# The Hamilton-Waterloo problem for Hamilton cycles and $C_{4k}$-factors *

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**Abstract**

In this paper we give a complete solution to the Hamilton-Waterloo problem for the case of Hamilton cycles and $C_{4k}$-factors for all positive integers $k$.

Keywords: 2-factorization; Hamilton-Waterloo problem; Hamilton cycle; cycle decompositions

## 1 Introduction

The Hamilton-Waterloo problem is a generalization of the well known Oberwolfach problem, which asks for a 2-factorization of the complete graph $K_n$ in which $r$ of its 2-factors are isomorphic to a given 2-factor $R$ and s of its 2-factors are isomorphic to a given 2-factor $S$ with $2(r+s)=n-1$. The most interesting case of the Hamilton-Waterloo problem is that $R$ consists of cycles of length $m$ and $S$ consists of cycles of length $k$, such a 2-factorization of $K_n$ is called uniform and denoted by $HW(n;r,s;m,k)$. The corresponding Hamilton-Waterloo problem is the problem for the existence of an $HW(n;r,s;m,k)$.

There exists no 2-factorization of $K_n$ when $n$ is even since the degree of each vertex is odd. In this case, we consider the 2-factorizations of $K_n - I_n$(where $I_n$ is a 1-factor of $K_n$) instead. The corresponding 2-factorization is also denoted by $HW(n;r,s;m,k)$. Obviously $2(r+s)=n-2$.

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It is easy to see that the following conditions are necessary for the existence of an $HW(n; r, s; m, k)$:

**Lemma 1.1.** If there exists an $HW(n; r, s; m, k)$, then

$n \equiv 0 \pmod m$ when $s = 0$;

$n \equiv 0 \pmod k$ when $r = 0$;

$n \equiv 0 \pmod m$ and $n \equiv 0 \pmod k$ when $r \neq 0$ and $s \neq 0$;

The Hamilton-Waterloo problem attracts much attention and progress has been made by several authors. Adams, Billington, Bryant and El-Zanati [1] deal with the case $(m, k) \in \{(3,5),(3,15),(5,15)\}$. Danziger, Quattrocchi and Stevens[3] give an almost complete solution for the case $(m, k) = (3, 4)$, which is stated below:

**Theorem 1.2.** [3] An $HW(n; r, s; 3, 4)$ exists if and only if $n \equiv 0 \pmod{12}$ and $(n, s) \neq (12, 0)$ with the following possible exceptions:

$n = 24$ and $s = 2, 4, 6$;

$n = 48$ and $s = 6, 8, 10, 14, 16, 18$.

The case $(m, k) = (n, 3)$, i.e. Hamilton cycles and triangle-factors, is studied by Horak, Nedela and Rosa [8], Dinitz and Ling [4, 5] and the following partial result obtained:

**Theorem 1.3.** [4, 5, 8]

(a) If $n \equiv 3 \pmod{18}$, then an $HW(n; r, s; n, 3)$ exists except possibly when $n = 93, 111, 129, 183, 201$ and $r = 1$;

(b) If $n \equiv 9 \pmod{18}$, then an $HW(n; r, s; n, 3)$ exists except $n = 9$ and $r = 1$, except possibly when $n = 153, 207$ and $r = 1$;

(c) If $n \equiv 15 \pmod{18}$ and $r \in \{1, \frac{(n+3)}{6}, \frac{(n+3)}{6}+2, \frac{(n+3)}{6}+3, \dots, \frac{(n-1)}{2}\}$, then an $HW(n; r, s; n, 3)$ exists except possibly when $n = 123, 141, 159, 177, 213, 249$ and $r = 1$.

For $n \equiv 0 \pmod 6$, the problem for the existence of an $HW(n; r, s; n, 3)$ is still open.

The cases $(m, k) \in \{(t, 2t) | t > 4\}$ and $(m, k) \in \{(4, 2t) | t > 3\}$ have been completely solved by Fu and Huang [6].

**Theorem 1.4.**[6]

(a) Suppose $t \geq 4$, an $HW(n; r, s; t, 2t)$ exists if and only if $n \equiv 0 \pmod{2t}$.

(b) For an integer $t \geq 3$, an $HW(n; r, s; 4, 2t)$ exists if and only if $n \equiv 0 \pmod 4$ and $n \equiv 0 \pmod{2t}$.

