Extended Binary Linear Codes from Legendre Sequences

T. Aaron Gulliver and Matthew G. Parker*

Abstract

A construction based on Legendre sequences is presented for a doubly-extended binary linear code of length $2p + 2$ and dimension $p + 1$. This code has a double circulant structure. For $p = 4k + 3$, we obtain a doubly-even self-dual code. Another construction is given for a class of triply extended rate 1/3 codes of length $3p + 3$ and dimension $p + 1$. For $p = 4k + 1$, these codes are doubly-even self-orthogonal.

1 Introduction

A binary $[n, K]$ code C is a K-dimensional vector subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field of two elements. The parameter n is called the length of C. The elements of a code C are called *codewords* and the *weight* of a codeword is the number of non-zero coordinates. Denote the weight of a codeword $\mathbf{c}$ as $wt(\mathbf{c})$. The *minimum weight* of C is the smallest weight among all non-zero codewords of C. An $[n, K, d]$ code is an $[n, K]$ code with minimum weight d. Two codes are *equivalent* if one can be obtained from the other by a permutation of coordinates.

*T.A. Gulliver is with the Dept. of Electrical and Computer Engineering, University of Victoria, P.O. Box 3055 STN CSC, Victoria, BC V8W 3P6 Canada. agullive@ece.uvic.ca. Web: http://www.ece.uvic.ca/~agullive/ M.G. Parker is with the Selmer Centre, Inst. for Informatikk, Høyteknologisenteret i Bergen, University of Bergen, Bergen 5020, Norway. E-mail: matthew@ii.uib.no. Web: http://www.ii.uib.no/~matthew/ This work was done while the first author was visiting the Selmer Centre.

The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{F}_2^n | (x,y) = 0 \text{ for all } y \in C\}$ where (x,y) denotes the inner product. A code C is called *self-dual* if $C = C^\perp$. A self-dual code C is called *doubly-even* or *singly-even* if all codewords have weight $\equiv 0 \pmod 4$ or if some codeword has weight $\equiv 2 \pmod 4$, respectively.

Let D_p and D_b be codes with generator matrices of the form

$$\begin{pmatrix} I_n & R \end{pmatrix} \tag{1}$$

and

$$\left(I_{n+1} \quad \begin{array}{cccc} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & R' & \\ 1 & & & \end{array} \right), \tag{2}$$

respectively, where I is the identity matrix of order n and R and R' are $n \times n$ circulant matrices. The codes D_p and D_b are called *pure double circulant* and *bordered double circulant*, respectively. The two families are collectively called double circulant codes. Many of the known self-dual codes are double circulant [2, 3, 5, 6, 9].

It was shown in [14] that the minimum weight d of a doubly-even self-dual code of length n is bounded by $d \leq 4[n/24]+4$. We call a doubly-even self-dual code meeting this upper bound *extremal*. The largest possible minimum weights of doubly-even self-dual codes of lengths up to 72 are given in [2, Table I]. This work was revised and extended to lengths up to 96 in [3, Table V]. We say that a doubly-even self-dual code with the largest possible minimum weight given in [2, Table I], [3, Table V] is *extremal*. Many extremal self-dual codes are double circulant [2, 3, 5, 6, 7, 9].

In this paper we employ a *Legendre sequence* [16] of length p, p an odd prime, to build a circulant matrix which is then used to construct a bordered double circulant code of length $n = 2p+2$ and dimension $K = p+1$. We show that these codes have good distance, in particular when 2 is a quadratic nonresidue, mod p. For $p = 4k+3$, we show that these codes are self-dual. Another construction based on these sequences is used to obtain a class of triply extended rate 1/3 codes of length $3p+3$ and dimension $p+1$. For $p = 4k+1$, these codes are doubly-even self-orthogonal.

2 The Construction

2.1 Legendre Sequences

Let a be a primitive integer root, mod p, where p is an odd prime. Let $\mathcal{A} = \{a^{2i}\}$ be the set of even powers of a, mod p, and $\mathcal{B} = \{a^{2i+1}\}$ be the set of odd powers of a, mod p.

