On Cayley graphs of normal bands *

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Abstract

In this paper, we describe Cayley graphs of rectangular bands and normal bands which are the strong semilattice of rectangular bands, respectively. In particular, we give the structure of Cayley graphs of rectangular bands and normal bands, and we determine which graphs are Cayley graphs of rectangular bands and normal bands.

Keywords: Cayley graph; Rectangular band; Normal band; Strong semilattice of rectangular bands

1 Introduction

The definition of Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. Many algebraic and combinatorial properties are extensively described on Cayley graphs of groups ([1], [2], [3], [5]). Cayley graphs of semigroups are generalization of Cayley graphs of groups. The concept of the Cayley graph of a semigroup was introduced by Bohdan Zelinka in [11]. It is natural to expect that the more general concept can be used to define various types of graphs with new combinatorial properties. Let S be a semigroup and A be a subset of S, the *Cayley graph* $Cay(S, A)$ of S relative to A is defined as the graph with vertex set S and edge set $\{(x, y) \mid y = ax \text{ for some } a \in A\}$. A is said to be the *connection set* of $Cay(S, A)$. There are two possible directions one can pursue to study the Cayley graphs of semigroups. The first is to infer properties of S from $Cay(S, A)$, and the second is to understand the possible combinatorial properties of the graph

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$Cay(S, A)$ as S ranges over the class of semigroups. All vertex transitive Cayley graphs produced by periodic semigroups were characterized in [8]. In [7], Kelarev studied Cayley graphs of inverse semigroups. Cayley graphs of bands were characterized in [4]. Cayley graphs of Clifford semigroups were investigated in [9].

The aim of this paper is to study Cayley graphs of rectangular bands and normal bands, respectively. In particular, the structure of the Cayley graphs of rectangular bands and normal bands is given, and which graphs are Cayley graphs of rectangular bands and normal bands are determined.

Graphs considered in this paper are directed graphs without multiple edges but possibly with loops. For a graph D, denote by $V(D)$ and $E(D)$ its vertex set and edge set, respectively. We regard an edge (x, ax) for $x \in S, a \in A$ as having color a.

Recall that an element x of a semigroup is an *idempotent* if $xx = x$. A *band* is a semigroup entirely consisting of idempotents. A band B is called a *left zero (right zero, rectangular) band* if it satisfies the identity $xy = x(resp., xy = y, xyx = x)$ for all $x, y \in B$. A band satisfying the identity $xyzx = xzyx$ is a *normal band.* A rectangular band can be represented as direct product $I \times \Lambda$ of a left zero band I and a right zero band Λ.

A semigroup S is said to be a *semilattice of semigroups* $(S_\alpha, \circ_\alpha), \alpha \in Y$, if Y is a semilattice, $S = \cup_{\alpha \in Y} S_\alpha$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, in notation $S = (Y; S_\alpha)$. Let $S = (Y; S_\alpha)$ and assume that: for any $\alpha, \beta, \gamma \in Y$,

(1) if $\alpha \geq \beta$, there exists a homomorphism $\Phi_{\alpha,\beta}$ from S_α to S_β, and $\Phi_{\alpha,\alpha} = 1_{S_\alpha}$,

(2) if $\alpha > \beta > \gamma$, then $\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = \Phi_{\alpha,\gamma}$,

(3) for any $a \in S_\alpha, b \in S_\beta$,

$$ab = \Phi_{\alpha,\alpha\beta}(a)\Phi_{\beta,\alpha\beta}(b),$$

where ab stands for the product of a and b in S. Then we say that S is a *strong semilattice* Y *of semigroups* S_α $(\alpha \in Y)$, and denote it by $S = S(Y; S_\alpha; \Phi_{\alpha,\beta})$. It is known that a band is a semilattice of rectangular bands and a band is a normal band if and only if it is a strong semilattice of rectangular bands.

For any element a in a rectangular band B, denote by i_a and λ_a the first and second components of a, respectively, that is $a = (i_a, \lambda_a)$. Let B_1, B_2 be two rectangular bands, and Φ be a map from B_1 to B_2. For any $a \in B_1$, denote by $\Phi^1(a)$ and $\Phi^2(a)$ the first and second components of $\Phi(a)$ in B_2, respectively, that is $\Phi(a) = (\Phi^1(a), \Phi^2(a))$.

For the terminology and notation not defined in this paper, the reader refers to [6] and [10].

2 Cayley Graphs of Rectangular bands

In this section, we shall explore Cayley graphs of rectangular bands.