For $r = 0$ or $s = 0$, the Hamilton-Waterloo problem is in fact the problem for the existence of resolvable cycle decompositions of the complete graph, which has been completely solved by Govzdjak [7].

**Theorem 1.5.**[7] There exists a resolvable $m$-cycle decomposition of $K_n$(or $K_n - I$ when n is even) if and only if $n \equiv 0 \pmod m$, $(n, m) \neq (6, 3)$ and $(n, m) \neq (12, 3)$.

The purpose of this paper is to give a complete solution to the Hamilton-Waterloo problem for the case of Hamilton cycles and $C_{4k}$-factors which is stated in the following theorem.

**Theorem 1.6.** For given positive integer $k$, an $HW(n; r, s; n, 4k)$ exists if and only if $r + s = [\frac{n-1}{2}]$ and $n \equiv 0 \pmod{4k}$ if $s > 0$ or $n \geq 3$ if $s = 0$.

## 2 Preliminaries

In this section, we provide some basic constructions.

For convenience, we introduce the following notations first. A $C_m$-factor of $K_n$ is a spanning subgraph of $K_n$ in which each component is a cycle of length $m$. Let $r+s=[(n-1)/2]$ and

$$HW^*(n;m,k)=\{r|an\ HW(n;r,s;m,k)\ exists\}.$$

We use HC to represent Hamilton cycle for short.

By Lemma 1.1, the necessary condition for the existence of $HW(n;r,s;n,4k)$ with $s>0$ is $n\equiv 0 \pmod{4k}$, we assume $n=4kt$ and the vertex set of $K_n$ is $Z_{2t}\times Z_{2k}$. We write $V_i=\{i\}\times Z_{2k}=\{i_0,i_1,\ldots,i_{2k-1}\}$ for $i\in Z_{2t}$. Let $K_{V_i,V_j}$ be the complete bipartite graph define on two partite sets $V_i$ and $V_j$, and $K_{V_i}$ be the complete graph of order $2k$ define on the vertex set $V_i$. Obviously,

$$E(K_{4kt})=\bigcup_{i=0}^{2t-1}E(K_{V_i})\cup\bigcup_{i\neq j}E(K_{V_i,V_j}).$$

Further for $d\in Z_{2k}$, we define sets of edges $(i,j)_d=\{(i_lj_{l+d})|l\in Z_{2k}\}$ for $i,j\in Z_{2t}$. Clearly, $(i,j)_d$ is a perfect matching in $K_{V_i,V_j}$. In fact,

$$E(K_{V_i,V_j})=\bigcup_{d=0}^{2k-1}(i,j)_d.$$

The following lemmas are useful in our constructions.

**Lemma 2.1.** [6] Let $I_{2n}=\{(v_0v_n)\}\cup\{(v_iv_{2n-i})|1\le i\le n-1\}$. Then $K_{2n}-I_{2n}$ can be decomposed into $n-1$ HCs, Each HC can be decomposed into two 1-factors. Moreover, by reordering the vertices of $K_{2n}$ if necessary, we may assume one of the HCs is $(v_0,v_1,\ldots,v_{2n-1})$.

The following lemma is a generalization of Lemma 1 in [8].

**Lemma 2.2.** Let $\pi$ be a permutation of $Z_{2t}$, $d_0,d_1,\ldots,d_{2t-1}$ be non-negative integers. Then the set of edges

$$(\pi(0),\pi(1))_{d_0}\cup(\pi(1),\pi(2))_{d_1}\cup\cdots\cup(\pi(2t-1),\pi(0))_{d_{2t-1}}$$

forms an HC of $K_n$ if $d_0+d_1+\cdots+d_{2t-1}$ and $2k$ are relatively prime.

