On the Crossing Number of the Generalized Petersen Graph $P(3k, k)$*

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Abstract

The generalized Petersen graph $P(n, k)$ is the graph whose vertex set is $U \cup W$, where $U = \{u_0, u_1, \ldots, u_{n-1}\}, W = \{v_0, v_1, \ldots, v_{n-1}\}$; and whose edge set is $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} | i = 0, 1, \ldots, n-1\}$, where n, k are positive integers, addition is modulo n, and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$. G.Exoo, F.Harary and J.Kabell have determined the crossing number of $P(n, 2)$; Richter and Salazar have determined the crossing number of the generalized Petersen graph $P(n, 3)$. In this paper, the crossing number of the generalized Petersen graph $P(3k, k)$ $(k \geq 4)$ is studied, and it is proved that $cr(P(3k, k)) = k$ $(k \geq 4)$.

1 Introduction

All graphs considered here are finite undirected graphs without loops or multiple edges. For definitions not explained here, readers are referred to [1] and [2].

A *graph* $G = (V, E)$ is a set V of vertices and a subset E of unordered pairs of vertices, called edges. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well known that the crossing number of a graph is attained only in *good drawings* of the graph, which are the drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. Let D be a good drawing of the graph G, we denote the number of crossings in D by

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$cr(D)$. If D is a good drawing of G satisfying $cr(D) = cr(G)$, then D is an *optimal drawing* of G.

The *generalized Petersen graph* $P(n, k)$ is the graph whose vertex set is $U \cup W$, where $U = \{u_0, u_1, \ldots, u_{n-1}\}, W = \{v_0, v_1, \ldots, v_{n-1}\}$; and whose edge set is $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} | i = 0, 1, \ldots, n-1\}$, where n, k are positive integers, addition is modulo n, and $2 \le k \le \lfloor \frac{n}{2} \rfloor$. It will be useful to call the subgraph induced by U the *principal cycle*. The edges $\{u_i v_i | i = 0, 1, \ldots n-1\}$ are the *spokes* of the graph.

The *circulant graph* $C(n; S)$ is the graph with vertex set $V(C(n; S)) = \{v_i | 0 \le i \le n-1\}$ and edge set $E(C(n; S)) = \{v_i v_j | 0 \le i \le n-1, 0 \le j \le n-1, (i-j) \bmod k \in S\}$, $S \subseteq \{1, 2, \ldots \lfloor \frac{n}{2} \rfloor\}$. It is clear that the circulant graph $C(n; \{1, k\})$ can be obtained by contracting the spokes of the generalized Petersen graph $P(n, k)$. Hence, the problem of determining the crossing number of $C(n; \{1, k\})$ is closely related to the problem of determining the crossing number of $P(n, k)$.

Calculating the crossing number of a given graph is, in general, an elusive problem. Garey and Johnson have proved that the problem of determining the crossing number of an arbitrary graph is NP-complete [3]. The crossing number of very few families of graphs are known exactly.

Yang, Y., and Lin, X., etc. investigated the crossing number of certain circulant graphs, in [4], they showed that

$$cr(C(n; \{1, 3\})) = \lfloor \frac{n}{3} \rfloor + n \bmod 3 \quad (n \ge 8)$$

and in [5], they gave an upper bound of $C(mk; \{1, k\})$ and proved that

$$cr(C(3k; \{1, k\})) = k \qquad (k \ge 3)$$

Ma, D., Ren, H., and Lu, J. determined that the crossing number of $C(2m + 2; \{1, m\})$ is $m + 1$ for $m \ge 3$, see [6].