For any nonempty subset A of a rectangular band $B = I \times \Lambda$, let

$$I_A = \{i \in I \mid (i, \mu) \in A \text{ for some } \mu \in \Lambda\}.$$

Lemma 2.1 *Let $B = I \times \Lambda$ be a rectangular band and A a subset of B. Then $(j, \mu) \to (i, \lambda)$ is an arc in Cay(B,A) if and only if $\lambda = \mu$ and $i \in I_A$.*

Proof. If $(j, \mu) \to (i, \lambda)$ is an arc in Cay(B,A), then there exists $(k, \xi) \in A$ such that $(i, \lambda) = (k, \xi)(j, \mu) = (k, \mu)$. So we get $\lambda = \mu$ and $i = k \in I_A$. Conversely, if $\lambda = \mu$ and $i \in I_A$, then there exists $\eta \in \Lambda$ such that $(i, \eta) \in A$, and hence $(i, \eta)(j, \mu) = (i, \mu) = (i, \lambda)$. Thus $(j, \mu) \to (i, \lambda)$ is an arc in $Cay(B, A)$.

The next theorem gives a necessary and sufficient condition to coincidence of two Cayley graphs of a rectangular band.

Theorem 2.2 *Let $B = I \times \Lambda$ be a rectangular band and A_1, A_2 two subsets of B. Then $Cay(B, A_1) = Cay(B, A_2)$ if and only if $I_{A_1} = I_{A_2}$.*

Proof. Sufficiency. It is clear that $Cay(B, A_1)$ and $Cay(B, A_2)$ have the same vertex set. We shall show that $E(Cay(B, A_1)) = E(Cay(B, A_2))$. Let $(j, \mu) \to (i, \lambda)$ be an arc in $Cay(B, A_1)$, then by Lemma 2.1, $\lambda = \mu$ and $i \in I_{A_1}$. Since $I_{A_1} = I_{A_2}$, we get $i \in I_{A_2}$. It follows that $(j, \mu) \to (i, \mu)$ is an arc in $Cay(B, A_2)$ by Lemma 2.1. So $E(Cay(B, A_1)) \subseteq E(Cay(B, A_2))$. Similar argument shows that $E(Cay(B, A_2)) \subseteq E(Cay(B, A_1))$. Therefore $E(Cay(B, A_1)) = E(Cay(B, A_2))$ and $Cay(B, A_1) = Cay(B, A_2)$.

Necessity. If $i \in I_{A_1}$, then for any $\lambda \in \Lambda$, we have that $(i, \lambda) \to (i, \lambda)$ is a loop in $Cay(B, A_1)$ by Lemma 2.1. Since $Cay(B, A_1) = Cay(B, A_2)$, we get that $(i, \lambda) \to (i, \lambda)$ is also a loop in $Cay(B, A_2)$. So by Lemma 2.1 we have $i \in I_{A_2}$ and hence $I_{A_1} \subseteq I_{A_2}$. Dually, we may show that $I_{A_2} \subseteq I_{A_1}$. Therefore $I_{A_1} = I_{A_2}$.

Given a family of graphs $D_i = (V_i, E_i)$ with $i \in I$, their union is the graph $D = \cup_{i \in I} D_i$ defined by $D = (\cup_{i \in I} V_i, \cup_{i \in I} E_i)$. For any connected graph D, let nD be the union of n disjoint copies of D, which is to say that nD is the graph with n connected components each of which is isomorphic to D. Denote by $\overrightarrow{K}_m$ the complete graph with m vertices, that is, a graph $\overrightarrow{K}_m$ with m vertices in which $(a, b) \in E(\overrightarrow{K}_m)$ for any $a, b \in V(\overrightarrow{K}_m)$. Hence each vertex has a loop in $\overrightarrow{K}_m$. Denote by H_n the graph with n isolated vertices. Denote by $S_{m,n}$ the graph obtained from the disjoint union $\overrightarrow{K}_m$ and H_n by adding the edges $\{y \to x \mid x \in V(K_m), y \in V(H_n)\}$.

Now we are ready to characterize the structure of Cayley graphs of rectangular bands.

Theorem 2.3 *Let $B = I \times \Lambda$ be a rectangular band, A a subset of B and $I_A = \{i \in I \mid (i, \lambda) \in A$ for some $\lambda \in \Lambda\}$. If $|I| = p$, $|\Lambda| = q$ and $|I_A| = m$, then $Cay(B, A) \cong qS_{m,p-m}$.*

Conversely, every graph $qS_{m,p-m}$ with $p, q \in \mathbb{N}$ and $0 \leq m \leq p$ is isomorphic to the Cayley graph of some rectangular band $B = I \times \Lambda$ relative to a connection set A of B with $|I| = p$, $|\Lambda| = q$ and $|I_A| = m$.

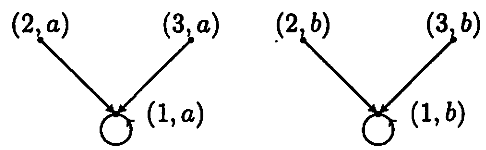
Proof. For any $\lambda \in \Lambda$, let $B_\lambda = I \times \{\lambda\}$. Then by Lemma 2.1, $Cay(B, A) = \cup_{\lambda \in \Lambda} Cay(B_\lambda, A)$. Also by Lemma 2.1, $I_A \times \{\lambda\}$ forms a complete subgraph of $Cay(B_\lambda, A)$ which is isomorphic to $\overrightarrow{K}_m$ and there is an arc from (i, λ) to (j, λ) for all $i \in I \setminus I_A$ and $j \in I_A$. Hence $Cay(B_\lambda, A) \cong S_{m,p-m}$ and so $Cay(B, A) \cong qS_{m,p-m}$.

