sqrt{(n-6)^2+64}}{2}}.$$

By Lemma 2.9, if $f(\lambda_n(G_8)) < 0$, then $\lambda_n(G_3) < \lambda_n(G_8)$.

Now we solve the inequality,

$$-n + 9 + 6\sqrt{\frac{n+2+\sqrt{(n-6)^2+64}}{2}} < 0.$$

After squaring and reordering we obtain the equivalent inequality

$$18\sqrt{(n-6)^2+64} < (n-18)^2 - 279.$$

Continuing squaring and putting in order we get the equivalent inequality

$$(n-18)^4 - 882(n-18)^2 - 7776(n-18) + 10449 > 0.$$

Let $g(x) = x^4 - 882x^2 - 7776x + 10449$. This function has exactly two positive roots: $x_1 \in (0,2), x_2 \in (33,34)$. Since $n \geq 52$, then $x = n - 18 \geq 34$, and clearly $g(x) > 0$. This completes the proof of Theorem 4.4. □

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References

[1] F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, I, Linear Algebra Appl. 429(2008)234-241.

[2] F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, II, Linear Algebra Appl. 429(2008)2168-2179.

[3] R.A. Brualdi, E.S. Solheid, On the spectral radius of complementary acyclic matrices of zeros and ones, SIAM J. Algebra. Discrete Method 7(1986)265-272.

[4] R.A. Brualdi, E.S. Solheid, On the spectral radius of connected graphs, Publ. Inst. Math. (Beograd)39(53)(1986)45-54.

[5] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third edition, Johann Abrosius Barth Verlag, 1995.

[6] D.M. Cvetković, M. Doob, I. Gutman, and A. Torgašev, Recent Results in the Theory of Graph Spectra, North-Holland, Amsterdam, 1988.

[7] D.M. Cvetković, M. Petrić, A table of connected graphs with six vertices, Discrete Math. 50(1984)37-49.

[8] Y.Z. Fan, Y. Wang, Y.B. Gao, Minimizing the least eigenvalue of unicyclic graphs with application to spectral spead, Linear Algebra Appl. 429(2008)577-588.

[9] X.Y. Geng, S.C. Li, The spectral radius of tricyclic graphs with n vertices and k pendant vertices, Linear Algebra Appl. 428(2008)2639-2653.

[10] X.Y. Geng, S.C. Li, X.C. Li, On the index of tricyclic graphs with perfect matchings, Linear Algebra Appl. 431(2009)2304-2316.

[11] S.G. Guo, On the spectral radius of bicyclic graphs with n vertices and diameter d, Linear Algebra Appl. 422(2007)119-132.

[12] S.G. Guo, The Laplacian spectral radius of tricyclic graphs with n vertices and k pendant vertices, Linear Algebra Appl. 431(2009)139-147.

[13] C.X. He, Y. Liu, J.Y. Shao, On the spectral radius of bicyclic graphs, J. Math. Res. Expos. 27(2007)445-454.

[14] A.J. Hoffman, J.H. Smith, in: Fiedler(Ed.), Recent Advances in Graph Theory, Academia Praha, New York, 1975, 273-281.

[15] P.L. Hammer, A.K. Kelmans, Laplacian spectral and spanning trees of threshold graphs, Discrete Appl. Math.65(1996)255-273.

[16] Y. Hong, On the spectra of unicyclic graphs, J. East China Norm. Univ. (Natur. Sci. Ed.)1(1986)31-34(in chinese).

[17] Q. Li, K.Q. Feng, On the largest eigenvalues of graphs, Acta Math. Appl. Sinica 2(1979)167-175(in Chinese).

[18] S.C. Li, X.C. Li, Z.X. Zhu, On tricyclic graphs of a given diameter with minimal energy, MATCH Commun. Math. Comput. Chem. 59(2008)397-419.

[19] M. Petrović, B. Borovićanin, T. Aleksić, Bicyclic graphs for which the least eigenvalue is minimum, Linear Algebra Appl. 430(2009)1328-1335.

[20] A. Schwenk, in: R. Bari, F. Harary(Eds.), Graphs and Combinatorics - Lecture Notes in Mathematics, vol.406, Springer-Verlag, Berlin, Heidelberg, New York, 1974, pp.153-172.

[21] S.K. Simić, F. Belardo, E.M.L. Marzi, D.V. ToŠić, Connected graphs of fixed order and size with maximal index: Some spectral bounds, Linear Algebra Appl. doi:10.1016/j.laa.2009.06.043.

[22] Z. Stanić, On nested split graphs whose second largest eigenvalue is less than 1, Linear Algebra Appl. 430(2009)2200-2211.

[23] B.F. Wu, Graft transformation of graphs and spectral radius, Master Thesis, East China Normal University, 2003.

[24] G.H. Xu, Q.F. Xu, S.K. Wang, A sharp lower bound on the least eigenvalue of unicyclic graphs, J. Ninbo Univ. (NSEE)16(3)(2003)225-227(in chinese).