Rainbow connection numbers of line graphs*

Xueliang Li, Yuefang Sun
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
E-mail: lxl@nankai.edu.cn; syf@cfc.nankai.edu.cn

Abstract

A path in an edge-coloring graph G, where adjacent edges may be colored the same, is called a rainbow path if no two edges of G are colored the same. A nontrivial connected graph G is rainbow connected if for any two vertices of G there is a rainbow path connecting them. The rainbow connection number of G, denoted $rc(G)$, is defined as the minimum number of colors by using which there is coloring such that G is rainbow connected. In this paper, we study the rainbow connection numbers of line graphs of triangle-free graphs, and particularly, of 2-connected triangle-free graphs according to their ear decompositions.

Keywords: edge-colored graph, rainbow path, rainbow connection number, line graph, triangle-free

AMS Subject Classification 2000: 05C15, 05C40

1 Introduction

All graphs in this paper are simple, finite and undirected. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, n\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path (trail) of G is called *rainbow* if no two edges of it are colored the same. An edge colored graph G is *rainbow connected* if for any two vertices there is a rainbow path connecting them. Clearly, if a graph is rainbow connected,

*Supported by NSFC, PCSIRT and the "973" program.

it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., the coloring such that each edge has a distinct color. Thus, we define the *rainbow connection number* of a connected graph G, denoted $rc(G)$, as the smallest number of colors for which there is an edge coloring of G such that G is rainbow connected. An easy observation is that if G has n vertices then $rc(G) \leq n-1$, since one may color the edges of a spanning tree with distinct colors, and color the remaining edges with one of the colors already used. Generally, if G_1 is a connected spanning subgraph of G, then $rc(G) \leq rc(G_1)$. We note the trivial fact that $rc(G) = 1$ if and only if G is complete, the fact that $rc(G) = n-1$ if and only if G is a tree, and the easy observation that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$ ([3]). As a Hamiltonian graph G has a Hamiltonian cycle which contains all n vertices, then G has rainbow connection number at most $\lceil \frac{n}{2} \rceil$. Also notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G.

Chartrand et al. in [3] determined that the rainbow connection numbers of some graphs including trees, cycles, wheels, complete bipartite graphs and complete multipartite graphs. Y. Caro et al. [4] observed that $rc(G)$ can be bounded by a function of $\delta(G)$, the minimum degree of G. They proved that if $\delta(G) \geq 3$ then $rc(G) \leq \alpha n$ where $\alpha < 1$ is a constant and $n = |V(G)|$. They conjectured that $\alpha = 3/4$ suffices and proved that $\alpha < 5/6$. Specifically, it was proved in [4] that if $\delta = \delta(G)$ then $rc(G) \leq \min\{\frac{\ln \delta}{\delta} n(1+o_\delta(1)), n\frac{4\ln\delta+3}{\delta}\}$. Their next two results give nontrivial sufficient conditions for $rc(G) = 2$, that is, any non-complete graph with $\delta(G) \geq n/2 + \log n$ has $rc(G) = 2$ and $p = \sqrt{\log n / n}$ is a sharp threshold function for the property $rc(G(n,p)) \leq 2$. Chakraborty et al. in [2] proved that for every $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that if G is a connected graph with n vertices and minimum degree at least εn, then $rc(G) \leq C$, and there is a polynomial time algorithm that constructs a corresponding coloring for a fixed ε. In that paper the authors mainly addressed the computational aspects of rainbow connection numbers. They solved, and extended, the complexity conjectures from [4]. It turns out that deciding whether $rc(G) = 2$ is an $\mathcal{NP}$-Complete problem. Krivelevich et al. in [6] also determined the behavior of $rc(G)$ as a function of $\delta(G)$: a connected graph G with n vertices has $rc(G) < 20n/\delta(G)$.

We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ be the subgraph induced by X, and $E[X]$ be the edge set of $G[X]$; similarly, for any subset E_1 of $E(G)$, let $G[E_1]$ be the subgraph induced by E_1. For any two disjoint subsets X, Y of $V(G)$, we use $G[X,Y]$ to denote the bipartite subgraph with vertex set $X \cup Y$ and edge set $E[X,Y] = \{uv \in E(G) | u \in X, v \in Y\}$. We define a *clique* in a graph G to be a complete subgraph of G, and a *maximal clique* is a clique that is not contained in a larger clique. The *clique graph* $K(G)$ of G is the intersection graph of the maximal cliques of G–that is, the vertices of $K(G)$ correspond to the maximal cliques of G, and two of these vertices are joined by an edge if and only if the corresponding maximal cliques intersect. A graph containing no triangle is a *triangle-free* graph. Let $[n] = \{1, \cdots, n\}$ denote the set of the first n natural numbers. For a set S, $|S|$ denotes the cardinality of S. We follow the notations and terminology of [1] for those not defined here.