Conversely, for any graph $qS_{m,p-m}$ with $p, q \in \mathbb{N}$ and $0 \leq m \leq p$, let $I = \{1, 2, \ldots, p\}$ be a left zero band, $\Lambda = \{1, 2, \ldots, q\}$ a right zero band and $A = \{(1, 1), (2, 1), \ldots, (m, 1)\}$. Then $B = I \times \Lambda$ is a rectangular band. By the proof of the direct part we have $Cay(B, A) \cong qS_{m,p-m}$.

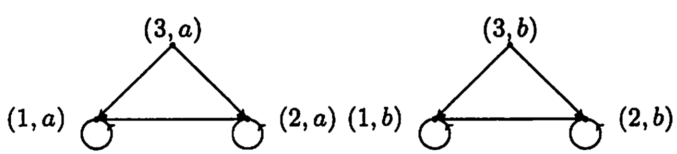
As a direct consequence of Theorem 2.3, we have

Corollary 2.4 *Let $B = I \times \Lambda$ be a rectangular band and A_1, A_2 two subsets of B. Then $Cay(B, A_1) \cong Cay(B, A_2)$ if and only if $|I_{A_1}| = |I_{A_2}|$. Furthermore, the number of isomorphic classes of the Cayley graphs of B is $|I| + 1$.*

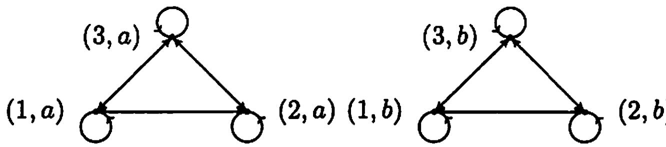
Example 2.5 *Let $B = I \times \Lambda$ be a rectangular band, where $I = \{1, 2, 3\}$ and $\Lambda = \{a, b\}$. Some of the Cayley graphs $Cay(B, A)$ of B relative to its subset A are indicated in the following:*



$Cay(B, \{(1, a)\})$



$Cay(B, \{(1, a), (2, a)\})$



$Cay(B, \{(1, a), (2, a), (3, a)\})$

Figure 1 The isomorphic classes of Cayley graphs of B.

Recall that a Cayley graph $Cay(B, A)$ is called vertex-transitive if for any two vertices $x, y \in B$, there exists an automorphism φ of $Cay(B, A)$ such that $\varphi(x) = y$. Using Theorem 2.3, we can easily get the following result which is Proposition 3.4 of [4].

Corollary 2.6 *Let $B = I \times \Lambda$ be a rectangular band and A a subset of B. Then $Cay(B, A)$ is vertex-transitive if and only if $A = \emptyset$ or $|I_A| = |I|$.*

Proof. Sufficiency. If $A = \emptyset$, then $Cay(B, A)$ is a graph with $|B|$ isolated vertices, this implies that $Cay(B, A)$ is vertex-transitive. If $|I_A| = |I|$, then by Theorem 2.3, we have $Cay(B, A) \cong q\overrightarrow{K}_{|I_A|}$, this implies that $Cay(B, A)$ is vertex-transitive.

Necessity. By Theorem 2.3, we have that $Cay(B, A) \cong qS_{m,p-m}$ with $p = |I|$, $q = |\Lambda|$ and $m = |I_A|$. If $Cay(B, A)$ is vertex-transitive, then $m = 0$ or $m = p$, that is $A = \emptyset$ or $|I_A| = |I|$.

3 Cayley Graphs of Normal Bands

In this section, we explore Cayley graphs of normal bands. The next theorem gives a result about the Cayley graph of a strong semilattice of semigroups.

Theorem 3.1 *Let Y be a finite semilattice, $S = S(Y; S_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of semigroups S_α ($\alpha \in Y$), and A a subset of S. For any $\delta \in Y$, let $A_\delta = \{\Phi_{\alpha,\delta}(a) \mid a \in A, a \in S_\alpha, \alpha \geq \delta\}$. Then*

(1) the Cayley graph $Cay(S, A)$ contains $|Y|$ disjoint induced subgraphs $Cay(S_\delta, A_\delta)$ ($\delta \in Y$), and

(2) if $\beta \neq \gamma$ and $b \in S_\beta, c \in S_\gamma$, then $b \to c$ is an arc in $Cay(S, A)$ if and only if $\beta > \gamma$ and there exist $\alpha \in Y$ and $a \in A \cap S_\alpha$ such that $\alpha\beta = \gamma$ and $\Phi_{\beta,\gamma}(b) \to c$ is an arc in $Cay(S_\gamma, A_\gamma)$ with color $\Phi_{\alpha,\gamma}(a)$.