Exoo began to investigate the crossing number of generalized Petersen graph in [7], he proved $cr(P(n, 2)) = 0$ if n is an even integer no less than 4, $cr(P(n, 2)) = 3$ if n is an odd integer no less than 7 and $cr(P(3, 2)) = 0$, $cr(P(5, 2)) = 2$. In [8], Fiorini determined that the crossing number of $P(9, 3)$ is 2, he claimed to have determined that the crossing number of $P(10, 3)$ is 4 and

(1) $cr(P(3h, 3)) = h \qquad (h \ge 4)$

(2) $h + 3 \ge cr(P(3h + 1, 3)) \ge h + 1 \qquad (h \ge 3)$

(3) $cr(P(3h + 2, 3)) = h + 2 \qquad (h \ge 2)$

In 1992, Mcquillan and Richter found Fiorini's claim about the crossing number of $P(10, 3)$ is false, and proved that the crossing number of $P(10, 3)$ is at least 5, see [9]. In [10], Richter and Salazar found Fiorini's paper contained one serious mistake that invalidates the principal results. By taking $cr(P(10,3)) = 6$, $cr(P(11,3)) = 5$, $cr(P(12,3)) = 4$ as the basis of induction, they proved that

(1) $cr(P(3h,3)) = h$ $\quad (h \geq 4)$

(2) $cr(P(3h+1,3)) = h+3$ $\quad (h \geq 3)$

(3) $cr(P(3h+2,3)) = h+2$ $\quad (h \geq 3)$

In this paper, we study the crossing number of the generalized Petersen graph $P(3k,k)$ when $k \geq 4$, and prove

Theorem. $cr(P(3k,k)) = k \quad (k \geq 4)$.

Our main proof is by induction on k. This paper is organized as follows. In section 2, we give some lemmas. In section 3, the proof of the induction basis, $cr(P(12,4)) = 4$, is given. In section 4, the final proof is presented.

2 Some Lemmas

In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D.

From [8], we have Lemma 2.1.

Lemma 2.1. *If there exists a crossed edge e in a drawing D and deleting it results a new drawing D^*, then $cr(D) \geq cr(D^*) + 1$.*

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings crossed by an edge in A and another edge in B is denoted by $cr_D(A,B)$. The number of crossings crossed by two edges in A is denoted by $cr_D(A)$, then $cr(D) = cr_D(E)$. By counting the number of crossings in D, we have Lemma 2.2.

Lemma 2.2. *Let A, B, C be mutually disjoint subsets of E. Then*

$$cr_D(A \cup B, C) = cr_D(A,C) + cr_D(B,C);$$

$$cr_D(A \cup B) = cr_D(A) + cr_D(B) + cr_D(A,B).$$

First we partite the edge set of $P(3k,k)$ $(k \geq 3)$ into two disjoint subsets, X and Y. Then we divide X into k mutually disjoint subsets as follows (subscripts modulo $3k$):

$$E_i = \{v_iv_{i+k}, v_{i+k}v_{i+2k}, v_{i+2k}v_i, u_iv_i, u_{i+k}v_{i+k}, u_{i+2k}v_{i+2k}\} \qquad (0 \leq i \leq k-1),$$

and divide Y into k mutually disjoint subsets (subscripts modulo $3k$):

$$H_i = \{u_iu_{i+1}, u_{i+k}u_{i+k+1}, u_{i+2k}u_{i+2k+1}\} \qquad (0 \leq i \leq k-1),$$

then

$$E(P(3k,k)) = X \cup Y,$$

$$X = \bigcup_{i=0}^{k-1} E_i, \qquad Y = \bigcup_{i=0}^{k-1} H_i,$$

$$E_i \cap E_j = \emptyset, \qquad H_i \cap H_j = \emptyset, \quad 0 \le i \ne j \le k-1.$$

It is clear that a graph obtained by deleting the edges of any E_i ($0 \le i \le k-1$) from $P(3k, k)$ is homeomorphic to $P(3(k-1), (k-1))$.

We define a function $f_D(H_i)$ ($0 \le i \le k-1$) counting the number of crossings related to H_i in a drawing D as follows:

$$f_D(H_i) = cr_D(H_i) + \sum_{0 \le j \le k-1,\, j \ne i} cr_D(H_i, H_j)/2.$$

With the above notations, we get

Lemma 2.3. $cr_D(Y) = \sum_{i=0}^{k-1} f_D(H_i)$.