**Proof.** Set $d=d_0+d_1+\cdots+d_{2t-1}$, then arrange the edges as

$$H=(\pi(0)_0,\pi(1)_{d_0},\pi(2)_{d_0+d_1},\cdots,\pi(0)_d,\pi(1)_{d+d_0},\cdots,\pi(2t-1)_{2kd-d_{2t-1}}).$$

Since $(d,2k)=1$, the vertices

$$\pi(i)_{d_0+d_1+\cdots+d_{i-1}},\pi(i)_{d+d_0+d_1+\cdots+d_{i-1}},\ldots,\pi(i)_{(2k-1)d+d_0+d_1+\cdots+d_{i-1}}$$

are mutually distinct for $i\in Z_{2t}$. Thus all vertices in $H$ are mutually distinct, so $H$ is an HC. □

**Lemma 2.3.** Let $d_1, d_2$ be nonnegative integers. If $d_1 - d_2$ and $2k$ are relatively prime, then the set of edges $(i,j)_{d_1} \cup (i,j)_{d_2}$ forms a cycle of length $4k$ on the vertex set $V_i \cup V_j$.

**Proof.** It's a direct consequence of Lemma 2.2. Arranging the edges as a cycle $(i_0, j_{d_1}, i_{d_1-d_2}, j_{2d_1-d_2}, \cdots, j_{2kd_1-(2k-1)d_2})$ completes the proof.□

# 3 Proof of the main theorem

With the above preparations, now we are ready to prove our main theorem.

Let $\widetilde{G}$ be a complete graph defined on $\{V_0, V_1, \ldots, V_{2t-1}\}$. By Lemma 2.1, $\widetilde{G}$ can be decomposed into $2t-1$ 1-factors, denoted by $\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_{2t-1}$, and $\widetilde{F}_{2i-1} \cup \widetilde{F}_{2i}$ forms an HC for $i = 1, 2, \ldots, t-1$. By reordering the vertices if necessary, we may assume

$$\widetilde{F}_1 = \{V_0V_1, V_2, V_3, \ldots, V_{2t-2}V_{2t-1}\},$$

$$\widetilde{F}_2 = \{V_1V_2, V_3V_4, \ldots, V_{2t-1}V_0\},$$

$$\widetilde{F}_{2t-1} = \{V_0V_t\} \cup \{V_iV_{2t-i} | i = 1, 2, \ldots, t-1\}.$$

Let

$$F_x = \bigcup_{V_iV_j \in E(\widetilde{F}_x)} E(K_{V_i,V_j}) \quad for\ x \in Z_{2t} \backslash \{0\}$$

and

$$H_l = (0,1)_l \cup (1,2)_{2k-l} \cup (2,3)_l \cup \cdots \cup (2t-1,0)_{2k-l} \quad for\ l \in Z_{2k}.$$

Then $F_1 \cup F_2 = H_0 \cup H_1 \cup \cdots \cup H_{2k-1}$.

**Lemma 3.1.** $F_{2i-1} \cup F_{2i} (i = 0, 1, \ldots, k-1)$ can be decomposed into $r_i \in \{0, 2, \ldots, 2k\}$ HCs and $2k - r_i$ $C_{4k}$-factors of $K_n$.

**Proof.** We only give the proof for the case $i = 1$, i.e. $F_1 \cup F_2$, the remaining cases are similar.

For $l = 0, 1, \ldots, k-1$, $H_{2l} \cup H_{2l+1}$ can be decomposed into two edge sets:

$$\bigcup_{j=0}^{t-1} ((2j, 2j+1)_{2l} \bigcup (2j, 2j+1)_{2l+1}),$$

$$\bigcup_{j=0}^{t-1} ((2j+1, 2j+2)_{2k-2l} \bigcup (2j+1, 2j+2)_{2k-2l-1}),$$

by Lemma 2.3, each forms a $C_{4k}$-factor of $K_n$.

Similarly, $H_{2l} \cup H_{2l+1}$ can be decomposed into another two edge sets:

$$(H_{2l} - (2t-1, 0)_{2k-2l}) \cup (2t-1, 0)_{2k-2l-1},$$

$$(H_{2l+1} - (2t-1, 0)_{2k-2l-1}) \cup (2t-1, 0)_{2k-2l},$$

by Lemma 2.2, each forms an HC of $K_n$.

Finally, by decomposing $H_{2l} \cup H_{2l+1}$ into two HCs when $l \in \{0, 1, \ldots, \frac{r_1}{2} - 1\}$ or into two $C_{4k}$-factors when $l \in \{\frac{r_1}{2}, \frac{r_1}{2} + 1, \ldots, k-1\}$, we have the proof.□

**Lemma 3.2.** For each $i \in Z_{2t} \backslash \{0\}$, $F_i \cup (\bigcup_{i \in Z_{2t}} K_{V_i})$ can be decomposed into $2k-1$ $C_{4k}$-factors and a 1-factor of $K_n$.