2 Results on line graphs of triangle-free graphs

2.1 Some basic observations

By deleting some edges of a rainbow trail connecting two vertices, we can obtain a rainbow path between these two vertices, that is the following simple remark:

Remark 2.1 *If there is a rainbow trail connecting vertices u and v in an edge colored graph G, then there is a rainbow path connecting them.* ∎

Using the above Remark, we have the following simple observation which will be used in the sequel.

Observation 2.2 *If G is a connected graph and $\{E_i\}_{i\in[t]}$ is a partition of the edge set of G into connected subgraphs $G_i = G[E_i]$ and $rc(G_i) = c_i$, then*

$$rc(G) \leq \sum_{i=1}^{t} c_i.$$

∎

We just give c_i fresh colors to subgraph G_i for each i such that it is rainbow connected, and find a rainbow $u - v$ trail satisfying that each section belongs to distinct G_is for any two vertices $u, v \in G$.

Let G be a connected graph, and X a proper subset of $V(G)$. To *shrink* X is to delete all edges between vertices of X and then identify the vertices of X into a single vertex, namely w. We denote the resulting graph by G/X.

Observation 2.3 *Let G' and G be two connected graphs, where G' is obtained from G by shrinking a proper subset X of $V(G)$, that is, $G' = G/X$, such that any two vertices of X have no common adjacent vertex in $V \setminus X$, then*

$$rc(G') \leq rc(G).$$

∎

2.2 The line graph of a general graph

The *line graph* of a graph G is the graph $L(G)$ whose vertex set $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G. The star, $S(v)$, at a vertex v of graph G, is the set of all edges incident to v. A *clique decomposition* of G is a collection $\mathscr{C}$ of cliques such that each edge of G occurs in exactly one clique in $\mathscr{C}$. The *clique decomposition number* $cp(G)$ of G is the minimum size of all clique decompositions of G. A *minimum* clique decomposition is a clique decomposition $\mathscr{C}_0$ with $|\mathscr{C}_0| = cp(G)$.

We now introduce new terminology: For a connected graph G, we call G a *clique-cycle-structure*, if it satisfies the following three conditions:

C_1. G has at least three maximal cliques;

C_2. Each edge belongs to exactly one maximal clique;

C_3. The clique graph $K(G)$ is a cycle.

By condition C_2, we know that any two maximal cliques of G have at most one common vertex. Furthermore, G is formed by its maximal cliques. An example is shown in Figure 2.1. The *size* of the clique-cycle-structure is the number of its maximal cliques. We call a clique-cycle-structure *odd* if its size is odd, otherwise, it is an *even* clique-cycle-structure. A clique-cycle-structure of size 5 is shown in Figure 2.1. Note that a triangle is not a clique-cycle-structure, but a cycle with length $l \geq 4$ is a clique-cycle-structure of size l.

Similarly, we call a connected graph G a *clique-path-structure* if

P_1. Each block is a maximal clique;

P_2. The clique graph $K(G)$ is a path.

By condition P_1, we know that any two maximal cliques of G have at most one common vertex. Similarly, the *size* of a clique-path-structure is the number of its maximal cliques. Clearly, the diameter of a clique-path-

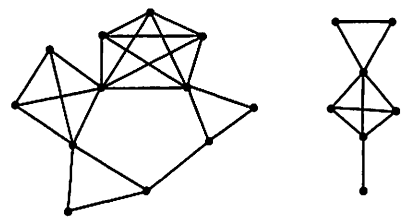


Figure 2.1 A clique-cycle-structure of size 5 and a clique-path-structure of size 3.

structure is equal to its size. A clique-path-structure of size 3 is shown in Figure 2.1.

An *inner vertex* of a graph is a vertex with degree at least two. For a graph G, we use V_2 to denote the set of all inner vertices of G. Let $n_1 = |\{v : deg_G(v) = 1\}|$, $n_2 = |V_2|$. $\langle S(v)\rangle$ is the subgraph of $L(G)$ induced by $S(v)$. Clearly, it is a clique of $L(G)$. Let $\mathscr{K}_0 = \{\langle S(v)\rangle : v \in V(G)\}$, $\mathscr{K} = \{\langle S(v)\rangle : v \in V_2\}$. It is easy to show that $\mathscr{K}_0$ is a clique decomposition of $L(G)$ ([7]) and each vertex of the line graph belongs to at most two elements of $\mathscr{K}_0$. We know each element $\langle S(v)\rangle$ of $\mathscr{K}_0 \setminus \mathscr{K}$, a single vertex of $L(G)$, is contained in the clique induced by u that is adjacent to v in G. $cp(L(G)) = n_2$([7]) when $G(\neq K_3)$. So $\mathscr{K}$ is a minimum clique decomposition of $L(G)$ for any $G \neq K_3$. We give each element of $\mathscr{K}$ a fresh color, and as the diameter of a clique-path-structure is just its size. We have the following theorem:

Theorem 2.4 *If G is a connected graph with n_2 inner vertices, then*

$$rc(L(G)) \leq n_2.$$

In particularly, if the induced subgraph $G[V_2]$ of G is a path, then the equality holds. ∎

2.3 The line graph of a triangle-free graph

Next, we will consider the rainbow connection number of the line graph of a triangle-free graph. We need to know the rainbow connection number of clique-cycle-structures:

Lemma 2.5 *Let G be a clique-cycle-structure of size l, then*

$$rc(G) = \begin{cases} \frac{l}{2} \text{ or } \frac{l}{2}+1 & l \text{ is even} \\ \frac{l+1}{2} & l \text{ is old} \end{cases}$$

Proof. As the conclusion clearly holds for each cycle of length at least 4, we only need to consider the case that G is not a cycle, that is, at least one maximal clique of G has order at least 3.

Case 1. $l = 2t$ where $t \geq 2$ is a positive integer.

Let the set of all maximal cliques be $\mathscr{C} = \{C_1, C_2, \cdots, C_{2t}\}$. v_i is the common vertex between C_i and C_{i+1} $(1 \leq i \leq 2t)$ where the subscripts are taken modulo $2t$. As shown in Figure 2.2, we give a $(t+1)$-edge-coloring of G as follows: For $1 \leq i \leq t$, we assign the edges of C_i which are incident to v_i with color i; for $t+1 \leq i \leq 2t$, we assign the edges of that C_i which are incident to v_i with color $i-t$. For other edges, we just give them color $t+1$. It is easy to show G is rainbow connected. As we used $t+1$ colors in total, $rc(G) \leq \frac{l}{2}+1$. On the other hand, the diameter of G is at least $\frac{l}{2}$, so $rc(G) = \frac{l}{2}$ or $\frac{l}{2}+1$.

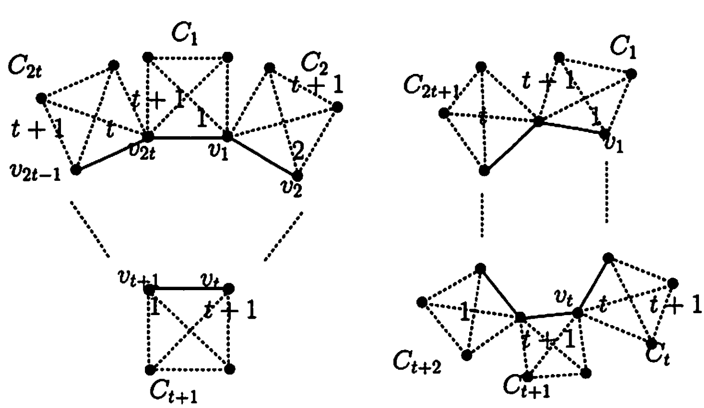


Figure 2.2 Rainbow edge-coloring of two cases of Lemma 2.5.

Case 2. $l = 2t+1$ where t is a positive integer.

Let the set of all maximal cliques be $\mathscr{C} = \{C_1, C_2, \cdots, C_{2t+1}\}$. v_i is the common vertex between C_i and C_{i+1} $(1 \leq i \leq t)$. As shown in Figure 2.2, for $1 \leq i \leq t$, we give color i to the edges of C_i which are incident to v_i and give color $t+1$ to the remaining edges of these cliques; we give all edges of

C_{t+1} with color $t+1$; for $t+2 \leq i \leq 2t+1$, we give all edges of clique C_i with the same color $i-(t+1)$. It is easy to show that with above coloring, G is rainbow connected. As we used $t+1$ colors in total, $rc(G) \leq \frac{l+1}{2}$. On the other hand, the diameter of G is $\frac{l+1}{2}$, so $rc(G) = \frac{l+1}{2}$. ∎

Now we only consider triangle-free graphs. There are some useful conclusions for a triangle-free graph and its line graph:

Theorem 2.6 *[5] A graph $L(G)$ is the line graph of a graph G without triangles if and only if $|C_i \cap C_j| \leq 1$ holds for any two maximal cliques of $L(G)$ and $K(L(G))$ has no triangles, where C_i, C_j are two maximal cliques of $L(G)$.* ∎

Theorem 2.7 *[5] For the line graph $L(G)$ of a connected triangle-free graph G, the set of all its maximal cliques are $\mathscr{C} = \mathscr{K} = \{\langle S(v_i)\rangle : v_i \in V_2\}$, where $\langle S(v_i)\rangle$ is a maximal clique and it corresponds to exactly one vertex v_i; for any two maximal cliques $\langle S(v_i)\rangle$ and $\langle S(v_j)\rangle$, they have at most one common vertex and they are adjacent (have a common vertex) if and only if v_i and v_j are adjacent in G.* ∎

We now introduce some new terminology: A set of maximal cliques of G is called its *clique-set*, denoted by CS. The *size* of a clique-set is the number of its elements. If the size of a clique-set is 1, it is a *trivial* clique-set; if the elements (maximal cliques) of some clique-set induce a clique-cycle-structure of size at least 4 in G, that is, the subgraph induced by those vertices contained in the elements of this clique-set is a clique-cycle-structure, it is called a *cyclic* clique-set.