Lemma 2.4. *Let D be a good drawing of $P(3k, k)$ for $k \ge 3$. If the edges in $\{E_i | i = 0, 1, \ldots, k-1\}$ are all clean in D, then $\forall i$, $0 \le i \le k-1$, $f_D(H_i) \ge 1$.*

Proof. We prove this by contradiction. Suppose that the edges in $\{E_i | i = 0, 1, \ldots, k-1\}$ are all clean in D, but there exists i ($0 \le i \le k-1$) such that $f_D(H_i) < 1$.

Let $C_i = v_i v_{i+k} v_{i+2k} v_i$, C_i divides the plane into two regions, *int* C_i and *ext* C_i. Since the edges in E_i are all clean, the edges $u_i v_i, u_{i+k} v_{i+k}, u_{i+2k} v_{i+2k}$ must lie in either *int* C_i or *ext* C_i. Without loss of generality, we may assume that they lie in *ext* C_i. Since the edges in E_i and E_{i+1} are all clean, the vertices of E_{i+1} must lie in *ext* C_i, otherwise C_i must be crossed by $u_i u_{i+1}, u_{i+k} u_{i+k+1}$ and $u_{i+2k} u_{i+2k+1}$, see Figure 1(a).

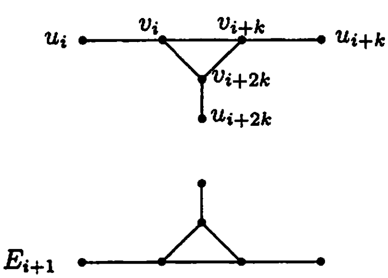

Figure 1 (a)

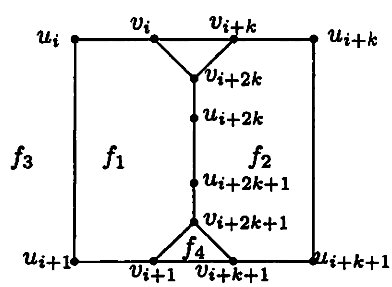

Figure 1 (b)

Because $f_D(H_i) < 1$, the edge $u_i u_{i+1}, u_{i+k} u_{i+k+1}, u_{i+2k} u_{i+2k+1}$ cannot cross each other, or else $f_D(H_i) \ge cr_D(H_i) \ge 1$. Up to isomorphism, the only possible way to label the vertices of E_{i+1} and draw edges u_iu_{i+1}, $u_{i+k}u_{i+k+1}$, $u_{i+2k}u_{i+2k+1}$ is shown in Figure 1(b), and the *ext* C_i is divided into 4 regions: f_1, f_2, f_3 and f_4.

It is clear that the vertices of E_{i+2} cannot lie in f_4, or else the 3-cycle $v_{i+1}v_{i+k+1}v_{i+2k+1}$ must be crossed. Without loss of generality, we may assume that the vertices of E_{i+2} lie in f_3. Since the edges in E_i and E_{i+1} are all clean, the edge $u_{i+2k+1}u_{i+2k+2}$ and the path $u_{i+k+2}u_{i+k+3}\ldots u_{i+2k-1}u_{i+2k}$ (which excludes vertices $u_i, u_{i+1}, u_{i+k}, u_{i+k+1}$) must cross H_i, so $f_D(H_i) \geq 1$, contradicts the previous assumption! □

By Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

Lemma 2.5. *Let D be a good drawing of $P(3k,k)$ for $k \geq 3$. If the edges in $\{E_i | i = 0, 1, \ldots, k-1\}$ are all clean in D, then $cr(D) \geq k$.*

Proof. By Lemma 2.2, Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} cr(D) &= cr_D(X \cup Y) \\ &\geq cr_D(Y) \\ &= \sum_{i=0}^{k-1} f_D(H_i) \\ &\geq k. \end{aligned}$$ □

In the following parts, we will prove the Theorem by induction on k ($k \geq 4$). First of all, the induction basis needs to be proved. So, the crossing number of $P(12,4)$ is studied in the next section.

3 The Crossing Number of $P(12,4)$

As we have referred to in the former section, $P(9,3)$ can be obtained from $P(12,4)$ by deleting the edges in E_i ($0 \leq i \leq 3$), see Figure 2. Some properties of $P(9,3)$ will be studied in the following paragraphs.