Proof. (1) Let $\delta \in Y$. Consider the subgraph (S_δ, E_δ) of $Cay(S, A)$ induced by S_δ. We shall show that $(S_\delta, E_\delta) = Cay(S_\delta, A_\delta)$. It is clear that (S_δ, E_δ) and $Cay(S_\delta, A_\delta)$ have the same vertex set. We only need to show that

$E((S_\delta, E_\delta)) = E(Cay(S_\delta, A_\delta))$. Let $b \to c$ be an arc in (S_δ, E_δ). Since (S_δ, E_δ) is an induced subgraph of $Cay(S,A)$, $b \to c$ is an arc in $Cay(S,A)$, and hence there exists $a \in A$ such that $c = ab$. Suppose $a \in S_\alpha$. Then $c = ab = \Phi_{\alpha,\alpha\delta}(a)\Phi_{\delta,\alpha\delta}(b) = \Phi_{\alpha,\alpha\delta}(a)b$. Therefore $\delta = \alpha\delta$ and $b \to c$ is an arc in $Cay(S_\delta, A_\delta)$ with color $\Phi_{\alpha,\alpha\delta}(a)$.

If $b \to c$ is an arc in $Cay(S_\delta, A_\delta)$. Then there exists $d \in A_\delta$ such that $c = db$. So there exist $\alpha \in Y$ and $a \in A \cap S_\alpha$ such that $\alpha \geq \delta$ and $d = \Phi_{\alpha,\delta}(a)$. Hence $c = db = ab$ and $b \to c$ is an arc in $Cay(S,A)$. Since (S_δ, E_δ) is an induced subgraph of $Cay(S,A)$, we have $b \to c$ is an arc in (S_δ, E_δ), as required.

(2) Let $b \in S_\beta, c \in S_\gamma$ with $\beta \neq \gamma$ and $b \to c$ be an arc in $Cay(S,A)$. Then there exists $a \in A$ such that $c = ab$. Suppose $a \in S_\alpha$. Then $c = ab = \Phi_{\alpha,\alpha\beta}(a)\Phi_{\beta,\alpha\beta}(b)$ and $\gamma = \alpha\beta$. Hence $\beta > \gamma$ and $\Phi_{\beta,\gamma}(b) \to c$ is an arc in $Cay(S_\gamma, A_\gamma)$ with color $\Phi_{\alpha,\gamma}(a)$.

Conversely, suppose $\beta > \gamma$ and and there exist $\alpha \in Y$ and $a \in A \cap S_\alpha$ such that $\alpha\beta = \gamma$ and $\Phi_{\beta,\gamma}(b) \to c$ is an arc in $Cay(S_\gamma, A_\gamma)$ with color $\Phi_{\alpha,\gamma}(a)$. Then $c = \Phi_{\alpha,\gamma}(a)\Phi_{\beta,\gamma}(b) = ab$ and so $b \to c$ is an arc in $Cay(S,A)$.

If $S_\alpha, \alpha \in Y$ are groups in Theorem 3.1, then we get Theorem 4.1 of [9]. If $S_\alpha, \alpha \in Y$ are rectangular bands in Theorem 3.1, then we have

Corollary 3.2 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ $(\alpha \in Y)$, and A a subset of B. For any $\delta \in Y$, let $A_\delta = \{\Phi_{\alpha,\delta}(a) \mid a \in A, a \in B_\alpha, \alpha \geq \delta\}$. Then*

(1) the Cayley graph $Cay(B,A)$ contains $|Y|$ disjoint induced subgraphs $Cay(B_\delta, A_\delta) \cong qS_{m,p-m}$, where $p = |I_\delta|$, $q = |\Lambda_\delta|$ and $m = |I_{A_\delta}|$, and

(2) if $\beta \neq \gamma$ and $b \in B_\beta, c \in B_\gamma$, then $b \to c$ is an arc in $Cay(B,A)$ if and only if $\beta > \gamma$ and there exist $\alpha \in Y$, $a \in A \cap B_\alpha$ such that $\alpha\beta = \gamma$ and $\Phi_{\beta,\gamma}(b) \to c$ is an arc in $Cay(B_\gamma, A_\gamma)$ with color $\Phi_{\alpha,\gamma}(a)$.

Proof. It follows from Theorems 2.3 and 3.1 immediately.

We need the following properties of normal bands.