**Proof.** Noticing that $F_i \cup (\bigcup_{i \in Z_{2t}} K_{V_i}) = tK_{4k}$ and these complete graphs of order $4k$ are edge-disjoint. By Lemma 2.1, each can be decomposed into $2k-1$ HCs and one 1-factor of $K_{4k}$. Hence, these HCs and 1-factors form $2k-1$ $C_{4k}$-factors and a 1-factor of $K_n$. This concludes the proof. □

For convenience in presentation, we use X to denote $\bigcup_{i \in Z_{2t}} K_{V_i}$ in what follows.

**Proposition 3.3.** $\{0, 2, 4, \ldots, \frac{n}{2} - 2k\} \subseteq HW^*(n; n, 4k)$ for all positive integers $n \equiv 0 \pmod{4k}$.

**Proof.** Since $K_n = F_1 \cup F_2 \cup \cdots \cup F_{2t-1} \cup \mathrm{X}$, applying Lemma 3.2 to $F_{2t-1} \cup \mathrm{X}$ and Lemma 3.1 to $F_{2i} \cup F_{2i-1} (1 \le i \le t-1)$ completes the proof. □

**Proposition 3.4.** $\{1, 3, 5, \ldots, \frac{n}{2} - 4k + 1\} \subseteq HW^*(n; n, 4k)$ for all positive integers $n \equiv 0 \pmod{4k}$.

**Proof.** First, by Lemma 3.2, we decompose $F_2 \cup \mathrm{X}$ into $2k-1$ $C_{4k}$-factors and a 1-factor. Without loss of generality, assume the 1-factor is $I_n' = (1,2)_0 \cup (3,4)_0 \cup \cdots \cup (2t-1, 0)_0$.

Since $E(F_1) = \bigcup_{i=0}^{2k-1} ((0,1)_i \bigcup (2,3)_i \cdots (2t-2, 2t-1)_i)$, we decompose $E(F_1) \cup I_n'$ into $k-1$ $C_{4k}$-factors, an HC and a 1-factor:

$$C_i = ((0,1)_{2i-1} \cup (0,1)_{2i}) \cup ((2,3)_{2i-1} \cup (2,3)_{2i}) \cup \cdots \cup ((2t-2, 2t-1)_{2i-1} \cup (2t-2, 2t-1)_{2i}), \quad i = 1, 2, \ldots, k-1,$$

$$HC_1 = (0,1)_{2k-1} \cup (1,2)_0 \cup (2,3)_0 \cup \cdots \cup (2t-2, 2t-1)_0,$$

$$I_n = (0,1)_0 \cup (2,3)_{2k-1} \cup (4,5)_{2k-1} \cdots \cup (2t-2, 2t-1)_{2k-1}.$$

It is straightforward to verify that $C_i$ is a $C_{4k}$-factor, $HC_1$ is an HC, $I_n$ is a 1-factor and they are edge-disjoint.

Finally, applying Lemma 3.1 to $F_{2i-1} \cup F_{2i} (2 \le i \le t-1)$ gives $\{1, 3, 5, \ldots, \frac{n}{2} - 4k + 1\} \subseteq HW^*(n; n, 4k)$. □

**Lemma 3.5.** If $r_1 \in \{2k, 2k+1, 2k+2, \ldots, 4k-1\}$, then $F_1 \cup F_2 \cup F_{2t-1} \cup \mathrm{X}$ can be decomposed into $r_1$ HCs, $4k - 1 - r_1$ $C_{4k}$-factors and a 1-factor of $K_n$.