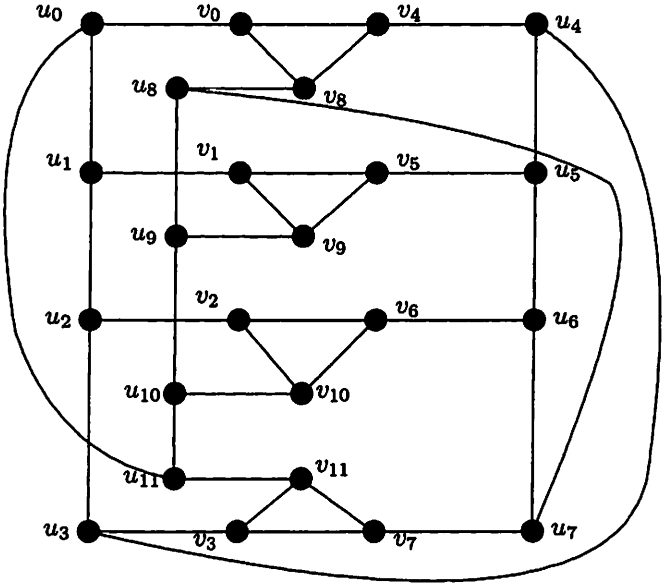

Figure 2: A good drawing of $P(12, 4)$

Lemma 3.1. *For any vertex v of $P(9,3)$, $cr(P(9,3) - v) \geq 1$.*
Proof. Figure 3(a) is a drawing of $P(9,3)$, for any vertex v of $P(9,3)$, Figure 3(b) shows that $P(9,3) - v$ contains a subgraph homeomorphic to $K_{3,3}$, so $cr(P(9,3) - v) \geq cr(K_{3,3}) = 1$. □

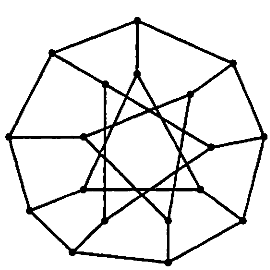

Figure 3(a): A drawing of $P(9,3)$

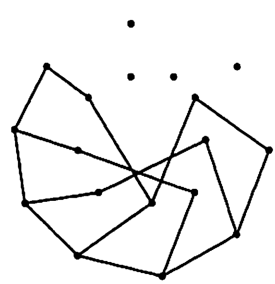

Figure 3(b): A subdivision of $K_{3,3}$

Corollary 3.2. *If D is a good drawing of $P(9,3)$ with $cr(D) = 2$, then the 4 edges involved forms a matching in $P(9,3)$.* □

Lemma 3.3. *If D is a good drawing of $P(9,3)$ with $cr(D) = 2$, then up to isomorphism, D must be one of the three possibilities shown in Figure 5 and Figure 7(a).*

Proof. First, we can assert that the principal cycle C has at most one internal crossing in D, otherwise $cr(D) \geq 3$ since the edges of E_i are all clean.

Case 1. Suppose that the principal cycle C has no internal crossing. C divides the plane into two regions, the interior region f_1 and the exterior region f_2. For $i = 0, 1, 2$, three vertices v_i, v_{i+3} and v_{i+6} must lie in the same region of C, otherwise, without loss of generality, we may assume that v_i lies in f_1 and v_{i+3}, v_{i+6} lie in f_2, then the edges v_iv_{i+3}, v_iv_{i+6} must be crossed, that contradicts with Corollary 3.2. Three vertices v_i, v_{i+3} and v_{i+6} must lie in the same region in D for the same reason. By the hypothesis of the lemma, we can also assert that there must exist i $(0 \leq i \leq 2)$ such that E_i doesn't have crossings with C. Without loss of generality, we may assume E_0 doesn't have crossings with C, and it lies in f_1.

Subcase 1.1. Suppose that E_0 has internal crossings. Then by Corollary 3.2, E_0 only have one internal crossing since the edges of E_0 cannot form a matching, see Figure 4(a) and Figure 4(b).

Subcase 1.1.1. Suppose that the edges of E_1 are all clean, they must lie in f_2, see Figure 4(a) and Figure 4(b). The drawing divides the plane into several regions with at most one vertex of u_2, u_5, u_8 on the boundary of every region. No matter which region do v_2, v_5, and v_8 lie in, the edges of E_2 must be crossed at least twice, which contradicts with $cr(D) = 2$.

