Fundamental relation on Γ-hyperrings

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Abstract

In this paper, we apply the concept of fundamental relation on Γ-hyperrings and obtain some related results. Specially, we show that there is a covariant functor between the category of Γ-hyperrings and the category of fundamental Γ/β^*-rings.

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1 Introduction

The theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by Marty in 1934 [14] when he defined the hypergroups and began to investigate their properties with applications to groups, rational functions and algebraic functions. Some review of the theory of hyperstructures can be found in [3, 4, 5, 18]. In a recent monograph of Corsini and Leoreanu [4], the authors have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Another monograph is devoted especially to the study of hyperring theory, written by Davvaz and Leoreanu-Fotea [5]. It begins with some basic results concerning ring theory and algebraic hyperstructures, which represent the most general algebraic context, in which the reality can be modelled. Several kinds of hyperrings are introduced and analyzed in this book.

The fundamental relation β^* was introduced on hypergroups by Koskas [11], for the first time and studied by many authors, for example see [3, 7, 8, 9, 20]. The fundamental relation is defined on hypergroups as the smallest equivalence relation so that the quotient would be a group. Vougiouklis in [19] defined the fundamental relation γ^* on a hyperring R as the smallest equivalence relation on R such that the quotient R/γ^* is a fundamental ring. Let R be a hyperring. Vougiouklis defined the relation γ as follows: $a\gamma b$ if and only if $\{a,b\} \subseteq u$, where u is a finite sum of finite products of elements of R (u may be a sum of only one element), and proved that γ^* is the transitive closure of γ. The fundamental equivalence relation extended to some classes of hyperrings by Anvariyeh, Davvaz, Hedayati, Mirvakili, Spartalis, Vougiouklis and others, for example see [1, 5, 10, 17, 18].

The notion of Γ-rings was introduced by N. Nobosawa in [15] and immediately after him in 1966, Barnes extended this notion and obtained more results [2]. Almost 10 years later Kyuno in [12, 13] investigated of new aspects of Γ-rings such as: prime Γ-rings and left and right unities of Γ-rings.

In this paper, we apply the concept of fundamental relation on Γ-hyperrings and obtain some related results. Specially, we show that there is a covariant functor between the category of Γ-hyperrings and the category of fundamental Γ/β^*-rings.

2 Preliminaries

In this section, we gather all definitions and simple properties of $\Gamma-$rings and hyperstructures and set the notions.

Definition 2.1. ([2, 15]) Let $(M,+)$ and $(\Gamma,+)$ be commutative groups. Then M is said to be a Γ-*ring*, if there exists a mapping $\cdot : M\times\Gamma\times M \longrightarrow M$ (the image is denoted by $x\alpha y$ for $x,y \in M$ and $\alpha \in \Gamma$) such that the following conditions are satisfied for all $x,y,z \in M$ and $\alpha,\gamma \in \Gamma$:

(1) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x\alpha(y+z) = x\alpha y + x\alpha z$;

(2) $x(\alpha+\beta)y = x\alpha y + x\beta y$;

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$;

(4) $0_M\alpha y = x\alpha 0_M = 0_M$;

where 0_M is the zero element of M. In this case, by (M,Γ) we mean M is a Γ-ring.

Let M_1 be a Γ_1-ring and M_2 a Γ_2-ring. Then $(f,\psi) : (M_1,\Gamma_1) \longrightarrow (M_2,\Gamma_2)$ is called a *homomorphism* if $f : M_1 \longrightarrow M_2$ and $\psi : \Gamma_1 \longrightarrow \Gamma_2$ are group homomorphisms and $f(x\gamma y) = f(x)\psi(\gamma)f(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. A homomorphism $(f,\psi) : (M_1,\Gamma_1) \longrightarrow (M_2,\Gamma_2)$ is called an *isomorphism* if $f : M_1 \longrightarrow M_2$ and $\psi : \Gamma_1 \longrightarrow \Gamma_2$ are group isomorphisms.

Let H be a non-empty set. A map $+ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* or *join operation*, where $P^*(H)$ denotes the set of all non-empty subsets of H.

Definition 2.2. A non-empty set M together with a hyperoperation $+$ is called a *polygroup* if the following conditions are satisfied:

(1) for all $x, y, z \in M$, $(x+y)+z = x+(y+z)$;

(2) for all $x \in M$, there exists an unique element $e \in M$ such that $e + x = x = x + e$ (we denote e by 0);

(3) for all $x \in M$, there exists a unique element $x' \in M$ such that $e \in x + x' \cap x' + x$ (we denote x' by $-x$);

(4) for all $x, y, z \in M$, $z \in x + y \Longrightarrow x \in z - y \Longrightarrow y \in z - x$.

A *canonical hypergroup* is a commutative polygroup. It is easy to see that every commutative group is a canonical hypergroup.

Let M_1 and M_2 be polygroups. Then $f : M_1 \longrightarrow M_2$ is called a *homomorphism (good homomorphism)* if $f(xy) \subseteq f(x)f(y)$ ($f(xy) = f(x)f(y)$) for all $x, y \in M$. A homomorphism (good homomorphism) $f : M_1 \longrightarrow M_2$ is called an *isomorphism (good isomorphism)* if f is one to one and onto.

Let $(M, +)$ be a polygroup. We define the relation β as follows:

$$a\beta b \Longleftrightarrow \{a, b\} \subseteq u, \ \exists u \in U_M,$$

where $U_M = U$ is the set of all finite sums of the elements of M. By β^* we mean the *transitive closure* of β. It is shown that β^* is an equivalence relation. We denote the equivalence class of $a \in M$ by $\beta^*(a)$. Thus every element $u \in U$ can be written as

$$u = \sum_{\text{finite}} x, \ x \in M.$$

Then β^* is the smallest equivalence relation on M such that M/β^* is a group (see [3, 18]). The relation β^* is called the *fundamental relation* on M.