Definition 1. *The binary* Legendre sequence, s, *of length p (see e.g. [1, 10]), satisfies*

$$\mathbf{s} = (s_0, s_1, \ldots, s_{p-1}) \quad | \quad s_0 = 0, s_t = 1 \text{ if } t \in \mathcal{A}, s_t = 0 \text{ if } t \in \mathcal{B}.$$

We have chosen in this case to assign $s_0 = 0$, but we retain the possibility to assign 0 or 1 to s_0.

Definition 2. *The* alternative Legendre sequence $\tilde{\mathbf{s}}$, *has* $\tilde{s}_0 = 1$, *and* $\tilde{s}_t = s_t$ *if* $t \neq 0$.

Define $\mathbf{u} = (u_0, u_1, \ldots, u_{p-1})$ as the *cyclic autocorrelation* of s with

$$u_j = \sum_{t=0}^{p-1} (-1)^{s_t - s_{t+j}},$$

where the index of s is taken mod p. Similarly, define $\tilde{\mathbf{u}}$ as the cyclic autocorrelation of $\tilde{\mathbf{s}}$. The following properties of s and $\tilde{\mathbf{s}}$ are well-known

Lemma 1. *[16]*

$$\begin{array}{ll} u_0 = \tilde{u}_0 = p, & \\ u_j, \tilde{u}_j = -1, & j \neq 0, p = 4k+3, \\ u_j, \tilde{u}_j \in \{1, -3\}, & j \neq 0, p = 4k+1, \\ u_j + \tilde{u}_j = -2, & j \neq 0. \end{array}$$

In the sequel we make particular use of the property that $u_j + \tilde{u}_j = -2$ when $j \neq 0$ or p to construct, for all odd primes p, a double circulant code of length $2p$. We illustrate the code construction by means of an example.

2.2 Example

Consider the length $p = 5$ Legendre sequence $\mathbf{s} = 01001$, where $s_t = 1$ for $t \in \mathcal{A} = \{1,4\}$ and $s_t = 0$ for $t \in \mathcal{B} = \{2,3\}$. The alternative Legendre sequence is $\tilde{\mathbf{s}} = 11001$. It follows that $\mathbf{u} = 5,-3,1,1,-3$ and $\tilde{\mathbf{u}} = 5,1,-3,-3,1$, and therefore $\mathbf{u}+\tilde{\mathbf{u}} = 10,-2,-2,-2,-2$. This suggests that appropriate bordering of the concatenation of the circulant matrices formed by $\mathbf{s}$ and $\tilde{\mathbf{s}}$ by two additional columns could give a matrix with orthogonal rows, and this proves to be the case for $p = 4k+3$.

For the example above, concatenating the circulant matrices formed from the Legendre and alternative Legendre sequences gives

$$\mathbf{D}' = \begin{matrix} 01001|11001 \\ 10100|11100 \\ 01010|01110 \\ 00101|00111 \\ 10010|10011 \end{matrix}$$

This is a double circulant generator matrix for a $[10,5,3]$ binary linear code ($\mathbf{D}'$ always generates a cyclic code). The above matrix can be bordered by the all-ones and all-zeroes columns, and then the all-ones row resulting in

$$\mathbf{D} = \begin{matrix} 11|11111|11111 \\ \hline 10|01001|11001 \\ 10|10100|11100 \\ 10|01010|01110 \\ 10|00101|00111 \\ 10|10010|10011 \end{matrix} .$$

$\mathbf{D}$ can be transformed into a bordered double circulant generator matrix for a $[12,6,4]$ optimal binary linear code, as will be shown later.

We generalise this construction to any length p Legendre sequence in the next section.

2.3 The Doubly-Extended Legendre Code Construction

Let $\mathbf{q} = \mathbf{s}|\tilde{\mathbf{s}}$.