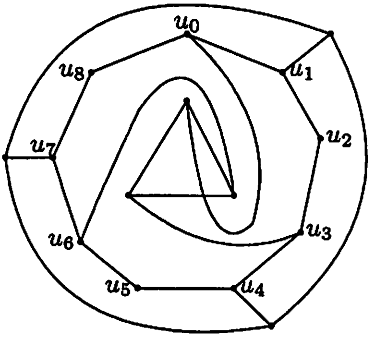

Figure 4 (a)

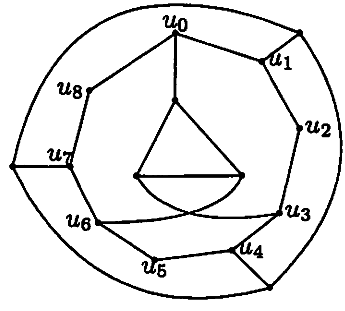

Figure 4 (b)

Subcase 1.1.2. Suppose that the edges of E_1 are crossed once. If the crossing is made by an edge of E_1 and an edge not belonging to E_2, then the edges of E_2 are all clean, we change the roles of E_1 and E_2 and the remaining arguments are similar to Subcase 1.1.1, so the crossing is made by an edge of E_1 and an edge of E_2. Then, both E_1 and E_2 don't have internal crossings and don't cross neither C nor E_0, and E_1 must lie in f_2, in Figure 4(a) and Figure 4(b) this is shown, but E_2 cannot be drawn with exactly one crossing with E_1, a contradiction.

Subcase 1.2. Suppose that E_0 does not have internal crossings, it divides f_1 into 4 regions, namely f_{11}, f_{12}, f_{13} and f_{14}, see Figure 5(a). By our earlier remark, for $i = 1, 2$, three vertices v_i, v_{i+3}, v_{i+6} must lie in the same region. And it is clear that the vertices of E_1 cannot lie in f_{14}, or the cycle $v_0v_3v_6v_0$ must be crossed at least three times by u_iv_i, for $i = 1, 4, 7$. The same holds for E_2.

Subcase 1.2.1. Suppose that the vertices of E_1 lie in one of the inner regions of C, without loss of generality, we may assume that the vertices of E_1 lie in f_{11}. Then the edges u_1v_1 and u_4v_4 must be crossed exactly once respectively, and the vertices of E_2 must lie in f_2 and the edges of E_2 are all clean. By the hypothesis that $cr(D) = 2$, u_4v_4 can only be crossed by u_6v_6 and u_1v_1 can be crossed by either u_0v_0 or u_8u_0. This is shown in Figure 5 (a) and Figure 5 (b) respectively.

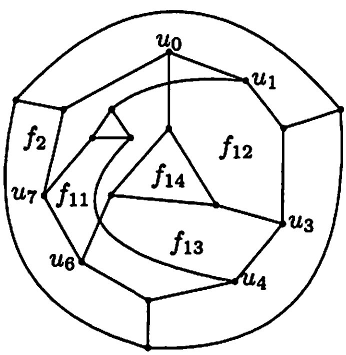

Figure 5 (a)

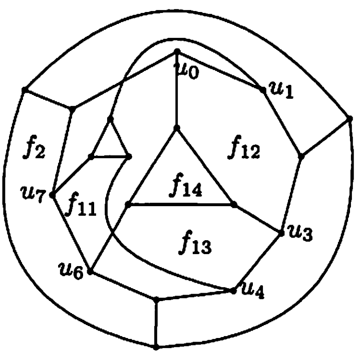

Figure 5 (b)

Subcase 1.2.2. Suppose that the vertices of E_1 lie in the outside region of C, f_2.

If the vertices of E_2 lie in f_{11}, f_{12} or f_{13}, the remaining arguments are similar to Subcase 1.2.1 by changing the roles of E_1 and E_2. Then we can suppose that the vertices of E_2 lie in f_2.