A *clique-cycle-structure decomposition* of a connected graph G, denoted $CCSP$, is an edge decomposition of G by a family of clique-sets, and each clique-set is either trivial or a cyclic one of size at least 4. Formally, let $CCSP=\{CS_1, \cdots, CS_l, \cdots, CS_t\}$ be a clique-cycle-structure decomposition of some graph G, where the former l elements are the cyclic clique-sets of G, and the remaining $t-l$ elements are trivial clique-sets. For any triangle-free graph G, if it contains no cycle, then it is a tree, and its line graph has no clique-cycle-structure (otherwise, there is a cycle in G by Theorem 2.7, a contradiction) and each element of its clique-cycle-structure partition is trivial. If G contains at least one cycle, we choose a minimal cycle (which has no chord), namely $C: v_1, v_2, \cdots, v_t, v_1$. Then by Theorem 2.7, in the line graph $L(G)$, vertices contained in all maximal cliques $< S(v_i) > (1 \leq i \leq t)$ induce a clique-cycle-structure, and so its line graph has at least one clique-cycle-structure decomposition containing at least one cyclic clique-set. Let $s = t - l$. Then we have the following conclusion:

Theorem 2.8 *If $L(G)$ is the line graph of a connected triangle-free graph with at least 3 vertices, $CCSP$ is a clique-cycle-structure decomposition of $L(G)$ with s trivial elements, then*

$$rc(L(G)) \leq \frac{3}{4}n_2 + \frac{s}{4}.$$

Furthermore, if the equality holds, then the size of each nontrivial element of $CCSP$ is 4.

Proof. Let $CCSP = \{CS_1, \cdots, CS_l, CS_{l+1}, \cdots, CS_{l+s}\}$, $|CS_i| = m_i$ $(1 \leq i \leq l+s)$, and the subgraph of $L(G)$ induced by vertices contained in the element(s) of CS_i be G_i. Then by the above definition, for $1 \leq i \leq l$, G_i is a clique-cycle-structure of size $m_i \geq 4$ and for $l+1 \leq i \leq l+s$, G_i is a maximal clique of $L(G)$. So by the definition of clique-cycle-structure decomposition, $\{E(G_i)\}_{i=1}^{l+s}$ is an edge partition of $L(G)$.

For $l+1 \leq i \leq l+s$, we assign each G_i a fresh color such that edges in the same G_i have the same color and edges in distinct cliques have distinct maximal colors. This procedure costs us s colors. For $1 \leq i \leq l$, each G_i is a clique-cycle-structure of size m_i. Without loss of generality, let the former p m_is are odd, by Lemma 2.5, for $1 \leq i \leq p$, we give each G_i a rainbow edge coloring using $\frac{m_i+1}{2}$ fresh colors; for $p+1 \leq i \leq l$, we give each G_i a rainbow edge coloring using $\frac{m_i}{2}+1$ fresh colors. So the number of colors we used is at most

$$\begin{aligned}
& s + \textstyle\sum_{i=1}^{p} \frac{m_i+1}{2} + \sum_{i=p+1}^{l}(\frac{m_i}{2}+1) \\
= \ & s + \textstyle\sum_{i=1}^{l} \frac{m_i}{2} + \frac{p}{2} + (l-p) \\
= \ & s + \frac{1}{2}(n_2 - s) + (l - \frac{p}{2}) \\
= \ & \frac{1}{2}n_2 + \frac{1}{2}s + l - \frac{p}{2} \\
\leq \ & \frac{1}{2}n_2 + \frac{n_2-s}{4} + \frac{s}{2} - \frac{p}{2} \\
\leq \ & \frac{3}{4}n_2 + \frac{s}{4},
\end{aligned}$$

So if the equality holds, then $m_i = 4$ for all $1 \leq i \leq l$ (in this case we have $p = 0$). ∎

There are infinitely many graphs whose rainbow connection numbers equal $\frac{3}{4}n_2 + \frac{s}{4}$. One example is the graph shown in Figure 2.3 which is formed by some paths and 4-cycles. In its line graph, the 4 vertices (and their adjacent edges) of each 4-cycle induce a clique-cycle-structure of size 4 and diameter 3; each 2-degree vertex (and their adjacent edges) in each path induces an edge in its line graph. It is easy to show that the diameter of $L(G)$ is just $\frac{3}{4}n_2 + \frac{s}{4}$.