Lemma 2.

$$wt(\mathbf{q}) = p.$$

Proof. From the definition of $\mathbf{s}, \mathrm{wt}(\mathbf{s}) = (p-1)/2$ and therefore $\mathrm{wt}(\tilde{\mathbf{s}}) = (p-1)/2+1$. Thus $\mathrm{wt}(\mathbf{q}) = 2(p-1)/2+1 = p$. □

Define $\rho = (\rho_0, \rho_1, \ldots, \rho_{2p-1})$ as the cyclic autocorrelation of $\mathbf{q}$, where

$$\rho_j = \sum_{t=0}^{2p-1} (-1)^{q_t - q_{t+j}},$$

and the index of q is taken mod $2p$.

Lemma 3.

$$\rho_j = -2, \qquad 0 < j < 2p,\ j \neq p.$$

Proof. Follows immediately from Lemma 1 as $\rho_j = u_j + \tilde{u}_j$. □

Define $\mathbf{w} = (w_0, w_1, \ldots, w_{p-1})$ as the $\{0,1\}$-cyclic autocorrelation of $\mathbf{q}$, where

$$w_j = \sum_{t=0}^{2p-1} q_t q_{t+j},$$

and the index of q is taken mod $2p$. Note that this is a shortened version of the complete autocorrelation as we are only concerned with the first p elements.

Theorem 1.

$$\begin{aligned} w_j &= 2k+1, \qquad p = 4k+3, \qquad 0 < j < p, \\ &= 2k, \qquad\quad\ \ p = 4k+1, \qquad 0 < j < p. \end{aligned}$$

Proof. We can alternatively define w_j by $w_j = |\{t | q_t = q_{t+j} = 1, 0 \leq t < 2p\}|$. Define the set $\mathbf{A} = \{t | q_t \neq q_{t+j}, 0 \leq t < 2p\}$.

Consider the set of bit pairs $\{(q_t, q_{t+j})\}$, $0 \leq t < 2p$. We have that $w_j = |\{t | (q_t, q_{t+j}) = (1,1)\}|$, and $\mathrm{wt}(q) = |\{t | (q_t, q_{t+j}) = (1,0)\}| = |\{t | (q_t, q_{t+j}) = (0,1)\}|$. It follows that $2 \times \mathrm{wt}(q) = |\{t | (q_t, q_{t+j}) = (1,0)\}| +$

$|\{t|(q_t, q_{t+j}) = (0,1)\}| = |\{t|(q_t, q_{t+j}) = (1,0) \text{ or } (0,1)\}| = |\mathbf{A}|$. Therefore it follows that

$$\mathrm{wt}(\mathbf{q}) = |\{t|q_t = 1\}| = w_j + \frac{|\mathbf{A}|}{2}. \qquad (3)$$

Lemma 3 implies that $|\mathbf{A}| = p + 1$ which, together with Lemma 2 and (3), gives $w_j = \frac{p-1}{2}$, and the theorem follows. □

Let $\mathbf{d}_i$ be the ith row of $\mathbf{D}'$. An immediate corollary of Theorem 1 is

Corollary 1.

$$wt(\mathbf{d}_i + \mathbf{d}_j) = p + 1.$$

Let $\mathbf{s}$ be a length p Legendre sequence, where p is a prime integer, and $\mathbf{S}$ and $\tilde{\mathbf{S}}$ be the $p \times p$ circulant matrices with $\mathbf{s}$ and $\tilde{\mathbf{s}}$ as their first rows, respectively. Then

$$\mathbf{D}' = \mathbf{S}|\tilde{\mathbf{S}}$$

is a length $2p$ double circulant binary linear code of dimension p. Let $\mathbf{1}$ be the $p \times 1$ all-ones vector and $\mathbf{0}$ be the $p \times 1$ all-zeroes vector. Then

$$\mathbf{D} = \begin{matrix} 1 1 \mathbf{1}^T & |\mathbf{1}^T \\ \mathbf{1} \mathbf{0} \mathbf{S} & |\tilde{\mathbf{S}} \end{matrix}$$

is a length $2p+2$ bordered double circulant binary linear code of dimension $p+2$.