If the edges of E_1 and E_2 have one crossing with C respectively, without loss of generality, we may assume that u_1v_1 is crossed by C, then it must cross u_2u_3, see Figure 6 (a), no matter which region do the vertices v_2, v_5 and v_8 lie in, E_2 cannot be drawn with one crossing with C and satisfying $cr(D) = 2$. Thus either E_1 or E_2 doesn't have crossings with C, without loss of generality, we may assume that E_1 doesn't cross C.

Subcase 1.2.2.1. Suppose E_1 has internal crossings. This subcase is similar to Subcase 1.1 by changing the roles of E_0 and E_1.

Subcase 1.2.2.2. Suppose E_1 doesn't have internal crossings, see Figure

6 (b). The edges of E_1 divide the region f_2 into 4 regions, f_{21}, f_{22}, f_{23} and f_{24}. E_2 can be drawn in f_{21}, f_{22} or f_{23} satisfying $cr(D) = 2$, this subcase is isomorphic to Subcase 1.2.1.

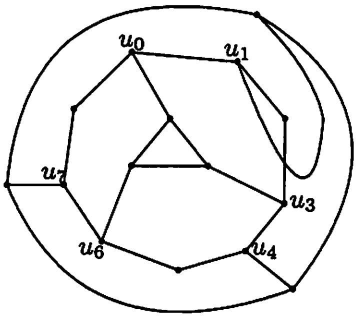

Figure 6 (a)

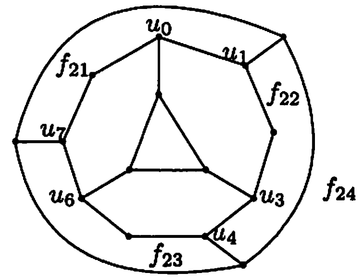

Figure 6 (b)

Case 2. Suppose that the principal cycle C has an internal crossing, the crossing point is named v. The principal cycle C divides the plane into three regions, namely, f_1, f_2 and f_3. By our earlier remark, for $i = 0, 1, 2$, three vertices v_i, v_{i+3}, v_{i+6} must lie in the same region. Up to isomorphism, we consider three subcases by the number of vertices on the boundary of f_1.

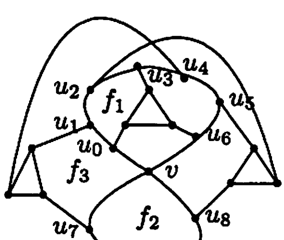

Figure 7(a)

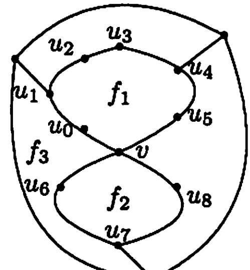

Figure 7(b)

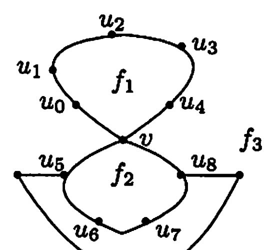

Figure 7(c)

Subcase 2.1. Suppose that the boundary of f_1 has 7 vertices. Without loss of generality, we may label the vertices $u_0, u_1, \ldots, u_8$ as shown in Figure 7(a). By Corollary 3.2, the edges adjacent to u_0, u_6, u_7 and u_8 except u_0u_8 and u_6u_7 are all clean, so for $i = 1, 2$, three vertices v_i, v_{i+3}, v_{i+6} should lie in f_3. And v_0, v_3, v_6 must lie in f_1, or the cycle $u_0vu_6v_6v_0u_0$ will be crossed at least twice by the two paths $u_1v_1v_7u_7$ and $u_2v_2v_8u_8$, a contraction!

The edge u_8v_8 is clean. If the edge u_5v_5 is crossed by C, then it must cross u_3u_4, and there must be one more crossing on the path $u_1v_1v_4u_4$, contradicts the previous assumption! Analogously, we can get that all the edges u_iv_i $(i = 1, 2, 4, 5)$ are not crossed by C. This possibility is as shown in Figure 7(a).