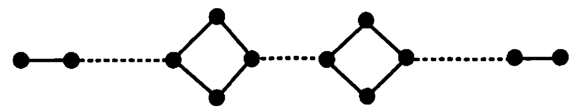


Figure 2.3 The figure for the example of Theorem 2.8.

3 The line graph of a 2-connected triangle-free graph

Let H be a subgraph of a graph G. An *ear* of H in G is a nontrivial path in G whose ends lie in H but whose internal vertices do not. A *nested sequence* of graphs is a sequence $(G_0, G_1, \cdots, G_k)$ of graphs such that $G_i \subset G_{i+1}$, $0 \leq i \leq k-1$. An *ear decomposition* of a 2-connected graph G is a nested sequence $(G_0, G_1, \cdots, G_k)$ of 2-connected subgraphs of G such that

1. G_0 is a cycle;
2. $G_{i+1} = G_i \cup P_{i+1}$ where P_{i+1} is an ear of G_i in G, $0 \leq i \leq k-1$;
3. $G_k = G$.

We now let G be a 2-connected triangle-free graph. We know that each 2-connected graph has an ear decomposition [1]. Furthermore, each vertex of a 2-connected graph has degree at least 2. So each vertex is an inner vertex, that is, $V_2 = V(G)$, $n_2 = n$. Let $(G_0, G_1, \cdots, G_k)$ be an ear decomposition of G. As G is a triangle-free graph, the length of G_0 is at least 4.

In [4], the authors observed that the lengths of the adding paths are *non-increasing*: at each step, just adding the path with the maximal length that can currently be added. Let l_{P_i} be the length of path P_i where P_i is the path added in the ear decomposition ($1 \leq i \leq k$). Then $l_{P_1} \geq l_{P_2} \geq \cdots \geq l_{P_t} \geq 2$, $l_{P_{t+1}} = \cdots = l_{P_k} = 1$ ($1 \leq t \leq k$) or $l_{P_1} = l_{P_2} = \cdots = l_{P_k} = 1$ (In this case, G is a Hamiltonian graph).

We first consider the case that there exists some $t \in [k]$ such that $l_{P_1} \geq l_{P_2} \geq \cdots \geq l_{P_t} \geq 2$, $l_{P_{t+1}} = \cdots = l_{P_k} = 1$ ($1 \leq t \leq k$).

Let $V_0' = V(G_0) = \{v_{0,1}, \cdots, v_{0,s_0}\}$ be the set of vertices of G_0 and $V_i' = V'(P_i) = \{v_{i,1}, \cdots, v_{i,s_i}\}$ be the set of inner vertices of path P_i $(1 \le i \le k)$. Clearly, $V_i' = \varnothing$ $(t+1 \le i \le k)$. By the definition of an ear-decomposition, $V_i' \cap V_j' = \varnothing$ $(0 \le i \ne j \le t)$ and $V(G) = \bigcup_{i=0}^{t} V_i'$, that is, $\{V_i', 0 \le i \le t\}$ is a partition of $V(G)$. We know that $\mathscr{K} = \{\langle S(v_i)\rangle : v_i \in V_2\} = \{\langle S(v_i)\rangle : v_i \in V(G)\} = \mathscr{K}_0 \cup \mathscr{K}_1 \cup \cdots \mathscr{K}_k$, where $\mathscr{K}_i = \{\langle S(v_{i,j})\rangle : 1 \le j \le s_i\}$ is the set of maximal cliques of $L(G)$ corresponding to the elements of V_i' of G, $0 \le i \le k$.

Let $\overline{G_i}$ be the graph whose vertex set and edge set are just those of $\bigcup_{a=0}^{i} \mathscr{K}_a$ where $0 \le i \le k$, and two maximal cliques $\langle S(v_i)\rangle$ and $\langle S(v_j)\rangle$ have (exactly) one common vertex (the common vertex corresponds the common adjacent edge of v_i and v_j in G_i) in $\overline{G_i}$ if and only if v_i and v_j are adjacent in G_i. Let $\overline{P_i}$ be the graph whose vertex set and edge set are just those of $\mathscr{K}_i$, and two maximal cliques $\langle S(v_i)\rangle$ and $\langle S(v_j)\rangle$ have a common vertex if and only v_i and v_j are adjacent in P_i. Note that $\overline{G_i}$ may not be the line graph of the subgraph G_i, $\overline{G_i}$ and $\overline{P_i}$ may not be the subgraphs of $L(G)$.