Lemma 3.3 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ $(\alpha \in Y)$, and $b, c \in B_\beta$ for some $\beta \in Y$. If $\lambda_b = \lambda_c$, then $\Phi^2_{\beta,\delta}(b) = \Phi^2_{\beta,\delta}(c)$ for any $\delta \in Y$ with $\delta \leq \beta$, and if $i_b = i_c$, then $\Phi^1_{\beta,\delta}(b) = \Phi^1_{\beta,\delta}(c)$ for any $\delta \in Y$ with $\delta \leq \beta$.*

Proof. If $\lambda_b = \lambda_c$, let $\Phi_{\beta,\delta}(b) = (s,\mu)$ and $\Phi_{\beta,\delta}(c) = (t,\nu)$. Then

$$\begin{aligned}(s,\mu) &= \Phi_{\beta,\delta}(b) = \Phi_{\beta,\delta}((i_b,\lambda_b)(i_c,\lambda_c)) \\ &= \Phi_{\beta,\delta}((i_b,\lambda_b))\Phi_{\beta,\delta}((i_c,\lambda_c)) \\ &= (s,\mu)(t,\nu) = (s,\nu).\end{aligned}$$

So $\mu = \nu$ and $\Phi^2_{\beta,\delta}(b) = \Phi^2_{\beta,\delta}(c)$. Similarly, if $i_b = i_c$, we may show that $\Phi^1_{\beta,\delta}(b) = \Phi^1_{\beta,\delta}(c)$.

Now we give the properties of the Cayley graph of a normal band.

Lemma 3.4 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ ($\alpha \in Y$), and A a subset of B. If b has a loop in $Cay(B, A)$ with $b \in B_\beta$ for some $\beta \in Y$, then there exists $a \in A$ with $a \in B_\alpha$ such that $\alpha \geq \beta$ and $\Phi^1_{\alpha,\beta}(a) = i_b$.*

Proof. Let b have a loop with $b \in B_\beta$ for some $\beta \in Y$. Then there exists $a \in A$ with $a \in B_\alpha$ for some $\alpha \in Y$ such that $b = ab$. So $\alpha \geq \beta$ and $b = \Phi_{\alpha,\beta}(a)b = (\Phi^1_{\alpha,\beta}(a), \lambda_b)$. Therefore $\Phi^1_{\alpha,\beta}(a) = i_b$, as required.

Lemma 3.5 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ ($\alpha \in Y$), and A a subset of B. Then*

(1) if $b \in B_\beta, c \in B_\gamma$ with $\beta > \gamma$ and $b \to c$ is an arc in $Cay(B, A)$ with color a for some $a \in A$, then for any $d \in B$, $d \to cd$ is an arc in $Cay(B, A)$, whenever ad and abd are in same rectangular component of B. In particular, if Y is a finite chain, then $d \to cd$ is an arc in $Cay(B, A)$ for any $d \in B$.

(2) if $b, c \in B_\beta$ for some $\beta \in Y$, $b \to c$ is an arc in $Cay(B, A)$, then for any $d \in B_\beta$, $d \to cd$ is an arc in $Cay(B, A)$.

Proof. (1) Let $b \in B_\beta, c \in B_\gamma$ with $\beta > \gamma$ and $b \to c$ be an arc in $Cay(B, A)$ with color a. Then $c = ab$. Let $a \in B_\alpha$ for some $\alpha \in Y$. Then $\gamma = \alpha\beta$. For any $d \in B$ with $d \in B_\theta$ for some $\theta \in Y$ and $ad, abd \in B_\delta$ for some $\delta \in Y$, we have $cd = abd = \Phi_{\alpha,\delta}(a)\Phi_{\beta,\delta}(b)\Phi_{\theta,\delta}(d) = (\Phi^1_{\alpha,\delta}(a), \Phi^2_{\theta,\delta}(d)) = ad$. So $d \to cd$ is an arc in $Cay(B, A)$. In particular, if Y is a finite chain, then $\gamma = \alpha\beta$ and $\beta > \gamma$ imply that $\alpha = \gamma$. So $ab, a \in B_\alpha$ and $abd, ad \in B_{\alpha\theta}$ for any $\theta \in Y$ and $d \in B_\theta$. Therefore, $d \to cd$ is an arc in $Cay(B, A)$ for any $d \in B$.

(2) Let $b, c \in B_\beta$ for some $\beta \in Y$. If $b \to c$ is an arc in $Cay(B, A)$, then there exists $a \in A$ such that $c = ab$. Let $a \in B_\alpha$ for some $\alpha \in Y$. Then $\alpha \geq \beta$. For any $d \in B_\beta$, we get $cd = abd = \Phi_{\alpha,\beta}(a)bd = (\Phi^1_{\alpha,\delta}(a), \lambda_d) = ad$. So $d \to cd$ is an arc in $Cay(B, A)$ for any $d \in B_\beta$.