**Proof.** It is well known that every complete graph with even order can be decomposed into Hamilton paths[2]. Noticing that

$$F_{2t-1} \cup \mathrm{X} = \{K_{V_0 \cup V_t}\} \cup \{K_{V_i \cup V_{2t-i}} | i = 1, 2, \ldots, t-1\} = tK_{4k}$$

and these complete graphs of order $4k$ have no common vertex. Let $P_{i,j}[u \ldots v]$ be the Hamilton path of $K_{V_i \cup V_j}$ with $u$ and $v$ as its end vertices. We may decompose $F_{2t-1} \cup X$ into $\{P_0, P_1, \ldots, P_{2k-1}\}$ where

$$P_j = \{P_{0,t}[0_j, \ldots, t_j]\} \cup \{P_{i,2t-i}[i_j, \ldots, (2t-i)_j] | i = 1, 2, \ldots, t-1\}.$$

For each $j$, connecting the Hamilton paths of $P_j$ with $t$ edges $(0_j 1_j)$, $(2_j 3_j), \ldots, ((2t-2)_j (2t-1)_j) \in (0,1)_0 \cup (2,3)_0 \cup \cdots \cup (2t-2, 2t-1)_0 \subseteq H_0$ which gives an $HC$. Then we have $2k$ Hamilton cycles $HC_j$, $j \in Z_{2k}$, when $t$ is odd,

$$\begin{aligned} HC_j = \quad & (0_j, 1_j, P_{1,2t-1}[1_j, \ldots, (2t-1)_j], (2t-1)_j, (2t-2)_j, \\ & P_{2t-2,2}[(2t-2)_j, \ldots, 2_j], \ldots, (t-1)_j, t_j, P_{t,0}[t_j, \ldots, 0_j]); \end{aligned}$$

when $t$ is even,

$$\begin{aligned} HC_j = \quad & (0_j, 1_j, P_{1,2t-1}[1_j, \ldots, (2t-1)_j], (2t-1)_j, (2t-2)_j, \\ & P_{2t-2,2}[(2t-2)_j, \ldots, 2_j], \ldots, (t+1)_j, t_j, P_{t,0}[t_j, \ldots, 0_j]). \end{aligned}$$

Then we can decompose $H_1 \cup (H_0 - (0,1)_0 \cup (2,3)_0 \cup \cdots \cup (2t-2, 2t-1)_0)$ into an HC and a 1-factor, or a $C_{4k}$-factor and a 1-factor. In the first case, let

$$HC_{2k} = H_1 \cup (2t-1, 0)_0 - (2t-1, 0)_{2k-1},$$

$$I_n = (1,2)_0 \cup (3,4)_0 \cup \cdots \cup (2t-3, 2t-2)_0 \cup (2t-1, 0)_{2k-1}.$$

By Lemma 2.2, $HC_{2k}$ forms an HC. $I_n$ is a 1-factor. In the second case, let

$$C = \bigcup_{j=0}^{t-1} \{(2j+1, 2j+2)_0 \bigcup (2j+1, 2j+2)_{2k-1}\},$$

$$I_n' = (0,1)_1 \cup (2,3)_1 \cup \cdots \cup (2t-2, 2t-1)_1.$$

By Lemma 2.3, $C$ is a $C_{4k}$-factor and $I_n'$ is a 1-factor.

Finally, in the same way as Lemma 3.1, for each $r_1 \in \{2k, 2k+2, 2k+4, \ldots, 4k-2\}$, we decompose each $H_{2l} \cup H_{2l+1}$ into two HCs for $l \in \{1, 2, \ldots, \frac{r_1}{2}\}$ or two $C_{4k}$-factors for $l \in \{\frac{r_1}{2}+1, \frac{r_1}{2}+2, \ldots, k-1\}$. Then we have the proof. □

**Proposition 3.6.** $\{2k, 2k+1, 2k+2, \ldots, \frac{n-2}{2}\} \subseteq HW^*(n; n, 4k)$ for all positive integers $n \equiv 0 \pmod{4k}$.

**Proof.** Let $r = p \cdot 2k + q$, where $0 \leq q < 2k$. If $2k \leq r \leq 2kt - 2k$ and $q$ is even, by Lemma 3.5, we may decompose $F_1 \cup F_2 \cup F_{2t-1} \cup X$ into $2k$ HCs, $2k-1$ $C_{4k}$-factors and a 1-factor. By Lemma 3.1, we may decompose $F_{2i-1} \cup F_{2i}$ into $2k$ HCs for each $2 \leq i \leq p$, $F_{2p+1} \cup F_{2p+2}$ into $q$ HCs and $2k-q$ $C_{4k}$-factors, and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+2 \leq j \leq t-1$. Then we have

$$\{2k, 2k+2, \ldots, 2kt-2k\} \subseteq HW^*(n; n, 4k).$$

If $2k \leq r \leq 2kt - 2k$ and $q$ is odd, by Lemma 3.5, we may decompose $F_1 \cup F_2 \cup F_{2t-1} \cup X$ into $2k+1$ HCs, $2k-2$ $C_{4k}$-factors and a 1-factor. By