Clearly, by Remark 2.7, $\overline{G_0}$ is a clique-cycle-structure of size $s_0 \ge 4$ and $\overline{P_i}$ is a clique-path-structure of size s_i. So by Lemma 2.5, $rc(\overline{G_0}) \le \lceil \frac{s_0+1}{2} \rceil$. For $0 \le i \le t-1$, by the definition of an ear decomposition,

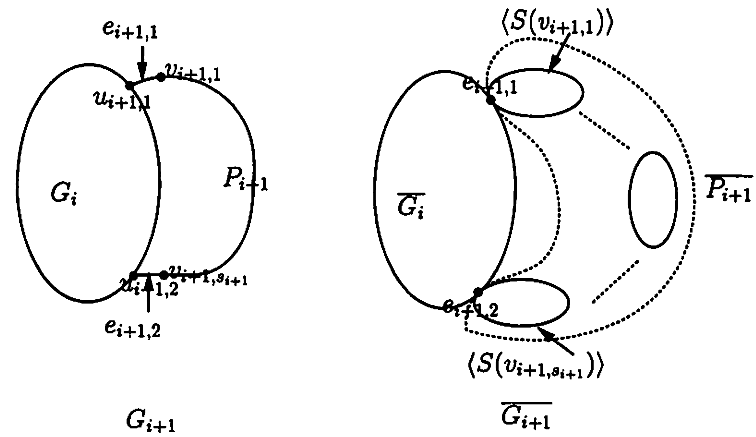


Figure 3.1 The figures of G_{i+1} and $\overline{G_{i+1}}$.

P_{i+1} is internally disjoint with G_i. Let the two common vertices of them be $u_{i+1,1}$, $u_{i+1,2}$, and let $v_{i+1,1}, v_{i+1,2}$ be the adjacent vertices of $u_{i+1,1}$, $u_{i+1,2}$ in P_{i+1} (as P_{i+1} $(0 \le i \le t-1)$ contains at least one inner vertex), respectively. Let $e_{i+1,1} = u_{i+1,1}v_{i+1,1}$, $e_{i+1,2} = u_{i+1,2}v_{i+1,s_{i+1}}$. Then by the definition of $\overline{G_{i+1}}$, $\overline{G_i}$ and $\overline{P_{i+1}}$ are subgraphs of $\overline{G_{i+1}}$; furthermore, by Theorem 2.7 and the definition of a line graph, in the graph $\overline{G_{i+1}}$, $\overline{G_i}$ and $\overline{P_{i+1}}$ have only two common vertices, namely $e_{i+1,1}$ and $e_{i+1,2}$, and $\{e_{i+1,1}\} = \langle S(u_{i+1,1})\rangle \cap \langle S(v_{i+1,1})\rangle$, $\{e_{i+1,2}\} = \langle S(u_{i+1,2})\rangle \cap \langle S(v_{i+1,s_{i+1}})\rangle$ (Figure 3.1).

Lemma 3.1 *For $0 \le i \le t-1$, we have*

$$rc(\overline{G_{i+1}}) \le rc(\overline{G_i}) + c_{i+1}$$

where $c_{i+1} = \lceil \frac{s_{i+1}}{2} \rceil$.

Proof. **Case 1.** s_{i+1} is even.

Let $s_{i+1} = 2b$ where b is a positive integer. We let the set of maximal cliques of $\overline{P_{i+1}}$ be $\{C_1, \cdots, C_{2b}\}$. The two common vertices between $\overline{G_i}$ and $\overline{P_{i+1}}$ are u and v as shown in Figure 3.2. For $1 \le j \le 2b-1$, the common vertex between C_j and C_{j+1} is v_j, and let $v_{2b} = v$. We give an

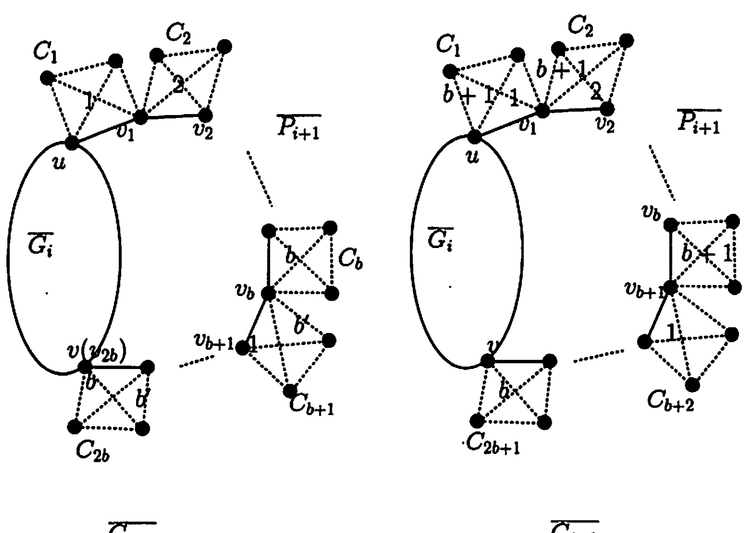


Figure 3.2 The figure of a rainbow edge coloring for the two cases of Lemma 3.1.

edge coloring as follows: We first give the subgraph $\overline{G_i}$ a rainbow $rc(\overline{G_i})$-edge coloring; for $1 \le j \le b$, we give the edges of C_j with the same fresh color i; for $b+1 \leq j \leq 2b$, we assign the edges in C_j incident to v_j with color $j-b$; for the rest edges of $\overline{P_{i+1}}$, we assign them with the same color b' where b' has been used in $\overline{G_i}$.