Definition 2.3. A triple $(R, +, \cdot)$ is called a *hyperring* if

(1) $(R, +)$ is a canonical hypergroup;

(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 = 0 \cdot x$;

(3) for all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

A hyperring R is said to be with *identity element*, if there exists an element $1 \in R$ such that $x \cdot 1 = x = 1 \cdot x$ for all $x \in R$.

Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows:

$$a\gamma b \iff \{a, b\} \subseteq u, \ \exists u \in U_R,$$

where $U_R = U$ is the set of all finite sum of finite products of the elements of R. By γ^* we mean the *transitive closure* of γ. It is shown that γ^* is an equivalence relation. We denote the equivalence class of $a \in R$ by $\gamma^*(a)$. Thus every element $u \in U$ can be written as

$$u = \sum_{j \in J} \prod_{i \in I_j} x_{i_j}, \ x_{i_j} \in R.$$

Hence, every element $u \in U$ is a polynomial of elements of R with coefficients in $\mathbb{N}$ (see [5, 18]). Then γ^* is the smallest equivalence relation on R such that R/γ^* is a ring. The relation γ^* is called the *fundamental relation* on R and has important role in the study of hyperstructure theory.

3 Fundamental relation on Γ-hyperrings

Definition 3.1. Let $(M, +)$ and $(\Gamma, +)$ be canonical hypergroups. Then M is said to be a Γ-*hyperring* if there exists a mapping $\cdot : M \times \Gamma \times M \longrightarrow P^*(M)$ such that the following conditions are satisfied for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$:

(1) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x\alpha(y + z) = x\alpha y + x\alpha z$;

(2) $x(\alpha + \beta)y = x\alpha y + x\beta y$;

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$;

(4) $0_M\alpha y = x\alpha 0_M = 0_M$;

where 0_M is the zero element of M. In this case, by (M, Γ) we mean M is a Γ-hyperring. Let M be a Γ-hyperring. We say M has an *identity element* if there exists an element 1_M such that $x \in 1_M\gamma x \cap x\gamma 1_M$ for all $x \in M$ and $\gamma \in \Gamma$.

Example 3.2. Let $(M,+,\cdot)$ be a hyperring and Γ be a hyperideal of M. We define $\circ : M\times\Gamma\times M \longrightarrow P^*(M)$ by $\circ(a,\gamma,b) = a\circ\gamma\circ b = a\cdot\alpha\cdot b$, for $a,b\in M$ and $\alpha\in\Gamma$. Then, it is easy to verify that M is a Γ-hyperring.

Example 3.3. Let R be a hyperring and $M_{m,n}(R)$ is the set of all $m\times n$ matrices with entries in R. We define $\circ : M_{m,n}(R)\times M_{n,m}(R)\times M_{m,n}(R) \longrightarrow P^*(M_{m,n}(R))$ by:

$$A\circ B\circ C = \{Z\in M_{m,n}(R) \mid Z\in ABC\},$$

for all $A,C\in M_{m,n}(R)$ and $B\in M_{n,m}(R)$. Then, it easy to verify that $M_{m,n}(R)$ is an $M_{n,m}(R)$-hyperring.

Example 3.4. Let $(P,\circ)$ be a canonical hypergroup and N a subpolygroup of P. Define $* : P\times N\times P \longrightarrow P^*(P)$ by $(x,n,y)\mapsto x*n*y = x\circ n\circ y$. It is easy to verify that P is an N-hyperring. By this example we obtain a large class of Γ-hyperrings.

In what follows, M is a Γ-hyperring, unless otherwise specified.

Definition 3.5. Let M be a Γ-hyperring and β^* the fundamental relation on Γ. We define the relation σ^* as the smallest equivalence relation such that the quotient M/σ^* is a Γ/β^*-ring. Then the relation σ^* is called the *fundamental equivalence relation* on Γ-hyperring M, and M/σ^* is called the *fundamental* Γ/β^**-ring.*

Let us denote the set $\mathcal{U}_{(M,\Gamma)} = \mathcal{U}$ as follows:

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} a_i\gamma_i b_i + \sum_{j=1}^{m} c_j \mid a_i,b_i,c_i\in M,\ \gamma_i\in\Gamma,\ m,n\in\mathbb{N} \right\}.$$

Now, we define the relation σ on M as follows:

$$x\sigma y \Longleftrightarrow \exists u\in\mathcal{U} :\ \{x,y\}\subseteq u.$$

Let $\overline{\sigma}$ be the transitive closure of σ. For all $a,b\in M$ and $\gamma\in\Gamma$, we define:

$$\overline{\sigma}(a)\oplus\overline{\sigma}(b) = \{\overline{\sigma}(c) \mid c\in\overline{\sigma}(a)+\overline{\sigma}(b)\},$$
$$\overline{\sigma}(a)\circ\beta^*(\gamma)\circ\overline{\sigma}(b) = \{\overline{\sigma}(c) \mid c\in\overline{\sigma}(a)\beta^*(\gamma)\overline{\sigma}(b)\}.$$

Lemma 3.6. *The sets* $\overline{\sigma}(a)\oplus\overline{\sigma}(b)$ *and* $\overline{\sigma}(a)\circ\beta^*(\gamma)\circ\overline{\sigma}(b)$ *are singleton, i.e.,*

$$|\overline{\sigma}(a)\oplus\overline{\sigma}(b)| = 1 = |\overline{\sigma}(a)\circ\beta^*(\gamma)\circ\overline{\sigma}(b)|.$$

