Decompositions of graphs into a given clique-extension

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Abstract

For $r \geq 3$, a *clique-extension of order* $r+1$ is a connected graph that consists of a K_r plus another vertex adjacent to at most $r-1$ vertices of K_r. In this paper we consider the problem of finding the smallest number t such that any graph G of order n admits a decomposition into edge disjoint copies of a fixed graph H and single edges with at most t elements. Here we solve the case when H is a fixed clique-extension of order $r+1$, for all $r \geq 3$ and will also obtain all extremal graphs. This work extends results proved by Bollobás [Math. Proc. Cambridge Philosophical Soc. **79** (1976) 19–24] for cliques.

1 Introduction

Given graphs G and H, an H*-decomposition* of G is a set of subgraphs $G_1, \dots, G_t$ such that any edge of G is an edge of exactly one of $G_1, \dots, G_t$ and each $G_1, \dots, G_t$ is either a single edge or forms a graph isomorphic to H.

We denote by K_r the complete graph of order r, and by $t_{r-1}(n)$ the number of edges in the Turán graph of order n, $T_{r-1}(n)$, which is the unique complete $(r-1)$-partite graph on n vertices that has maximum number of edges and contains no complete graph of order r.

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Erdös, Goodman and Pósa [?] showed that the edges of any graph on n vertices can be decomposed into at most $\lfloor n^2/4 \rfloor$ edge disjoint K_3's and single edges and that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only extremal graph. Later Bollobás [?] generalized this result for $r \geq 4$ by showing that a graph of order n can be decomposed into at most $t_{r-1}(n)$ edge disjoint K_r's and single edges and that $T_{r-1}(n)$ is the unique extremal graph.

For $r \geq 3$, *a clique-extension of order* $r+1$ is a connected graph that consists of a K_r plus another vertex, say x, adjacent to at most $r-1$ vertices of K_r. For $i = 1, \cdots, r-1$ the $H_{r,i}$ be the clique-extension of order $r+1$ that has $\deg x = i$.

Let $r \geq 3$ and let H be a fixed clique-extension of order $r+1$. In Section 2 we prove that any graph of order n admits a decomposition into edge disjoint copies of H and single edges with at most $t_{r-1}(n)$ elements. Furthermore, a complete characterization of all extremal graphs is also obtained.

2 Clique-extension decompositions of graphs

In this section we will prove our results about decompositions of graphs into single edges and clique-extensions of order $r+1$, for all $r \geq 3$. Theorem ?? solves the case when $r = 3$ while Theorem ?? deals with the case $r \geq 4$. The reason for having two different theorems is that the proofs require different approaches and techniques. The case $r = 3$, stated in Theorem ??, was first obtained by the author in [?]. However, at that time the extremal graphs were not obtained. Here we conclude that work by giving the characterization of all extremal graphs. Therefore, for the sake of completeness, the proof of Theorem ?? will be entirely reproduced here with the necessary modifications and additions to fulfil our needs.

Before stating our main results we need to define three graphs and state some auxiliary Lemmas.

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be the graphs on five and six vertices shown if Figure ??.

Lemma 2.1. *[?, Proposition 1.3.1] Let G be a graph of order n with minimum degree k. Then G contains a path of length k.*

Lemma 2.2. *Given a graph H, denote by $\mathrm{ex}(n, H)$ the maximum number of edges in a graph of order n without containing a copy of H.*

(i) Let $r \geq 4$ and let K_r^- denote the complete graph on r vertices minus one edge. Then for all $n \geq r+1$, $\mathrm{ex}(n; K_r^-) = \mathrm{ex}(n; K_{r-1})$;

(ii) $\frac{r-2}{r-1}\binom{n}{2} \leq \mathrm{ex}(n; K_r) = t_{r-1}(n) \leq \frac{1}{2}n^2\frac{r-2}{r-1}$, for all $r \geq 3$.

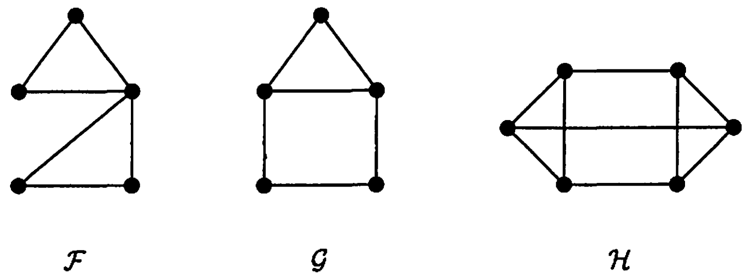

Figure 1: The graphs $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$.