For any nonempty subset A of a normal band B, let

$$L_A = \{i \in \cup_{\alpha \in Y} I_\alpha \mid (i, \lambda) \in A \text{ for some } \lambda \in \cup_{\alpha \in Y} \Lambda_\alpha\}.$$

Lemma 3.6 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ ($\alpha \in Y$), and A_1, A_2 two subsets of B. If $L_{A_1} = L_{A_2}$, then $Cay(B, A_1) = Cay(B, A_2)$.*

Proof. It is clear that $Cay(B, A_1)$ and $Cay(B, A_2)$ have the same vertex set. We shall show that $E(Cay(B, A_1)) = E(Cay(B, A_2))$. Let $b \in B_\beta, c \in B_\gamma$ with $\beta \geq \gamma$ and $b \rightarrow c$ be an arc in $Cay(B, A_1)$. Then there exists $a \in A_1$ such that $c = ab$. Since $L_{A_1} = L_{A_2}$, there is $a' \in B_\alpha \cap A_2$ such that $i_{a'} = i_a$. So $a'b = ab = c$ and $b \rightarrow c$ is an arc in $Cay(B, A_2)$. Hence $E(Cay(B, A_1)) \subseteq E(Cay(B, A_2))$. Similar argument shows that $E(Cay(B, A_2)) \subseteq E(Cay(B, A_1))$. Therefore $E(Cay(B, A_1)) = E(Cay(B, A_2))$ and $Cay(B, A_1) = Cay(B, A_2)$.

The next example illustrates that the converse of Lemma 3.6 is not true.

Example 3.7 *We consider the strong semilattice of rectangular bands G_α (top left), G_β (top right) and G_γ (bottom), where $G_\alpha = \{(1, a)\}$, $G_\beta = \{(2, b)\}$, $G_\gamma = \{(3, c)\}$. Let $A_1 = \{(2, b), (3, c)\}$ and $A_2 = \{(1, a), (2, b), (3, c)\}$. Then $Cay(B, A_1) = Cay(B, A_2)$ (see Figure 2). But $L_{A_1} \neq L_{A_2}$.*

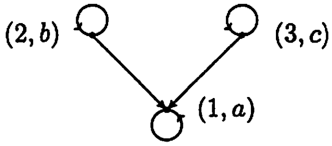


Figure 2 $Cay(B, A_1) = Cay(B, A_2)$.

If Y is a chain, then the converse of Lemma 3.6 is true.

Lemma 3.8 *Let Y be a finite chain, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong chain of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ ($\alpha \in Y$), and A_1, A_2 two subsets of B. Then $Cay(B, A_1) = Cay(B, A_2)$ if and only if $L_{A_1} = L_{A_2}$.*

Proof. Necessity. Let $i \in L_{A_1}$. Then there exists $a \in A_1$ with $a \in B_\alpha$ for some $\alpha \in Y$ such that $i = i_a$. Hence $d \rightarrow ad$ is an arc in $Cay(B, A_1)$ for any $d \in B$. Since $Cay(B, A_1) = Cay(B, A_2)$, $d \rightarrow ad$ is also an arc in $Cay(B, A_2)$. So there exists $a' \in A_2$ with $a' \in B_\gamma$ for some $\gamma \in Y$ such that $ad = a'd$. If α is not the greatest element of Y, then there exists $\beta \in Y$ such that $\beta > \alpha$. Let $d \in B_\beta$. Then $ad = a'd$ implies that $\gamma = \alpha$ since Y is a chain. Hence $i_a = i_{a'} \in L_{A_2}$. If α is the greatest element of Y, then $a = a^2 = a'a$ and so $i_a = i_{a'} \in L_{A_2}$. Consequently, $L_{A_1} \subseteq L_{A_2}$. Similar argument will show that $L_{A_2} \subseteq L_{A_1}$. Therefore $L_{A_1} = L_{A_2}$.

Sufficiency. It follows directly from Lemma 3.6.

To determine a connection set of a Cayley graph, we need the following result which is true for any Cayley graphs of semigroups.

Lemma 3.9 *Let S be a semigroup, A a subset of S, and let $A_1 = \{a \in S \mid (d, ad) \in E(Cay(S, A))$ for any $d \in S\}$. Then $Cay(S, A) = Cay(S, A_1)$.*

Proof. It is clear that $Cay(S, A)$ and $Cay(S, A_1)$ have the same vertex set. We shall show that $E(Cay(S, A)) = E(Cay(S, A_1))$. Let $b \to c$ be an arc in $Cay(S, A)$. Then there exists $a \in A$ such that $c = ab$. Since $(d, ad) \in Cay(S, A)$ for any $d \in S$, by the definition of A_1, we have $a \in A_1$. Hence $b \to c$ is also an arc in $Cay(S, A_1)$ and so $E(Cay(S, A)) \subseteq E(Cay(S, A_1))$.

On the other hand, let $b \to c$ be an arc in $Cay(S, A_1)$. Then there exists $a \in A_1$ such that $c = ab$. By the definition of A_1, we get that $(d, ad) \in Cay(S, A)$ for any $d \in S$. In particular, take $d = b$, we have $(b, c) = (b, ab) \in Cay(S, A)$. Therefore $E(Cay(S, A_1)) \subseteq E(Cay(S, A))$ and $Cay(S, A) = Cay(S, A_1)$.