Subcase 2.2. Suppose that the boundary of f_1 has 6 vertices. Without loss of generality, we may label the vertices $u_0, u_1, \ldots, u_8$ as shown in Figure 7(b). By Corollary 3.2, the edges adjacent to u_0, u_5, u_6 and u_8 except u_0u_8 and u_5u_6 are all clean, so the vertices v_i $(i \neq 1, 4, 7)$ should lie in f_3.

Furthermore, we can conclude that v_1, v_4 and v_7 should lie in f_3 too. It is clear that they cannot lie in f_2, or the cycle $u_6u_7u_8vu_6$ will be crossed at least twice by the edges u_1v_1 and u_4v_4, a contradiction! If the vertices lie in f_1, then the edge u_7v_7 has a crossing with C, and it must cross one of the three edges, u_1u_2, u_2u_3 or u_3u_4. If u_7v_7 is crossed by u_1u_2, then the cycle $u_1v_1v_7u_7u_8vu_0u_1$ divides the vertices u_3 and u_6 in two regions, there will be at least one more crossing on the path $u_3v_3v_6u_6$, a contradiction! And we can get that u_7v_7 cannot cross neither u_2u_3 nor u_3u_4 by the analogous arguments, which implies that v_1, v_4 and v_7 cannot lie in f_1.

If the edge u_7v_7 is crossed by C, then it will be crossed at least twice since v_7 lies in f_3 and the edge u_7v_7 cannot cross u_5u_6, u_6u_7, u_7u_8 and u_8u_0, contradicts the previous assumption! If the edge u_1v_1 is crossed by C, then it must cross one of the three edges of u_2u_3, u_3u_4 and u_4u_5. If u_1v_1 crosses u_2u_3, then the cycle $u_1v_1v_7u_7u_6vu_0u_1$ divides the vertices u_3 and u_0 in two regions, there will be at least one more crossing in path $u_0v_0v_3u_3$, a contraction. And u_1v_1 cannot cross neither u_3u_4 nor u_4u_5 by the similar arguments, which implies that u_1v_1 has no crossing with C. Analogously, u_4v_4 has no crossing with C neither, this is shown in Figure 7(b). It can be seen from Figure 7(b) that there will be at least one crossing on the path $u_2v_2v_5u_5$ and $u_3v_3v_6u_6$ respectively, contradicts the previous assumption!

Subcase 2.3. Suppose that the boundary of f_1 has 5 vertices. Without loss of generality, we may label the vertices $u_0, u_1, \ldots, u_8$ as shown in Figure 7(c). By Corollary 3.2, the edges adjacent to u_0, u_4, u_5 and u_8 except u_0u_8 and u_4u_5 are all clean. Using the analogous arguments in Subcase 2.1 and Subcase 2.2, we can assert that the vertices v_i $(i = 0, 1, \ldots, 8)$ should lie in f_3, and the edge v_5v_8 cannot have a crossing with C. Thus the cycle $u_5v_5v_8u_8vu_5$ divides the vertices u_6 and u_0, u_7 and u_1 in different regions, there will be at least one crossing on the path $u_0v_0v_6u_6$ and $u_1v_1v_7u_7$ respectively, contradicts the previous assumption!

In all, if D is a drawing of $P(9,3)$ with $cr(D) = 2$, then up to isomorphism, the only three possibilities of D are shown in Figure 5 and Figure 7(a). □

Theorem 3.4. $cr(P(12,4)) = 4$.

Proof. Figure 2 shows that $cr(P(12,4)) \leq 4$. And we get that $cr(P(12,4)) \geq cr(P(9,3)) = 2$ since $P(12,4)$ contains $P(9,3)$ as a subgraph. Let D be an optimal drawing of $P(12,4)$.

If $cr(D) = 2$, then it is clear that there exists i $(0 \le i \le 3)$ such that E_i is crossed, or $cr(D) \ge 4$ by Lemma 2.5. By deleting the edges of E_i, we can obtain a new drawing D_1 and the graph corresponding to D_1 is homeomorphic to $P(9,3)$, then

$$cr(D_1) \le cr(D) - 1 = 1$$

a contradiction!