Proof. Part *(i)* follows easily from the definition of $\text{ex}(n, H)$ and exercise 19 in [?, section IV.7]. The upper bound in part *(ii)* can be found in [?, Section 7.1] and the lower bound follows from simple calculations and can be found in [?, Section IV.2]. □

We are now able to state and prove our results.

Theorem 2.3. *Let H be a fixed clique-extension of order 4. Then every graph of order $n \geq 4$ can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ edge disjoint copies of H and single edges. Furthermore, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only graph that cannot be decomposed with fewer edge disjoint copies of H and single edges, except for $H = H_{3,2}$ where the graphs $H_{3,1}, \mathcal{F}, \mathcal{G}, K_5, \mathcal{H}$ are also extremal.*

Proof. By induction on the number of vertices in a graph.

We will see first that the result holds for all graphs of order n with $4 \leq n \leq 6$. To help us with this task we will use Harary's [?] atlas of all graphs of order at most 6. Observe first that the graphs stated as extremal are in fact extremal. Therefore, it remains to see that all other graphs admit a decomposition into edge disjoint copies of H and single edges with less than $\lfloor \frac{n^2}{4} \rfloor$ elements. Clearly, we only need to consider graphs with at least $\lfloor \frac{n^2}{4} \rfloor$ edges.

For $n = 4$ there are exactly 4 graphs that need to be studied. In this case we can easily check that the theorem holds. Let $n = 5$ and let G be a graph of order 5. If $e(G) = 6$ then, by inspection, we can see that the theorem holds. Let $e(G) = 7, 8$. Lemma ?? implies that G contains a copy of K_4^-. Since $H \subseteq K_4^-$ the result clearly holds. If $e(G) = 9$ or $e(G) = 10$ then the theorem easily holds.

Finally, let $n = 6$ and let G be a graph with 6 vertices and at least 9 edges. If $e(G) = 9$ then, by inspection, we can see that the theorem holds. So, suppose that $e(G) \geq 10$. We consider first the case $H = H_{3,1}$.

If $10 \le e(G) \le 11$ then G must contain a copy of $H_{3,1}$. Therefore, we can decompose G into one copy of $H_{3,1}$ and the remaining edges as single edges and we are done. If $12 \le e(G) \le 14$ then, we can easily see that G contains 2 edge disjoint copies of $H_{3,1}$ and we are done. Finally, for $G = K_6$ the result also holds. To conclude consider the case $H = H_{3,2}$. Observe that $H_{3,2} = K_4^-$ and that $e(G) > \mathrm{ex}(6, K_4^-) = 9$. So if $10 \le e(G) \le 12$ we can decompose G into one copy of $H_{3,2}$ and the remaining edges as single edges and we are done. If $13 \le e(G) \le 15$ we can see by inspection that the theorem holds.

Assume that it is true for all graphs of order less than n and note that for any positive integer n

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .$$

Let G be a graph of order n, where $n \ge 7$, and let v be a vertex of minimum degree, say $\deg_G(v) = d + m$ where $d := \lfloor \frac{n}{2} \rfloor$ and m is an integer.

If $m < 0$ or if $m = 0$ and there exists a copy of H in G containing at least two edges incident with v, then the edges incident with v and some other edges of G can be decomposed with at most $d-1$ edge disjoint copies of H and single edges, so the induction hypothesis implies the result.

Let $m = 0$ and suppose that there is no copy of H containing at least two edges incident with v. By induction either $G - v = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ or we have $n = 6, H = H_{3,2}$ and $G = \mathcal{H}$. Suppose the latter happens. Since $\deg v = 3$ it follows that at least 2 edges incident with v will also be adjacent with 2 vertices of $G - v$ that form a triangle in $G - v$. This creates in G a copy of H that has at least 2 edges incident with v, which contradicts our assumption. Hence $G - v = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Suppose v is adjacent to vertices in both parts of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Then, we can easily find a copy of H containing at least two edges incident with v. Therefore, v can only have neighbors in one part of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ and since $\deg v = d$ it follows that $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ as required.