The next theorem gives some conditions under which a graph, whose vertex set can be associated with elements of a normal band, is a Cayley graph of a normal band for some appropriate connection set. Theorem 3.11 dose the same for the Cayley graph of a strong chain of rectangular bands.

Theorem 3.10 *Let Y be a finite semilattice, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong semilattice of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ $(\alpha \in Y)$, and let (B, E) be a graph such that:*

(1) the graph (B, E) contains $|Y|$ disjoint induced subgraphs $(B_\delta, E_\delta) = Cay(B_\delta, A_\delta)$ $(\delta \in Y)$, where A_δ is the set of vertices in B_δ which have a loop, and

(2) if $c \in B_\gamma$ has a loop, then there exist $\alpha \in Y$ and $a \in B_\alpha$ such that $\alpha \geq \gamma$, $\Phi^1_{\alpha,\gamma}(a) = i_c$ and $(d, ad) \in E$ for any $d \in B$, and

(3) if $b \in B_\beta, c \in B_\gamma$ with $\beta \neq \gamma$ and $(b, c) \in E$, then $\beta > \gamma$ and there exist $\alpha \in Y$, $a \in B_\alpha$ such that $\alpha\beta = \gamma$, $c = ab$ and $(d, ad) \in E$ for any $d \in B$.

Let $A = \{a \in B \mid (d, ad) \in E \text{ for any } d \in B\}$. Then $(B, E) = Cay(B, A)$.

Proof. It is clear that (B, E) and $Cay(B, A)$ have the same vertex set. We shall show that $E = E(Cay(B, A))$. Let $b \to c$ be an arc in $Cay(B, A)$. Then there exists $a \in A$ such that $c = ab$. So by the definition of A, we have $(b, c) = (b, ab) \in E$, and therefore $E(Cay(B, A)) \subseteq E$.

Let $b \in B_\beta, c \in B_\gamma$ and $(b, c) \in E$. If $\beta \neq \gamma$, then by (3), $\beta > \gamma$ and there exist $\alpha \in Y$, $a \in B_\alpha$ such that $\alpha\beta = \gamma$, $c = ab$ and $(d, ad) \in E$ for any $d \in B$. Therefore $a \in A$ and $(b, c) = (b, ab) \in E(Cay(B, A))$. If $\beta = \gamma$, then by (1) and Lemma 2.1, $\lambda_b = \lambda_c$ and c has a loop in (B, E). So by (2), there exist $\alpha \in Y$ and $a \in B_\alpha$ such that $\alpha \geq \beta$, $\Phi^1_{\alpha,\beta}(a) = i_c$ and $(d, ad) \in E$ for any $d \in B$. It follows that $a \in A$ and $ab = \Phi_{\alpha,\beta}(a)b = (\Phi^1_{\alpha,\beta}(a), \lambda_b) = (i_c, \lambda_c) = c$. Hence $b \to c$ is an arc in $Cay(B, A)$. Therefore $E \subseteq E(Cay(B, A))$ and $(B, E) = Cay(B, A)$.

Theorem 3.11 *Let Y be a finite chain, $B = S(Y; B_\alpha; \Phi_{\alpha,\beta})$ a strong chain of rectangular bands $B_\alpha = I_\alpha \times \Lambda_\alpha$ $(\alpha \in Y)$, and let (B, E) be a graph such that:*

(1) *the graph* (B,E) *contains* $|Y|$ *disjoint induced subgraphs* $(B_\delta, E_\delta) = Cay(B_\delta, A_\delta)$ $(\delta \in Y)$, *where* A_δ *is the set of vertices in* B_δ *which have a loop, and*

(2) *if* $c \in B_\gamma$ *has a loop, then there exist* $\alpha \in Y$ *and* $a \in B_\alpha$ *such that* $\alpha \geq \gamma$, $\Phi^1_{\alpha,\gamma}(a) = i_c$ *and* $(d, ad) \in E$ *for all* $d \in B$, *and*

(3) *if* $b \in B_\beta, c \in B_\gamma$ *with* $\beta \neq \gamma$ *and* $(b,c) \in E$, *then* $\beta > \gamma$, $\lambda_c = \Phi^2_{\beta,\gamma}(b)$ *and* $(d, cd) \in E$ *for any* $d \in B$.

For any $\theta \in Y$, *let* $C_\theta = A_\theta$ *if* θ *is the greatest element of* Y *and* $C_\theta = \{a \in A_\theta \mid (d, ad) \in E \text{ for some } d \in B_\delta \text{ with } \delta > \theta\}$ *otherwise, and let* $A = \cup_{\theta \in Y} C_\theta$. *Then* $(B,E) = Cay(B,A)$.