Proof. Let $\overline{\sigma}(c) \in \overline{\sigma}(a) \oplus \overline{\sigma}(b)$. Then $c \in \overline{\sigma}(a) + \overline{\sigma}(b)$ so there exist $a' \in \overline{\sigma}(a)$ and $b' \in \overline{\sigma}(b)$ such that $c \in a' + b'$. It is enough we prove $\overline{\sigma}(z) = \overline{\sigma}(z')$ for all $z \in a + b$ and $z' \in a' + b'$. We know that $a'\overline{\sigma}a$ if and only if there exist $x_1, ..., x_{m+1} \in M$ with $x_1 = a'$ and $x_{m+1} = a$ and there exist $u_1, ..., u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $i = 1, 2, ..., m$. Also $b'\overline{\sigma}b$ if and only if there exist $y_1, ..., y_{n+1} \in M$ with $y_1 = b'$ and $y_{n+1} = b$ and there exist $v_1, ..., v_n \in \mathcal{U}$ such that $\{y_j, y_{j+1}\} \subseteq v_j$ for $j = 1, 2, ..., n$. Therefore, we obtain

$$\begin{cases} \{x_i, x_{i+1}\} + y_1 \subseteq u_i + v_1, & i = 1, 2, \ldots, m-1 \\ x_{m+1} + \{y_j, y_{j+1}\} \subseteq u_m + v_j, & j = 1, 2, \ldots, n. \end{cases} \qquad (*)$$

Therefore, $u_i + v_1 = t_i \in \mathcal{U}$ for $i = 1, 2, \ldots, m-1$ and $u_m + v_j = t_{m+j-1} \in \mathcal{U}$ for $j = 1, 2, \ldots, n$.

Now, choose the elements $z_1, z_2, \ldots, z_{m+n}$ such that $z_i \in x_i + y_1$ for $i = 1, ..., m$ and $z_{m+j} \in x_{m+1} + y_{j+1}$ for $j = 1, \ldots, n$. By using $(*)$, we have $\{z_k, z_{k+1}\} \subseteq t_k$ for $k = 1, \ldots, m+n-1$. So every element $z_1 \in x_1 + y_1 = a' + b'$ is equivalent to every element $z_{m+n} \in x_{m+n} + y_{m+n} = a + b$ with respect to the $\overline{\sigma}$. Therefore $|\overline{\sigma}(a) \oplus \overline{\sigma}(b)| = 1$ and we can write $\overline{\sigma}(a) \oplus \overline{\sigma}(b) = \overline{\sigma}(c)$ for all $c \in \overline{\sigma}(a) + \overline{\sigma}(b)$.

Now, we prove $|\overline{\sigma}(a) \circ \beta^*(\gamma) \circ \overline{\sigma}(b)| = 1$. Let $\overline{\sigma}(c) \in \overline{\sigma}(a) \circ \beta^*(\gamma) \circ \overline{\sigma}(b)$. Then $c \in \overline{\sigma}(a)\beta^*(\gamma)\overline{\sigma}(b)$. So there exist $a' \in \overline{\sigma}(a)$, $\gamma' \in \beta^*(\gamma)$ and $b' \in \overline{\sigma}(b)$ such that $c \in a'\gamma'b'$. It is enough we prove that for all $z \in a\gamma b$ and $z' \in a'\gamma'b'$, $\overline{\sigma}(z) = \overline{\sigma}(z')$.

We have $a'\overline{\sigma}a$ if and only if there exist $x_1, \ldots, x_{m+1} \in M$ with $x_1 = a'$ and $x_{m+1} = a$ and there exist $u_1, \ldots, u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $1 \leq i \leq m$. Also, $b'\overline{\sigma}b$ if and only if there exist $y_1, \ldots, y_{n+1} \in M$ with $y_1 = b'$ and $y_{n+1} = b$ and there exist $v_1, \ldots, v_n \in \mathcal{U}$ such that $\{y_j, y_{j+1}\} \subseteq v_j$ for $1 \leq j \leq n$. Also $\gamma'\beta^*\gamma$ if and only if there exist $\gamma_1, \ldots, \gamma_{k+1} \in \Gamma$ with $\gamma_1 = \gamma'$ and $\gamma_{k+1} = \gamma$ and there exist $\Delta_1, \ldots, \Delta_k \in U_\Gamma$ such that $\{\gamma_l, \gamma_{l+1}\} \subseteq \Delta_l$ for $1 \leq l \leq k$. Thus, we have

$$\begin{cases} \{x_i, x_{i+1}\}.\{\gamma_l, \gamma_{l+1}\}.y_1 \subseteq u_i\Delta_l v_1, & 1 \leq i \leq m-1,\ 1 \leq l \leq k \\ x_{m+1}.\{\gamma_l, \gamma_{l+1}\}.\{y_j, y_{j+1}\} \subseteq u_m\Delta_l v_j, & 1 \leq j \leq n,\ 1 \leq l \leq k. \end{cases} \qquad (**)$$

Now, set $u_i\Delta_l v_1 = t_{il} \in \mathcal{U}$ for $1 \leq i \leq m-1$ and $1 \leq l \leq k$, and also $u_m\Delta_l v_j = t_{(m+j-1)l} \in \mathcal{U}$ for $1 \leq j \leq n$ and $1 \leq l \leq k$. So we can say $t_{ql} \in \mathcal{U}$ for $1 \leq q \leq m+n-1$ and $1 \leq l \leq k$. Now choose $z_{1l}, \ldots, z_{(m+n)l} \in M$ such that $z_{il} \in x_i\gamma_l y_1$ for $1 \leq i \leq m$ and $1 \leq l \leq k$, and also $z_{(m+j)l} \in x_{m+1}\gamma_l y_{j+1}$ for $1 \leq j \leq n$ and $1 \leq l \leq k$. Now, by using $(**)$, we have $\{z_{ql}, z_{(q+1)(l+1)}\} \subseteq t_{ql}$ for $1 \leq q \leq m+n-1$ and $1 \leq l \leq k$. Hence for all $z_{11} \in x_1\gamma_1 y_1 = a'\gamma'b'$ and $z_{(m+n)(k+1)} \in x_{m+1}\gamma_{k+1}y_{n+1} = a\gamma b$ we have $z_{11}\overline{\sigma}z_{(m+n)(k+1)}$. Therefore, $|\overline{\sigma}(a) \circ \beta^*(\gamma) \circ \overline{\sigma}(b)| = 1$. □