Theorem 2. *The code with generator matrix* $\mathbf{D}$ *for* $p = 4k+3$ *is a doubly-even self-dual code.*

Proof. Since $4|2p+2$ when p is an odd prime, the first row of $\mathbf{D}$ has weight a multiple of 4. The rows of $\mathbf{S}$ have weight $(p-1)/2$ and the rows of $\tilde{\mathbf{S}}$ have weight $(p+1)/2$. Adding these together gives $2p/2 = p$. The all-ones column adds weight 1 to each row, so all rows of $\mathbf{D}$ have weight $p+1$. From Corollary 1, the weight of the sum of any two rows of $\mathbf{D}'$ is even, and this also holds for the rows of $\mathbf{D}$, so the rows are orthogonal. When $p = 4k+3$, $p + 1 = 4k + 4$ so the weight of all rows is divisible by 4. Therefore from [12], the code is doubly-even self-dual. □

It is obvious that the minimum distance of the code generated by $\mathbf{D}$ is upperbounded by $p + 1$.

2.4 Reduced Echelon Form

It is often desirable to have a code in systematic or reduced echelon form

$$\mathbf{I}|\mathbf{P}$$

where $\mathbf{I}$ is the $p \times p$ identity matrix. The double circulant form of our construction should then be converted to the form (1). To achieve this, it is necessary that $\mathbf{S}$ or $\tilde{\mathbf{S}}$ be invertible. This in turn implies that s or $\tilde{\mathrm{s}}$, when viewed as polynomials, $s(x)$ or $\tilde{s}(x)$, should be invertible, mod x^p+1, mod 2. It turns out that, for $p = 8k \pm 1$, $s(x)$ and $\tilde{s}(x)$ are never invertible, for $p = 8k+3$ $s(x)$ is always invertible, and for $p = 8k-3$ $\tilde{s}(x)$ is always invertible. These conditions reflect the fact that 2 is a quadratic residue for $p = 8k \pm 1$ and a quadratic nonresidue for $p = 8k \pm 3$. Therefore a row echelon form for the doubly-extended Legendre code, $\mathbf{D}$, with the identity in the first p+2 or last p+2 columns, can only be achieved when $p = 8k \pm 3$, i.e. neither columns 0 to p+1, or columns p+2 to 2p+3 are information sets). Let $\overline{s(x)}$ denote that every coefficient of $s(x)$ is negated. Then, when $p = 8k \pm 3$, it can be shown that

$$\begin{array}{ll}
\tilde{s}(x)^{-1} = \tilde{s}(x)^2 = \overline{s(x)} & \bmod x^p+1, \bmod 2, \quad p = 8k-3 \\
s(x)^{-1} = s(x)^2 = \overline{\tilde{s}(x)} & \bmod x^p+1, \bmod 2, \quad p = 8k+3 \\
\tilde{s}(x)^{-1}s(x) = \overline{\tilde{s}(x)} & \bmod x^p+1, \bmod 2, \quad p = 8k-3 \\
s(x)^{-1}\tilde{s}(x) = \overline{s(x)} & \bmod x^p+1, \bmod 2, \quad p = 8k+3.
\end{array}$$

Therefore, when 2 is a quadratic nonresidue, mod p, we obtain a $p \times p$ circulant matrix, $\mathbf{P}$, whose first row is the negation of $\tilde{\mathrm{s}}$ for $p = 8k-3$, and the negation of s for $p = 8k+3$. In this case, we obtain a double circulant code having the first row of the circulant matrix as defined above. When $p = 8k+3$, the codes (bordered or pure) are equivalent to those given in [15, 13, 8, 11].