Proof. It is clear that (B,E) and $Cay(B,A)$ have the same vertex set. We shall show that $E = E(Cay(B,A))$. Let $b \in B_\beta, c \in B_\gamma$ with $\beta \geq \gamma$ and $b \to c$ be an arc in $Cay(B,A)$. Then there exists $a \in A$ such that $c = ab$. Let $a \in B_\alpha$, then $\alpha\beta = \gamma$ and $c = ab = \Phi_{\alpha,\gamma}(a)\Phi_{\beta,\gamma}(b) = (\Phi^1_{\alpha,\gamma}(a), \Phi^2_{\beta,\gamma}(b))$. If α is not the greatest element of Y, then there exist $\delta > \alpha$ and $d \in B_\delta$ such that $(d, ad) \in E$ by the definition of A. Since $ad \in B_\alpha$ and $\Phi^1_{\alpha,\gamma}(ad) = \Phi^1_{\alpha,\gamma}(a)$ by Lemma 3.3, we have

$$adb = \Phi_{\alpha,\gamma}(ad)\Phi_{\beta,\gamma}(b) = (\Phi^1_{\alpha,\gamma}(ad), \Phi^2_{\beta,\gamma}(b)) = (\Phi^1_{\alpha,\gamma}(a), \Phi^2_{\beta,\gamma}(b)) = c.$$

Since $(d, ad) \in E$, by (3) we have $(b,c) = (b, adb) \in E$. If α is the greatest element of Y, then a has a loop in (B,E) by the definition of A. By (2) there exists $a' \in B_\alpha$ such that $i_a = i_{a'}$ and $(d, a'd) \in E$ for all $d \in B$. Since $\Phi^1_{\alpha,\gamma}(a) = \Phi^1_{\alpha,\gamma}(a')$ by Lemma 3.3, we have

$$a'b = \Phi_{\alpha,\gamma}(a')\Phi_{\beta,\gamma}(b) = (\Phi^1_{\alpha,\gamma}(a'), \Phi^2_{\beta,\gamma}(b)) = (\Phi^1_{\alpha,\gamma}(a), \Phi^2_{\beta,\gamma}(b)) = c.$$

It follows that $(b,c) = (b, a'b) \in E$. Therefore $E(Cay(B,A)) \subseteq E$.

Let $b \in B_\beta, c \in B_\gamma$ and $(b,c) \in E$. If $\beta \neq \gamma$, by (3), $\beta > \gamma$ and $\lambda_c = \Phi^2_{\beta,\gamma}(b)$. So $cb = c\Phi_{\beta,\gamma}(b) = (i_c, \Phi^2_{\beta,\gamma}(b)) = (i_c, \lambda_c) = c$ and $(b, cb) = (b,c) \in E$. Hence $c \in A$ by the definition of A and $b \to c$ is an arc in $Cay(B,A)$. If $\beta = \gamma$, by (1) and Lemma 2.1, $\lambda_b = \lambda_c$ and c has a loop in (B,E). By (2), there exist $\alpha \in Y$ and $a \in B_\alpha$ such that $\alpha \geq \gamma, \Phi^1_{\alpha,\gamma}(a) = i_c$ and $(d, ad) \in E$ for all $d \in B$. Next we show that $a \in A$. If α is not the greatest element of Y, then $(d, ad) \in E$ for any $d \in B_\delta$ with $\delta > \alpha$ and so $a \in A$ by the definition of A. If α is the greatest element of Y, it is clear that a has a loop in (B,E) and $a \in A$. Since $ab = \Phi_{\alpha,\gamma}(a)b = (\Phi^1_{\alpha,\gamma}(a), \lambda_b) = (i_c, \lambda_c) = c$, we conclude that $b \to c$ is an arc in $Cay(B,A)$. Therefore $E \subseteq E(Cay(B,A))$ and $(B,E) = Cay(B,A)$.

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References

[1] A. Abdollahi, 1-factorizations of Cayley graphs, Ars Combin. 86(2008), 129–131.

[2] I. Broere, J.H. Hattingh, The chromatic index of an abelian Cayley graph, Ars Combin. 31(1991), 312–316.

[3] N. Celniker, Finding the chromatic polynomial of Cayley graphs using the Tutte polynomial, Ars Combin. 82(2007), 33–40.

[4] S.H. Fan and Y.S. Zeng, On Cayley graphs of bands, Semigroup Forum 74(2007) 99–105.

[5] C.D. Godsil, On Cayley graph isomorphisms, Ars Combin. 15(1983), 231–246.

[6] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.

[7] A.V. Kelarev, On Cayley graphs of inverse semigroups, Semigroup Forum 72(3)(2006) 411–418.

[8] A.V. Kelarev and C.E. Praeger, On transitive Cayley graphs of groups and semigroups, European J. Combin. 24(1)(2003) 59–72.

[9] S. Panma, N.Na Chiangmai, U. Knauer, Sr. Arworn, Characterizations of Clifford semigroup digraphs, Discrete Math. 306(2006) 1247–1252.

[10] R.J. Wilson, Introduction to Graph Theory, Longman Inc., New York, Third Edition, 1982.

[11] B. Zelinka, Graphs of semigroups, Casopis. Pest. Mat. 106(1981) 407–408.