Lemma 3.7. *$(M/\overline{\sigma}, \oplus)$ is a commutative group.*

Proof. Clearly, $\overline{\sigma}(0_M)$ is the zero element of $(M/\overline{\sigma}, \oplus)$, and for all $\overline{\sigma}(a) \in M/\overline{\sigma}$, $\overline{\sigma}(-a)$ is the opposite element such that $\overline{\sigma}(a) \oplus \overline{\sigma}(-a) = \overline{\sigma}(0_M)$. Also, associativity is obtained from associativity of $(M, +)$. □

In the next lemma, by the help of a Γ-hyperring, we construct a fundamental Γ/β^*−ring.

Lemma 3.8. *$M/\overline{\sigma}$ is a Γ/β^*−ring.*

Proof. For any $\overline{\sigma}(x), \overline{\sigma}(y), \overline{\sigma}(z) \in M/\overline{\sigma}$ and $\beta^*(\alpha), \beta^*(\gamma) \in \Gamma/\beta^*$ we prove:

(i) $\overline{\sigma}(x) \circ \beta^*(\alpha) \circ [\overline{\sigma}(y) \oplus \overline{\sigma}(z)] = \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y) \oplus \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(z)$ and $[\overline{\sigma}(y) \oplus \overline{\sigma}(z)] \circ \beta^*(\alpha) \circ \overline{\sigma}(x) = \overline{\sigma}(y) \circ \beta^*(\alpha) \circ \overline{\sigma}(x) \oplus \overline{\sigma}(z) \circ \beta^*(\alpha) \circ \overline{\sigma}(x)$.

(ii) $\overline{\sigma}(x) \circ [\beta^*(\alpha) \oplus \beta^*(\gamma)] \circ \overline{\sigma}(y) = \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y) \oplus \overline{\sigma}(x) \circ \beta^*(\gamma) \circ \overline{\sigma}(y)$.

(iii) $\overline{\sigma}(x) \circ \beta^*(\alpha) \circ [\overline{\sigma}(y) \circ \beta^*(\gamma) \circ \overline{\sigma}(z)] = [\overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y)] \circ \beta^*(\gamma) \circ \overline{\sigma}(z)$.

We only prove (i). The proofs of (ii) and (iii) are similar. Suppose that $\overline{\sigma}(a) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ [\overline{\sigma}(y) \oplus \overline{\sigma}(z)]$. By Lemma 3.6, we have $\overline{\sigma}(a) = \overline{\sigma}(a_1)$, where $a_1 \in x\alpha(y+z)$. So

$$\begin{aligned}
\overline{\sigma}(a) &= \overline{\sigma}(a_1) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ [\overline{\sigma}(y) \oplus \overline{\sigma}(z)] \\
&\Leftrightarrow a_1 \in x\alpha(y+z) \\
&\Leftrightarrow a \in x\alpha y + x\alpha z \\
&\Rightarrow \overline{\sigma}(a_1) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y) \oplus \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(z).
\end{aligned}$$

Therefore, $\overline{\sigma}(a) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y) \oplus \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(z)$.

Conversely, suppose that $\overline{\sigma}(a) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(y) \oplus \overline{\sigma}(x) \circ \beta^*(\alpha) \circ \overline{\sigma}(z)$. Then $\overline{\sigma}(a) = \overline{\sigma}(a_1) \oplus \overline{\sigma}(a_2)$, where we can assume that $a_1 \in x\alpha y$ and $a_2 \in x\alpha z$. By Lemma 3.6, we have $\overline{\sigma}(a) = \overline{\sigma}(b)$, where $b \in a_1 + a_2$. So, $b \in x\alpha y + x\alpha z = x\alpha(y+z)$. Therefore, $\overline{\sigma}(a) = \overline{\sigma}(b) \in \overline{\sigma}(x) \circ \beta^*(\alpha) \circ [\overline{\sigma}(y) \oplus \overline{\sigma}(z)]$. □

Lemma 3.9. *$\overline{\sigma}$ is the smallest equivalence relation on M such that $M/\overline{\sigma}$ is a Γ/β^*-ring. In other word $\overline{\sigma} = \sigma^*$.*

Proof. Let μ be an equivalence relation on M such that M/μ is a Γ/β^*-ring. We denote the equivalence class of $a \in M$ by $\mu(a)$. Then we have:

$$\mu(a) \oplus \mu(b) = \mu(c), \quad \forall c \in \mu(a) + \mu(b),$$

and

$$\mu(a) \circ \beta^*(\gamma) \circ \mu(b) = \mu(d), \quad \forall d \in \mu(a)\beta^*(\gamma)\mu(b).$$

Thus, for every $A \subseteq \mu(a)$, $B \subseteq \mu(b)$ and $\theta \subseteq \beta^*(\gamma)$ we can write $\mu(a) \oplus \mu(b) = \mu(a+b) = \mu(A+B)$ and $\mu(a) \circ \beta^*(\gamma) \circ \mu(b) = \mu(a\gamma b) = \mu(A\theta B)$.