2.4.1 Example

For $p = 5$, $\tilde{s} = 11001$ and $\tilde{s}(x) = x^4 + x + 1$ has multiplicative order 3 mod $x^5 + 1$ (mod 2). Moreover $\tilde{s}(x)^{-1} = x^3 + x^2 + 1$. Thus

$$\tilde{\mathbf{S}} = \begin{matrix} 11001 \\ 11100 \\ 01110 \\ 00111 \\ 10011 \end{matrix} \qquad \text{and } \tilde{\mathbf{S}}^{-1} = \begin{matrix} 10110 \\ 01011 \\ 10101 \\ 11010 \\ 01101 \end{matrix}$$

since $\tilde{\mathbf{S}}^{-1}\tilde{\mathbf{S}} = \mathbf{I}$. Thus

$$\tilde{\mathbf{S}}^{-1}\mathbf{D}' = \mathbf{P}|\mathbf{I}$$

where

$$\mathbf{P} = \begin{matrix} 00110 \\ 00011 \\ 10001 \\ 11000 \\ 01100 \end{matrix}$$

since

$$\tilde{s}(x)^{-1}s(x) = (x^3 + x^2 + 1)(x^4 + x) \bmod x^5 + 1 = x^3 + x^2$$

The generator matrix then has the form

$$\mathbf{G} = \begin{matrix} 100000|011111 \\ 010000|100011 \\ 001000|110001 \\ 000100|111000 \\ 000010|101100 \\ 000001|100110 \end{matrix}$$

This is a bordered double circulant generator matrix for a $[12, 6, 4]$ binary linear code.

3 The Double Circulant Codes

The most well-known case is $p = 11$ as the $[24, 12, 8]$ Golay code is obtained. Note that $p = 7$ is the first case where both $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are singular, but in this case we obtain an extremal code. Table 1 shows the Hamming distances for the first 40 codes ($n \leq 180$) constructed from $\mathbf{D}$. The extremal codes are denoted by a '*'. For large n, it was not possible to find the minimum distance, so in these cases bounds are given. Of particular interest is when $p = 8k \pm 1$, since in these cases it is not possible to obtain a bordered double circulant code. Such primes are marked in table 1 with a '#'.

Table 1: Hamming Distances for the Doubly-Extended Double Circulant Codes

p	d	p	d	p	d	p	d	p	d
3	4*	29	12	61	20	101	20 – 30	139	20 – 44
5	4	31#	8	67	24*	103#	20	149	18 – 50
7#	4*	37	12	71#	12	107	20 – 36	151#	20
11	8*	41#	10	73#	14	109	20 – 36	157	16 – 52
13	8	43	16*	79#	16	113#	16	163	16 – 56
17#	6	47#	12	83	24	127#	20	167#	16 – 24
19	8*	53	20	89#	18	131	20 – 44	173	16 – 62
23#	8	59	20	97#	16	137#	18 – 22	179	16 – 60

From Table 1 one observes that, in general, the codes for $p = 8k \pm 1$ have lower minimum Hamming distance than those for $p = 8k \pm 3$. A lower bound on the minimum Hamming distance of the unextended form of the codes (given by $\mathbf{D}'$), when $p = 8k \pm 3$, can be obtained from the lower bound on Hamming distance for double circulant codes [11]

$$d \geq \frac{2(p + \sqrt{p})}{\sqrt{p} + 3}.$$

The corresponding bound for $\mathbf{D}$ (when $p = 8k \pm 3$) is

$$d \geq \frac{2p + 3\sqrt{p} + 3}{\sqrt{p} + 3}.$$

However, the bounding technique of [11] cannot easily adapted to the case when $p = 8k \pm 1$. This is because in this case $n(x)^i = n(x)$ and $q(x)^i = q(x)$

for all i where $q(x)$ are the quadratic residues and $n(x)$ are the quadratic nonresidues.