It is not hard to show that with above edge coloring, $\overline{G_{i+1}}$ is rainbow connected. As we used $rc(\overline{G_i})+b$ colors in total, $rc(\overline{G_{i+1}}) \leq rc(\overline{G_i}) + \frac{s_{i+1}}{2}$.

Case 2. s_{i+1} is odd.

Let $s_{i+1} = 2b+1$ where b is a nonnegative integer. We let the set of maximal cliques of $\overline{P_{i+1}}$ be $\{C_1, \cdots, C_{2b+1}\}$. The two common vertices between $\overline{G_i}$ and $\overline{P_{i+1}}$ are u and v as shown in Figure 3.2. For $1 \leq j \leq b$, the common vertex between C_j and C_{j+1} is v_j. We now give an edge coloring as follows: For $1 \leq j \leq b$, the edges of C_j incident to v_j are assigned color j, the rest edges of C_j are assigned color $b+1$; for $j = b+1$, we assign the edges of C_j with the same color $b+1$; for $b+2 \leq j \leq 2b+1$, we assign the edges of the maximal clique C_j with the same color $j-(b+1)$; for remaining edges, that is, the edges in the graph $\overline{G_i}$, we assign them with $rc(\overline{G_i})$ fresh colors such that $\overline{G_i}$ is rainbow connected.

It is not hard to show that with the above edge coloring, $\overline{G_{i+1}}$ is rainbow connected. As we used $rc(\overline{G_i})+b+1$ colors in total, $rc(\overline{G_{i+1}}) \leq rc(\overline{G_i}) + \lceil \frac{s_{i+1}}{2} \rceil$. ∎

For $t \leq i \leq k-1$, as the length of P_{i+1} is just 1, it has no inner vertex, $V'_{i+1} = \varnothing$, and so $\mathscr{K}_{i+1} = \varnothing$, that is, $\bigcup_{a=0}^{i+1} \mathscr{K}_a = \bigcup_{a=0}^{t} \mathscr{K}_a$.

Lemma 3.2 *For $t \leq i \leq k-1$, we have*

$$rc(\overline{G_{i+1}}) \leq rc(\overline{G_i}).$$

Proof. We know that for $t \leq i \leq k-1$, G_{i+1} is obtained from G_i by just adding an edge between two nonadjacent vertices, namely $e_{i+1} = v'_{i+1,1}v'_{i+1,2}$. So in the graph $\overline{G_i}$, the maximal cliques $\langle S(v'_{i+1,1})\rangle$ and $\langle S(v'_{i+1,2})\rangle$ have no common vertex. Then by the definitions of $\overline{G_{i+1}}$ and a line graph, in the graph $\overline{G_{i+1}}$, vertex e_{i+1} is the common vertex of the maximal cliques $\langle S(v'_{i+1,1})\rangle$ and $\langle S(v'_{i+1,2})\rangle$. So graph $\overline{G_{i+1}}$ is obtained from $\overline{G_i}$ by shrinking two nonadjacent vertices in $\langle S(v'_{i+1,1})\rangle$ and $\langle S(v'_{i+1,2})\rangle$, respectively, and this procedure produces a new vertex, e_{i+1} (shown in the Figure 3.3). By Observation 2.3, $rc(\overline{G_{i+1}}) \leq rc(\overline{G_i})$. ∎

From the definition of $\overline{G_k}$, $\overline{G_k} = L(G)$. So by Lemmas 3.1 and 3.2, we have the following theorem:

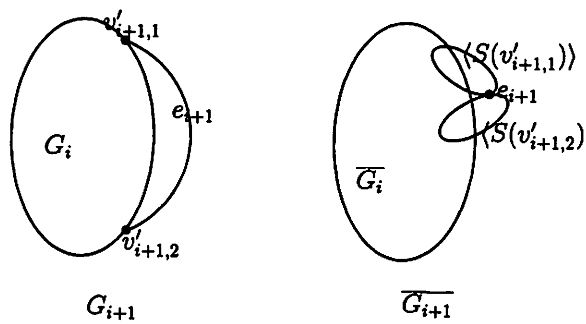


Figure 3.3 The figures for Lemma 3.2.

Theorem 3.3 *Let G be a 2-connected triangle-free graph of order n, $(G_0, G_1, \cdots, G_k)$ be an ear decomposition of G, where $G_{i+1} = G_i \cup P_{i+1}$, and P_i is an ear of G_i in G, $0 \le i \le k-1$, and let m_1 denote the number of P_is with length of positive even number, and $s_0 = |V(G_0)|$. Then*

$$rc(L(G)) \le \frac{n}{2} + c_e.$$

where

$$c_e = \begin{cases} \frac{m_1}{2} + 1 & s_0 \text{ is even} \\ \frac{m_1+1}{2} & s_0 \text{ is odd} \end{cases}$$

In particular, if $s_0 > 6$, and the length of each added path is at least 3, then $rc(L(G)) < \frac{2}{3}n$.