By induction we can extend these relations on finite sums. Then for all $u \in \mathcal{U}$ and $x \in u$ we have $\mu(x) = \mu(u)$. Hence for all $m \in M$, $x \in \sigma(m)$ implies $x \in \mu(m)$. Also μ is transitivity closed, so if $(x,m) \in \overline{\sigma}$ implies $(x,m) \in \mu$. Therefore, $\overline{\sigma}$ is the smallest equivalence relation such that $M/\overline{\sigma}$ is a Γ/β^*-ring. □

Theorem 3.10. *The fundamental relation σ^* is the transitive closure of the relation σ.*

Proof. It is concluded by Lemmas 3.6, 3.7, 3.8 and 3.9. □

Definition 3.11. Let M_1 be a Γ_1-hyperring and M_2 a Γ_2-hyperring. Then $(f,\psi) : (M_1,\Gamma_1) \longrightarrow (M_2,\Gamma_2)$ is called a *homomorphism (good homomorphism)* if $f : M_1 \longrightarrow M_2$ and $\psi : \Gamma_1 \longrightarrow \Gamma_2$ are polygroups homomorphism (good homomorphism) and $f(x\gamma y) \subseteq f(x)\psi(\gamma)f(y)$ ($f(x\gamma y) = f(x)\psi(\gamma)f(y)$) for all $x,y \in M$ and $\gamma \in \Gamma$. A homomorphism (good homomorphism) $(f,\psi) : (M_1,\Gamma_1) \longrightarrow (M_2,\Gamma_2)$ is called an *isomorphism (good isomorphism)* if $f : M_1 \longrightarrow M_2$ and $\psi : \Gamma_1 \longrightarrow \Gamma_2$ are polygroup isomorphisms.

Lemma 3.12. *Let M be a Γ-hyperring, σ^* the fundamental relation on M and β^* the fundamental relation on Γ. Then $(\pi_M,\pi_\Gamma) : (M,\Gamma) \to (M/\sigma^*,\Gamma/\beta^*)$ is a good epimorphism, where $\pi_M : M \longrightarrow M/\sigma^*$ is defined by $\pi_M(x) = \sigma^*(x)$, and $\pi_\Gamma : \Gamma \longrightarrow \Gamma/\beta^*$ is defined by $\pi_\Gamma(\gamma) = \beta^*(\gamma)$.*

Proof. Clearly π_M and π_Γ are well-defined and onto. We prove that (π_M,π_Γ) is a good homomorphism. It is easy to verify that $\pi_M(x+y) = \pi_M(x) \oplus \pi_M(y)$ and $\pi_\Gamma(\alpha+\gamma) = \pi_\Gamma(\alpha) \oplus \pi_\Gamma(\gamma)$ for all $x,y \in M$ and $\alpha,\gamma \in \Gamma$. We prove $\pi_M(x\gamma y) = \pi_M(x) \circ \pi_\Gamma(\gamma) \circ \pi_M(y)$. Let $z \in x\gamma y \subseteq \pi_M(x)\pi_\Gamma(\gamma)\pi_M(y)$. Then $\pi_M(z) \in \pi_M(x) \circ \pi_\Gamma(\gamma) \circ \pi_M(y)$. By Lemma 3.6, we know that $|\pi_M(x) \circ \pi_\Gamma(\gamma) \circ \pi_M(y)| = 1$, hence $\pi_M(z) = \pi_M(x) \circ \pi_\Gamma(\gamma) \circ \pi_M(y)$, consequently $\pi_M(x\gamma y) = \pi_M(x) \circ \pi_\Gamma(\gamma) \circ \pi_M(y)$. Therefore, (π_M,π_Γ) is a good epimorphism from (M,Γ) to $(M/\sigma^*,\Gamma/\beta^*)$. □

Theorem 3.13. *Let M be a Γ-hyperring, σ^* the fundamental relation on M and β^* the fundamental relation on Γ.*

(i) *If there exist $A,A' \subseteq \sigma^*(a)$ and $B,B' \subseteq \sigma^*(b)$ for some $a,b \in M$ such that $x + A \subseteq B$ and $x' + A' \subseteq B'$, then $x\sigma^*x'$.*

(ii) *If M has identity element 1_M and $x\sigma^*x'$, then there exist $A,A' \subseteq \sigma^*(a)$, $B,B' \subseteq \sigma^*(b)$, $C,C' \subseteq \sigma^*(c)$ and $\Delta,\Delta' \subseteq \beta^*(\gamma)$ for some $a,b,c \in M$ and $\gamma \in \Gamma$ such that $x\Delta C \subseteq B \supseteq x + A$ and $x'\Delta'C' \subseteq B' \supseteq x' + A'$.*

Proof. (i) Since $A \subseteq \sigma^*(a)$ and $B \subseteq \sigma^*(b)$, so for all $t \in A$ and $s \in B$ we have $\sigma^*(a) = \sigma^*(t)$ and $\sigma^*(b) = \sigma^*(s)$. Hence $\pi_M(x) \oplus \pi_M(a) = \pi_M(b)$. Thus $\sigma^*(x) \oplus \sigma^*(a) = \sigma^*(b)$. Then $\sigma^*(x) = \sigma^*(b) \oplus \sigma^*(-a) = \sigma^*(x')$, which implies that $x\sigma^*x'$.