4 A Construction for Rate 1/3 Codes

Now consider the $[3p, p, d]$ codes with generator matrices

$$\mathbf{E}' = \mathbf{I}|\mathbf{D}' = \mathbf{I}|\mathbf{S}|\tilde{\mathbf{S}}.$$

These can be extended to $[3p+3, p+1, d]$ codes with generator matrices

$$\mathbf{E} = \begin{array}{cc} 0 & \mathbf{0}^T \\ \mathbf{1} & \mathbf{I} \end{array} \; \mathbf{D} = \begin{array}{cc|cc|c|c} 0 & \mathbf{0}^T & 1 & 1 & \mathbf{1}^T & \mathbf{1}^T \\ \mathbf{1} & \mathbf{I} & \mathbf{1} & \mathbf{0} & \mathbf{S} & \tilde{\mathbf{S}} \end{array}$$

Theorem 3. *The code with generator matrix* $\mathbf{E}$ *for* $p = 4k+1$ *is a doubly-even self-orthogonal code.*

Proof. Since $4|2p+2$ when p is an odd prime, the first row of $\mathbf{E}$ has weight a multiple of 4. The rows of $\mathbf{S}$ have weight $(p+1)/2$ and the rows of $\tilde{\mathbf{S}}$ have weight $(p-1)/2$. Adding these together gives $2p/2 = p$. The remaining columns in $\mathbf{E}$ add 3 to the weight of each of these rows, so they have weight $p+3$. From Corollary 1, the weight of the sum of any two rows of $\mathbf{D}'$ is even, so the rows are orthogonal. The inner product of the first row of $\mathbf{D}'$ with any other row is 1, therefore the first column makes the first row of $\mathbf{E}$ orthogonal to the others. When $p = 4k+1$, $p+3 = 4k+4$ so the weight of all rows is divisible by 4. Therefore from [12], the code is doubly-even self-orthogonal. □

Deleting the first row and 3 columns in $\mathbf{E}$ we obtain the following.

Corollary 2. *The code with generator matrix* $\mathbf{E}'$ *for* $p = 4k+1$ *is a singly-even self-orthogonal code.*

4.1 Example

Consider as before the length $p = 5$ Legendre sequence. The circulant matrices formed from the Legendre and alternative Legendre sequences give

$$\mathbf{E}' = \begin{array}{c} 10000|01001|11001 \\ 01000|10100|11100 \\ 00100|01010|01110 \\ 00010|00101|00111 \\ 00001|10010|10011 \end{array}$$

This is the generator matrix for a $[15, 5, 6]$ self-orthogonal quasi-cyclic code. This leads to the following extended code

$$\mathbf{E} = \begin{array}{c} 110|00000|11111|11111 \\ \hline 101|10000|01001|11001 \\ 101|01000|10100|11100 \\ 101|00100|01010|01110 \\ 101|00010|00101|00111 \\ 101|00001|10010|10011 \end{array}.$$

$\mathbf{E}$ is a bordered generator matrix for an $[18, 6, 8]$ optimal self-orthogonal binary linear code.

Table 2 gives the distances of the first few codes generated from $\mathbf{E}'$, and Table 3 gives distances and bounds for $\mathbf{E}$ up to $p = 151$. Note that the code from $\mathbf{E}'$ has distance 2 less than the corresponding code from $\mathbf{E}$. Several of these codes attain the lower bound on the maximum minimum distance for a binary linear code [4]. For large n, it was not possible to find the minimum distance, so in these cases bounds are given.

Table 2: Hamming Distances for Codes Generated Using $\mathbf{E}'$

p	d	p	d
5	6	17	10
7	6	19	14
11	10	29	22
13	10		

Table 3: Hamming Distances for Rate 1/3 Codes Generated by **E**

p	d	p	d	p	d	p	d	p	d
3	6	29	24	61	20	101	32 – 56	139	24 – 86
5	8	31	16	67	36	103	30 – 40	149	24 – 88
7	8	37	24	71	24	107	30 – 66	151	24 – 40
11	12	41	20	73	28	109	32 – 64		
13	12	43	28	79	32	113	28 – 32		
17	12	47	24	83	34 – 48	127	26 – 40		
19	16	53	32	89	32 – 36	131	28 – 78		
23	16	59	30	97	32	137	28 – 44		

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