If $cr(D) = 3$, then there must exist i $(0 \le i \le 3)$ such that E_i is crossed. According to Lemma 2.1, it is easy to see that for each i, $0 \le i \le 3$, E_i can be crossed at most once. Without loss of generality, we may assume that E_0 is crossed exactly once. A new drawing D_2 can be obtained by deleting the edges of E_0, and the graph corresponding to D_2 is homeomorphic to $P(9,3)$ with $cr(D_2) = 2$. Then D_2 must be one of the three possibilities shown in Figure 5 and Figure 7(a). In any one of the three possibilities, it is impossible to insert 3 vertices of $E_0 \cap U$ in the edge segments u_iu_{i+1}, $u_{i+3}u_{i+4}$, $u_{i+6}u_{i+7}$ $(i = 0, 1, 2)$ of $P(9,3)$ and draw 6 edges of E_0 with only one crossing increased. This impossibility shows that $cr(D) \ne 3$.

Since D is an optimal drawing of $P(12,4)$ and the above arguments show that $cr(D) \ne 2$ and $cr(D) \ne 3$, the crossing number of $P(12,4)$ in D can only be equal to 4, that is $cr(P(12,4)) = 4$. □

4 The Proof of the Theorem

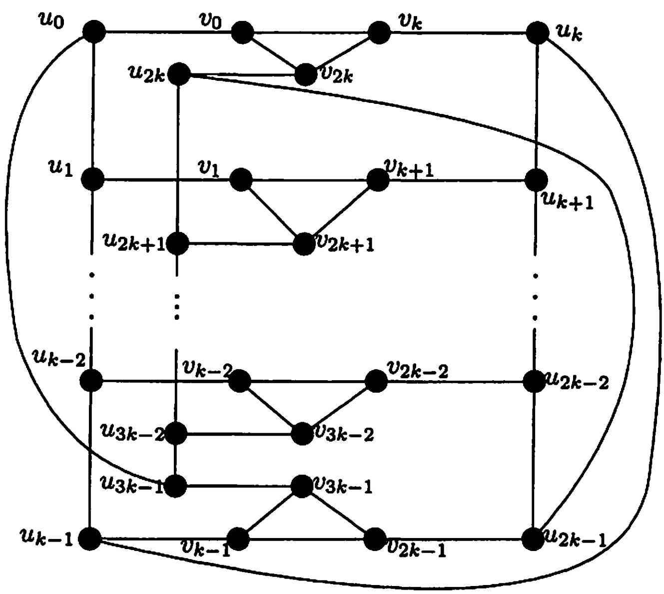

Figure 8: A good drawing of $P(3k, k)$

proof. The drawing in Figure 8 shows that $cr(P(3k, k)) \leq k$ for $k \geq 4$. We prove the reverse inequality by induction on k.

(i) By Theorem 3.4, $cr(P(12, 4)) = 4$, the result is true for $k = 4$.

(ii) Suppose that for $k = l - 1$ $(l \geq 5)$, $cr(P(3(l-1), (l-1))) = l - 1$, consider $P(3l, l)$. Let D be any good drawing of $P(3l, l)$.

Case 1. Suppose that there is at least one crossing in the edges of $\{E_i | 0 \leq i \leq l-1\}$ in D. Without loss of generality, we may assume that there is at least a crossing in E_0. We can get a drawing D_0 by deleting E_0 in D, then $cr(D) \geq cr(D_0) + 1$ by Lemma 2.1. Since the graph corresponding to D_0 is homeomorphic to $P(3(l-1), (l-1))$, and $cr(P(3(l-1), (l-1))) = l-1$, we have

$$cr(D) \geq cr(D_0) + 1 \geq cr(P(3(l-1), (l-1))) + 1 = l.$$

Case 2. Suppose that the edges in $\{E_i | 0 \leq i \leq l-1\}$ are all clean in D. Then $cr(D) \geq l$ by Lemma 2.5.

According to Case 1 and Case 2, for any good drawing D of $P(3l, l)$, we have $cr(D) \geq l$, so $cr(P(3l, l)) \geq l$.

According to (i) and (ii), we have $cr(P(3k, k)) \geq k$ for $k \geq 4$. So, the crossing number of $P(3k, k)$ is k for $k \geq 4$. □

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