(ii) It is enough we take $A = A' = \sigma^*(0_M)$, $B = B' = \sigma^*(x) = \sigma^*(x')$, $C = C' = \sigma^*(1_M)$ and $\Delta = \Delta' = \beta^*(\gamma)$ for arbitrary $\gamma \in \Gamma$. Now, let $z \in x\Delta C \subseteq \sigma^*(x)\beta^*(\gamma)\sigma^*(1_M)$, then $\sigma^*(z) \in \sigma^*(x) \circ \beta^*(\gamma) \circ \sigma^*(1_M)$. But we know that $\sigma^*(x) \in \sigma^*(x) \circ \beta^*(\gamma) \circ \sigma^*(1_M)$ (since $\sigma^*(1_M)$ is the identity element of M/σ^*). In other hand, $|\sigma^*(x) \circ \beta^*(\gamma) \circ \sigma^*(1_M)| = 1$, so $\sigma^*(z) = \sigma^*(x)$, hence $z \in \sigma^*(x) = B$, thus $x\Delta C \subseteq B$.

Now, let $z \in x + A$. Then there exists $a \in A$ such that $z \in x + a$. Hence $\pi_M(z) = \pi_M(x) \oplus \pi_M(a)$. It follows that $\sigma^*(z) = \sigma^*(x) \oplus \sigma^*(0_M)$, which implies that $\sigma^*(z) = \sigma^*(x)$, i.e $z \in \sigma^*(x) = B$. Therefore, $x + A \subseteq B$. Similarly, we can prove that $x'\Delta'C' \subseteq B' \supseteq x' + A'$. □

Theorem 3.14. *Let M be a Γ-hyperring, σ^* the fundamental relation on M and β^* the fundamental relation on Γ.*

(i) *$x \in \sigma^*(0_M)$ if and only if there exists $A \subseteq \sigma^*(a)$ for some $a \in M$ such that $x + A \subseteq A$.*

(ii) *If M has the identity element 1_M, then $y \in \sigma^*(1_M)$ if and only if there exist $B \subseteq \sigma^*(b)$ and $\Delta \subseteq \beta^*(\gamma)$ for some $b \in M$ and $\gamma \in \Gamma$ such that $y\Delta B \subseteq B$.*

Proof. (i) Let $x \in \sigma^*(0_M)$, $a \in M$ and $A = \sigma^*(a)$. We have

$$\begin{aligned} z \in x + A &\Rightarrow \exists t \in A, z \in x + t \\ &\Rightarrow \sigma^*(z) = \sigma^*(x) \oplus \sigma^*(t) = \sigma^*(0_M) \oplus \sigma^*(a) = \sigma^*(a) \\ &\Rightarrow z \in \sigma^*(a) = A. \end{aligned}$$

Thus $x + A \subseteq A$.

Conversely, if there exists $A \subseteq \sigma^*(a)$ for some $a \in M$ such that $x + A \subseteq A$, then

$$\begin{aligned} \sigma^*(x) \oplus \sigma^*(a) = \sigma^*(a) &\Rightarrow \sigma^*(x) = \sigma^*(a) \oplus \sigma^*(-a) = \sigma^*(0_M) \\ &\Rightarrow x \in \sigma^*(0_M). \end{aligned}$$

(ii) Let $y \in \sigma^*(1_M)$, $b \in M$, $\gamma \in \Gamma$, $B = \sigma^*(b)$ and $\Delta = \beta^*(\gamma)$. Let $z \in y\Delta B$, we have

$$\sigma^*(z) \in \sigma^*(y) \circ \beta^*(\gamma) \circ \sigma^*(b) = \sigma^*(1_M) \circ \beta^*(\gamma) \circ \sigma^*(b) = \sigma^*(b),$$

so $z \in \sigma^*(b) = B$, which implies that $y\Delta B \subseteq B$.

Conversely, suppose that there exist $B \subseteq \sigma^*(b)$ and $\triangle \subseteq \beta^*(\gamma)$ for some $b \in M$ and $\gamma \in \Gamma$ such that $y\triangle B \subseteq B$. Then

$$\sigma^*(y) \circ \beta^*(\gamma) \circ \sigma^*(b) = \sigma^*(b) = \sigma^*(1_M) \circ \beta^*(\gamma) \circ \sigma^*(b).$$

In other hand $\sigma^*(1_M)$ is unique and hence $\sigma^*(y) = \sigma^*(1_M)$, which implies that $y \in \sigma^*(1_M)$. □

Proposition 3.15. *Let M be a Γ-hyperring. If $u = \sum_{i=1}^{n} a_i\gamma_i b_i + \sum_{j=1}^{k} c_j \in \mathcal{U}$, then*

$$\sigma^*(u) = [\oplus_{i=1}^{n} \sigma^*(a_i) \circ \beta^*(\gamma_i) \circ \sigma^*(b_i)] \oplus [\oplus_{j=1}^{n} \sigma^*(c_j)] = \sigma^*(z)$$

for all $z \in u$.

Proof. We have

$$\begin{aligned} z \in u &= \sum_{i=1}^{n} a_i\gamma_i b_i + \sum_{j=1}^{k} c_j \\ &\Rightarrow \sigma^*(z) \in [\oplus_{i=1}^{n} \sigma^*(a_i) \circ \beta^*(\gamma_i) \circ \sigma^*(b_i)] \oplus [\oplus_{j=1}^{n} \sigma^*(c_j)] \\ &\Rightarrow \sigma^*(z) = [\oplus_{i=1}^{n} \sigma^*(a_i) \circ \beta^*(\gamma_i) \circ \sigma^*(b_i)] \oplus [\oplus_{j=1}^{n} \sigma^*(c_j)] . \end{aligned}$$

In other hand, clearly

$$\sigma^*(u) = \sigma^*(z) = [\oplus_{i=1}^{n} \sigma^*(a_i) \circ \beta^*(\gamma_i) \circ \sigma^*(b_i)] \oplus [\oplus_{j=1}^{n} \sigma^*(c_j)] .$$

□

In the next theorem, by the help of Γ-rings, we construct Γ-hyperrings.