Lemma 3.1, we may decompose $F_{2i-1} \cup F_{2i}$ into $2k$ HCs for each $2 \le i \le p$, $F_{2p+1} \cup F_{2p+2}$ into $q-1$ HCs and $2k-q+1$ $C_{4k}$-factors, and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+2 \le j \le t-1$. Then we have

$$\{2k+1, 2k+3, \ldots, 2kt-2k-1\} \in HW^*(n; n, 4k).$$

If $2kt-2k < r \le \frac{n-2}{2}$ and $q$ is even, by Lemma 3.5, we may decompose $F_1 \cup F_2 \cup F_{2t-1} \cup X$ into $4k-2$ HCs, a $C_{4k}$-factor and a 1-factor. When $q+2 < 2k$, by Lemma 3.1, we may decompose $F_{2i-1} \cup F_{2i}$ into $2k$ HCs for each $2 \le i \le p-1$, $F_{2p-1} \cup F_{2p}$ into $q+2$ HCs and $2k-q-2$ $C_{4k}$-factors, and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+1 \le j \le t-1$; when $q+2 = 2k$, we decompose $F_{2i-1} \cup F_{2i}$ into $2k$ HCs for each $2 \le i \le p$ and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+1 \le j \le t-1$. Then we have

$$\{2kt-2k+2, 2kt-2k+4, \ldots, 2kt-2\} \in HW^*(n; n, 4k).$$

If $2kt-2k < r \le \frac{n-2}{2}$ and $q$ is odd, by Lemma 3.5, we may decompose $F_1 \cup F_2 \cup F_{2t-1} \cup X$ into $4k-1$ HCs and a 1-factor. When $q+1 = 2k$,by Lemma 3.1, we may decompose each $F_{2i-1} \cup F_{2i}$into $2k$ HCs for each $2 \le i \le p$ and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+1 \le i \le t-1$; when $q+1 \ne 2k$, we decompose $F_{2i-1} \cup F_{2i}$into $2k$ HCs for each $2 \le i \le p-1$, $F_{2p-1} \cup F_{2p}$ into $q+1$ HCs and $2k-q-1$ $C_{4k}$-factors, and $F_{2j-1} \cup F_{2j}$ into $2k$ $C_{4k}$-factors for each $p+1 \le j \le t-1$. Then we have

$$\{2kt-2k+1, 2kt-2k+3, \ldots, 2kt-1\} \in HW^*(n; n, 4k). \square$$

Combining Proposition 3.3, Proposition 3.4 and Proposition 3.6, we have the main result of this paper.

**Theorem 3.7.** $\{0, 1, 2, \ldots, \frac{n-2}{2}\} = HW^*(n; n, 4k)$ for all positive integers $n \equiv 0 \pmod{4k}$.

**Proof.** For $n = 4k$, the theorem is obvious by Theorem 1.5. For $n = 8k$, the result is also correct by Theorem 1.4. When $n > 8k$, we have $\frac{n}{2} - 2k > 2k$ and $\frac{n}{2} - 4k + 1 \ge 2k + 1$, then combining with Proposition 3.3, Proposition 3.4 and Proposition 3.6 completes the proof. $\square$

## 4 Concluding remarks

It would be interesting to determine the necessary and sufficient conditions for the existence of an $HW(n; r, s; n, k)$ for any even integer $k$. As a first step, we proved in this paper that for any integer $k \equiv 0 \pmod 4$ the necessary condition for the existence of $HW(n; r, s; n, k)$ is $n \equiv 0 \pmod k$, and the necessary condition is also sufficient. The next step is for the case when $k \equiv 2 \pmod 4$, we conjecture that for $k \equiv 2 \pmod 4$ and $s > 0$ there exists an $HW(n; r, s; n, k)$ if and only if $n \equiv 0 \pmod k$.

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