Proof. The terminology is the same as above, and as discussed in above paragraphs. We let $|P_1| \ge \cdots |P_k|$, and $s_i (1 \le i \le k)$ denote the number of inner vertices of added path P_i. We distinguish the following two cases:

Case 1. There exists some $t \in [k]$, such that $l_{P_1} \ge l_{P_2} \ge \cdots \ge l_{P_t} \ge 2$, $l_{P_{t+1}} = \cdots = l_{P_k} = 1$ $(1 \le t \le k)$.

Then by Lemmas 3.1 and 3.2, we have
$rc(G) = rc(\overline{G_k}) \le rc(\overline{G_{k-1}}) \le \cdots \le rc(\overline{G_t}) \le rc(\overline{G_{t-1}}) + c_t \le rc(\overline{G_{t-2}}) + c_{t-1} + c_t \le \cdots \le rc(\overline{G_0}) + \sum_{i=1}^{t} c_i$, where

$$c_i = \begin{cases} \frac{s_i}{2} & s_i \text{ is even} \\ \lceil \frac{s_i}{2} \rceil & s_i \text{ is old} \end{cases}$$

So $rc(G) \leq rc(G_0)+\sum_{i=1}^{t}\frac{s_i}{2}+\frac{m_1}{2} \leq \lceil\frac{s_0+1}{2}\rceil+\sum_{i=1}^{t}\frac{s_i}{2}+\frac{m_1}{2}=\frac{n}{2}+c_e$, where

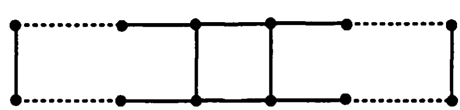
$$c_e = \begin{cases} \frac{m_1}{2}+1 & s_0 \text{ is even} \\ \frac{m_1+1}{2} & s_0 \text{ is odd} \end{cases}$$

Case 2. $l_{P_1} = \cdots = l_{P_k} = 1$.

This means that G is obtained from G_0 by adding k edges, $V(G) = V(G_0) = s_0$, and $m_1 = m_2 = 0$. So by Lemma 3.2, we have $rc(L(G)) = rc(\overline{G_k}) \leq rc(\overline{G_{k-1}}) \leq \cdots \leq rc(\overline{G_0}) = \lceil\frac{s_0+1}{2}\rceil = \lceil\frac{n+1}{2}\rceil$, and the conclusion holds.

We know $\frac{\lceil\frac{s_i}{2}\rceil}{s_i} \leq \frac{2}{3}$ for all $s_i \geq 2$, and only when $s_i = 3$, the equality holds. So by the above discussion, the worst case for counting this upper bound is when each path is length 4 ($s_i = 3$). And as $s_0 > 6$, $rc(L(G) \leq \lceil\frac{s_0+1}{2}\rceil + \frac{n-s_0}{3} \times 2 < \frac{2}{3}n$. ∎

From Theorem 3.3, while the order of G is large enough, and the lengths of the added paths in an ear decomposition of G are large (at least 4), then the rainbow connection number of the line graph $L(G)$ is very close to half of the order of graph G. There are many graphs whose rainbow connection numbers are very close to the bound given in the above theorem. For example, as shown in Figure 3.4, G is formed by a 4-cycle and two disjoint paths with an even number of inner vertices. It is easy to show that the diameter of the line graph $L(G)$ is equal to half of the order n of G, and so $rc(L(G)) = \frac{n}{2}$ is very close to the bound as $m_1 = 0$.



G

Figure 3.4 The figure for the last example.

Acknowledgement: The authors would like to thank the referee for helpful comments and suggestions.

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, *Hardness and algorithms for rainbow connectivity*, 26th International Symposium on Theoretical Aspects of Computer Science STACS 2009 (2009), 243-254.

[3] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133**(2008), 85-98.

[4] Y. Caro, Y. Roditty, Z. Tuza, R. Yuster, *on rainbow connection*, Electron. J. Combin. **15** (2008), Paper R57.

[5] S.T. Hedetniemi, P.J. Slater, *Line graphs of triangleless graphs and iterated clique graphs*, in *Graph Theory and Applications*, Lecture Notes in Mathematics **303** (ed. Y. Alavi *et al.*), Springer-Verlag, Berlin, Heidelberg, New York, 1972, pp. 139-147; *MR*49#151.

[6] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, Preprint 2009.

[7] Bo-Jr Li, G.J. Chang, *Clique coverings and partitions of line graphs*, Discrete Math. **308**(2008), 2075-2079.