Theorem 3.16. *Let M be a Γ-ring and I a non-empty subset of M. Then M is a Γ-hyperring with the map*

$$\circ_I : M \times \Gamma \times M \longrightarrow P^*(M)$$

defined by $x \circ_I \gamma \circ_I y = x\Gamma I\gamma y$ for all $x, y \in M$ and $\gamma \in \Gamma$.

Proof. It is easy to verify that $\circ_I$ is well-defined. Then for all $x, y, z \in M$ and $\gamma, \gamma' \in \Gamma$ we have

$$\begin{aligned} x \circ_I \gamma \circ_I (y + z) &= x\Gamma I\gamma(y + z) \\ &= x\Gamma I\gamma y + x\Gamma I\gamma z \\ &= x \circ_I \gamma \circ_I y + x \circ_I \gamma \circ_I z. \end{aligned}$$

Similarly we can prove that

$$(x+y)\circ_I \gamma \circ_I z = x \circ_I \gamma \circ_I z + y \circ_I \gamma \circ_I z.$$

Also we have

$$\begin{aligned} x \circ_I (\gamma+\gamma') \circ_I y &= x\Gamma I(\gamma+\gamma')y \\ &= x\Gamma I\gamma y + x\Gamma I\gamma' y \\ &= x \circ_I \gamma \circ_I y + x \circ_I \gamma' \circ_I y. \end{aligned}$$

Finally, it is easy to prove that $0_M \circ_I \gamma \circ_I x = 0_M = x \circ_I \gamma \circ_I 0_M$. Therefore M is a Γ-hyperring. □

In the sequel, we prove that there exists a covariant functor between the category of Γ-hyperrings and the category of fundamental Γ/β^*-rings. For this we need the following theorem.

Theorem 3.17. *Let M_1 be a Γ_1-hyperring, M_2 a Γ_2-hyperring and σ_1^*, β_1^*, σ_2^* and β_2^* the fundamental relations on M_1, Γ_1, M_2 and Γ_2, respectively. If $(f,g):(M_1,\Gamma_1)\to(M_2,\Gamma_2)$ is a homomorphism, then there is a unique homomorphism $(f^*,g^*):(M_1/\sigma_1^*,\Gamma_1/\beta_1^*)\to(M_2/\sigma_2^*,\Gamma_2/\beta_2^*)$ such that the following diagram commutes:*

$$\begin{array}{ccccc} & (M_1,\Gamma_1) & \xrightarrow{(f,g)} & (M_2,\Gamma_2) & \\ (\pi_{M_1},\pi_{\Gamma_1}) & \downarrow & & \downarrow & (\pi_{M_2},\pi_{\Gamma_2}) \\ & (M_1/\sigma_1^*,\Gamma_1/\beta_1^*) & \xrightarrow{(f^*,g^*)} & (M_2/\sigma_2^*,\Gamma_2/\beta_2^*) & \end{array}$$

Moreover, if (f,g) is an isomorphism, then (f^*,g^*) is an isomorphism.

Proof. We define $f^*: M_1/\sigma_1^* \to M_2/\sigma_2^*$ by $f^*(\sigma_1^*(x)) = \sigma_2^*(f(x))$ for all $\sigma_1^*(x)\in M_1/\sigma_1^*$ and $g^*:\Gamma_1/\beta_1^*\to\Gamma_2/\beta_2^*$ by $g^*(\beta_1^*(\gamma)) = \beta_2^*(g(\gamma))$ for all $\beta_1^*(\gamma)\in\Gamma_1/\beta_1^*$. Clearly $f^*\circ\pi_{M_1} = \pi_{M_2}\circ f$ and $g^*\circ\pi_{\Gamma_1} = \pi_{\Gamma_2}\circ g$. Therefore, the diagram commutes. We prove (f^*,g^*) is a homomorphism.

Let $\sigma_1^*(x)=\sigma_1^*(y)$, i.e., $x\sigma_1^* y$. Then there exist $a_1,\cdots,a_{m+1}\in M$ and $u_1,\cdots,u_m\in\mathcal{U}_{(M_1,\Gamma_1)}$ by $x=a_1$ and $y=a_{m+1}$ such that $\{a_i,a_{i+1}\}\subseteq u_i$ for all $1\le i\le m$. Now, since f is a homomorphism we have

$$\begin{aligned} f(u_i)\in\mathcal{U}_{(M_2,\Gamma_2)} &\Rightarrow \{f(a_i),f(a_{i+1})\}\subseteq f(u_i)\in\mathcal{U}_{(M_2,\Gamma_2)} \\ &\Rightarrow f(x)\sigma_2^* f(y) \\ &\Rightarrow \sigma_2^*(f(x))=\sigma_2^*(f(y)) \\ &\Rightarrow f^*(\sigma_1^*(x)) = f^*(\sigma_1^*(y)). \end{aligned}$$

Therefore, f^* is well-defined. Similarly, we can show that g^* is well-defined. Now, we prove

$$f^*(\sigma_1^*(x) \oplus \sigma_1^*(y)) \subseteq f^*(\sigma_1^*(x)) \oplus f^*(\sigma_1^*(y).$$

Let $f^*(\sigma_1^*(z)) \in f^*(\sigma_1^*(x) \oplus \sigma_1^*(y))$ for $z \in \sigma_1^*(x) + \sigma_1^*(y)$. We have

$$\begin{aligned} \forall t \in x+y, \sigma_1^*(t) = \sigma_1^*(z) &\Rightarrow f(t) \in f(x) + f(y) \\ &\Rightarrow \sigma_2^*(f(t)) \in \sigma_2^*(f(x)) \oplus \sigma_2^*(f(y)) \\ &\Rightarrow f^*(\sigma_1^*(t)) \in f^*(\sigma_1^*(x)) \oplus f^*(\sigma_1^*(y)) \\ &\Rightarrow f^*(\sigma_1^*(z)) \in f^*(\sigma_1^*(x)) \oplus f^*(\sigma_1^*(y)). \end{aligned}$$