To complete the proof we shall show that if $m \ge 1$ then the edges incident with v and some other edges of G can be decomposed with at most $d - 1$ edge disjoint copies of H and single edges. In this case such graph G cannot be extremal.

Let G_v be the subgraph spanned by the vertices adjacent to v. Then G_v has $d+m$ vertices and each vertex has degree at least $d+m-(n-d-m) = 2d + 2m - n \ge 2m - 1$, since v was chosen to have minimum degree.

Then by Lemma ?? G_v contains a path of length $2m-1$, say P. Observe that every three vertices of P give rise to one copy of H containing three edges incident with v. Therefore, it suffices to show that

$$\left\lfloor \frac{2m}{3} \right\rfloor + \left(d + m - 3 \left\lfloor \frac{2m}{3} \right\rfloor \right) \le d - 1, \tag{2.1}$$

which holds for all values of m, except for $m = 1, 2, 4$.

In what follows let $\overline{N}(v) := V(G) - (N(v) \cup \{v\})$, where $N(v)$ is the set of vertices adjacent to v and let P_k denote the path of length k.

Let $m = 1$. If G_v contains a path of length two then we can always find a copy of H containing three edges incident with v and we are done. If not, then G_v contains only independent edges. To prove our result it suffices to find two copies of H each having two edges incident with v. Observe that all vertices of G_v must be adjacent to all vertices in $\overline{N}(v)$ and since $n \ge 7$ the result follows easily.

Let $m = 2$. If $n = 7$ then G_v contains P_4 and $y \in \overline{N}(v)$ is adjacent to all vertices of G_v. Therefore, we can find two copies of H that contain all edges incident with v and we are done. Assume $n \ge 8$. In this case it suffices to find two copies of H each containing three edges incident with v. If G_v contains a path of length five we are done. Assume first that the longest path in G_v is P_4 and let $y \in G_v$ be a vertex not in P_4. Since $\deg_{G_v} y \ge 3$ and G_v contains no P_5 it follows that y must have at least two neighbors not in P_4. But then G_v contains a P_4 and a P_2 with disjoint vertex sets and we are done. Finally, assume that the longest path in G_v has length three. Then its vertices induce a K_4. Since G_v has at least six vertices then it must have at least two K_4's and we are done.

To complete the proof let $m = 4$. It suffices to find three copies of H containing at least eight edges incident with v. If $d = 4$ then $n = 9$ and thus $G = K_9$ and we are done. Assume $d \ge 5$ then G_v has order at least nine. If G_v contains a path of length eight we are done. Suppose that the longest path in G_v has length seven, but then its endpoints have to be adjacent to all vertices of the path. In this situation it is not hard too see that we can always find three copies of H containing at least eight edges incident with v and the proof is complete. □

Theorem 2.4. *Let $r \ge 4$ and let H be a fixed clique-extension of order $r+1$. Then, every graph of order $n \ge r+1$ can be decomposed into at most $t_{r-1}(n)$ edge disjoint copies of H and single edges. Moreover, $T_{r-1}(n)$ is the only graph that cannot be decomposed with fewer edge disjoint copies of H and single edges, except for $n = r+1$ and $H = H_{r,r-1}$ when the graph $H_{r,r-2}$ is also extremal.*

Proof. First observe that H must have a vertex of degree r, say v. Furthermore, since $H \ne K_{r+1}$ we must have $H - v \subseteq K_r^-$. Recall that K_r^- denotes the complete graph on r vertices minus one edge.

We now proceed by induction on n.

Let G be a graph of order $r+1$. If $e(G) < t_{r-1}(r+1)$ then it suffices to decompose G into single edges. Assume that $e(G) \geq t_{r-1}(r+1)+1$. Then G is either K_{r+1}^- or K_{r+1}. Therefore, G contains a copy of H and thus it can be decomposed into at most $t_{r-1}(r+1)-1$ parts. Finally, if $e(G) = t_{r-1}(r+1)$ then $\overline{G}$ has two edges. If the edges of $\overline{G}$ are adjacent then $G = H_{r,r-2}$, otherwise $G = T_{r-1}(r+1)$. So the theorem holds for $n = r+1$.

Let G be a graph of order $n \geq r+2$ and let v be a vertex of minimum degree, say $\deg_G(v) = d+m$ where $d := t_{r-1}(n) - t_{r-1}(n-1) = \left\lfloor \frac{r-2}{r-1} n \right\rfloor$ is the minimum degree in $T_{r-1}(n)$ and m is an integer.