Similarly, we can prove that

$$g^*\left(\beta_1^*(\gamma) \oplus \beta_1^*(\alpha)\right) \subseteq g^*\left(\beta_1^*(\gamma)\right) \oplus g^*\left(\beta_1^*(\alpha)\right).$$

Now, we prove

$$f^*\left(\sigma_1^*(x) \circ \beta_1^*(\gamma) \circ \sigma_1^*(y)\right) \subseteq f^*\left(\sigma_1^*(x)\right) \circ g^*\left(\beta_1^*(\gamma)\right) \circ f^*\left(\sigma_1^*(y)\right).$$

Let $f^*\left(\sigma_1^*(z)\right) \in f^*\left(\sigma_1^*(x) \circ \beta_1^*(\gamma) \circ \sigma_1^*(y)\right)$ for $z \in \sigma_1^*(x)\beta_1^*(\gamma)\sigma_1^*(y)$. We have

$$\begin{aligned} \forall t \in x\gamma y, \sigma_1^*(z) = \sigma_1^*(t) &\Rightarrow f(t) \in f(x)g(\gamma)f(y) \\ &\Rightarrow \sigma_2^*(f(t)) \in \sigma_2^*(f(x)) \circ \beta_2^*(g(\gamma)) \circ \sigma_2^*(f(y)) \\ &\Rightarrow f^*(\sigma_1^*(z)) \in f^*(\sigma_1^*(x)) \circ g^*(\beta_1^*(\gamma)) \circ f^*(\sigma_1^*(y)). \end{aligned}$$

Moreover, if (f, g) is an isomorphism, we show that (f^*, g^*) is an isomorphism. It is enough we prove that (f^*, g^*) is one to one and onto.

Let $f^*(\sigma_1^*(x)) = f^*(\sigma_1^*(y))$. Then, $\sigma_2^*(f(x)) = \sigma_2^*(f(y))$. Hence, there exist $t_1, \cdots, t_{m+1} \in M_2$ and $w_1, \cdots, w_m \in \mathcal{U}_{(M_2,\Gamma_2)}$ by $f(x) = t_1$ and $f(y) = t_{m+1}$ such that $\{t_i, t_{i+1}\} \subseteq w_i$ for all $1 \leq i \leq m$. Now, since f is onto, so there exists $r_i \in M_1$ such that $f(r_i) = t_i$ for all $2 \leq i \leq m$, and hence there exists $u_i \in \mathcal{U}_{(M_1,\Gamma_1)}$ such that $f(u_i) = w_i$. Thus $\{f(r_i), f(r_{i+1})\} \subseteq f(u_i)$. Since f is one to one, then $\{r_i, r_{i+1}\} \subseteq u_i$. It concludes that $x\sigma_1^* y$, i.e., $\sigma_1^*(x) = \sigma_1^*(y)$. Therefore f^* is one to one. Similarly, we can show that g^* is one to one. Also, clearly f^* and g^* are onto. This proves that (f^*, g^*) is an isomorphism. □

Theorem 3.18. *Let $\Gamma - HR$ be the category of Γ-hyperrings and $\Gamma/\beta^* - R$ be the category of Γ/β^*-rings. Then there is a covariant functor between $\Gamma - HR$ and $\Gamma/\beta^* - R$.*

Proof. We define $F : \Gamma - HR \to \Gamma/\beta^* - R$ by $F(M) = M/\sigma^*$ and $F(f,g) = (f^*, g^*)$, where M is a Γ-hyperring, σ^* the fundamental relation on M and (f,g) is a homomorphism between Γ-hyperrings. Let $(\psi, \varphi) : (M_1, \Gamma) \to (M_2, \Gamma)$ and $(f,g) : (M_2, \Gamma) \to (M_3, \Gamma)$ be homomorphisms. We have

$$(f,g) \circ (\psi, \varphi) = (f \circ \psi, g \circ \varphi) : (M_1, \Gamma) \to (M_3, \Gamma).$$

We prove $(f \circ \psi)^* = f^* \circ \psi^*$ and $(g \circ \varphi)^* = g^* \circ \varphi^*$. We know that $(f \circ \psi)^* : M_1/\sigma_1^* \to M_3/\sigma_3^*$ and $f^* \circ \psi^* : M_1/\sigma_1^* \to M_3/\sigma_3^*$. By Theorem 3.17, we have

$$\begin{aligned} (f \circ \psi)^*(\sigma_1^*(x)) &= \sigma_3^*(f \circ \psi(x)) \\ &= \sigma_3^*(f(\psi(x))) \\ &= f^*(\sigma_2^*(\psi(x))) \\ &= f^* \circ \psi^*(\sigma_1^*(x)). \end{aligned}$$

Thus, $(f \circ \psi)^* = f^* \circ \psi^*$. Similarly, we can prove that $(g \circ \varphi)^* = g^* \circ \varphi^*$. Therefore

$$F\left[(f,g) \circ (\psi, \varphi)\right] = F(f,g) \circ F(\psi, \varphi).$$

Let $(I_M, I_\Gamma) : (M, \Gamma) \to (M, \Gamma)$ be the identity homomorphism. We have

$$F(I_M, I_\Gamma) = (I_M^*, I_\Gamma^*) = \left(I_{M/\sigma^*}, I_{\Gamma/\beta^*}\right),$$

because (I_M^*, I_Γ^*) and $\left(I_{M/\sigma^*}, I_{\Gamma/\beta^*}\right)$ are identity homomorphisms of $(M/\sigma^*, \Gamma/\beta^*)$. Therefore, F is a covariant functor. □

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