If $m < 0$ or if $m = 0$ and there exists a copy of H in G containing at least two edges incident with v, then the edges incident with v and some other edges of G can be decomposed with at most $d-1$ edge disjoint copies of H and single edges, so the induction hypothesis shows that the result holds and that G cannot be extremal.

Let $m = 0$ and suppose that v is not contained in any copy of H that has at least two edges incident with v.

By induction either $G-v = T_{r-1}(n-1)$ or we have $n = r+2, H = H_{r,r-1}$ and $G - v = H_{r,r-2}$. If the latter happens then we can find a copy of H in G containing at least two edges incident with v, which contradicts our assumption. Thus $G - v = T_{r-1}(n-1)$. Suppose that $G \neq T_{r-1}(n)$, then G must contain a K_r. Furthermore, v must be a vertex of K_r since $G - v = T_{r-1}(n-1)$ does not contain any copy of K_r. Moreover, $\deg v = \lfloor (r-2)n/(r-1) \rfloor \geq r$. Then v must have a neighbor not in K_r, say x. But then, by definition of $T_{r-1}(n-1)$, the vertex x must be adjacent to $r-2$ vertices of $K_r - v$. So v belongs to a copy of H, which is a contradiction. Therefore $G = T_{r-1}(n)$.

To complete the proof we shall show that if $m \geq 1$ then the edges incident with v and some other edges of G can be decomposed with at most $d-1$ edge disjoint copies of H and single edges.

Let G_v be the subgraph spanned by the vertices adjacent to v. Then G_v has $d+m$ vertices and each vertex of G_v has degree at least $d+m-(n-d-m) = 2d+2m-n =: f$, since v was chosen to have minimum degree.

Let h denote the maximum number of independent (i.e. vertex disjoint) K_r^-'s in G_v. Thus we can find h edge disjoint copies of H containing vertex v. Furthermore, we can ensure that each copy of H contains r edges incident with v. Therefore, the edges incident with v can be decomposed into at most

$$h + d + m - rh = h(1-r) + d + m \leq d - 1$$

copies of H and single edges, provided that $h \geq \frac{m+1}{r-1}$. So, to complete the proof it suffices to show that

$$h \geq \frac{m+1}{r-1}. \tag{2.2}$$

Let F be the subgraph obtained from G_v by omitting h independent K_r^-'s (vertices and edges) and all incident edges. Then $F \not\supseteq K_r^-$, F has $d+m-rh$ vertices and each vertex has degree at least $f-rh = 2d+2m-n-rh$. Consequently,

$$\frac{1}{2}\delta(F)v(F) \leq e(F) \leq \text{ex}(v(F); K_r^-).$$

We now need to prove the following claim.

Claim 1. *Let d, m and G_v be as before. Then G_v contains a copy of $H-v$ for all $m \geq 1$.*

Proof of Claim ??. It suffices to prove that G_v contains a copy of K_r^-. We have,

$$e(G_v) \geq \frac{1}{2}(d+m)(2d+2m-n).$$

Since $d+m \geq r+1$, Lemma ?? implies that

$$\text{ex}(d+m; K_r^-) = \text{ex}(d+m; K_{r-1}) \leq \frac{1}{2}\frac{r-3}{r-2}(d+m)^2.$$

Easy calculations and (??) imply that

$$\frac{1}{2}(d+m)(2d+2m-n) > \frac{1}{2}\frac{r-3}{r-2}(d+m)^2$$

and thus $e(G_v) > \text{ex}(d+m; K_r^-)$, so G contains a copy of K_r^- as required. □

We are now able to complete the proof of the theorem.

Case 1: $v(F) \geq r+1$.
Using Lemma ?? we have

$$\frac{1}{2}\delta(F)v(F) \leq e(F) \leq \text{ex}(v(F); K_{r-1}) \leq \frac{1}{2}[v(F)]^2\frac{r-3}{r-2}$$

that is

$$rh \geq (r-1)d + (r-1)m - (r-2)n.$$

As

$$(r-1)d \geq (r-2)n - (r-2), \tag{2.3}$$

this implies

$$rh \geq (r-1)m - (r-2).$$

For $m \geq 2$ we have $(r-1)m-(r-2) \geq \frac{r(m+1)}{r-1}$, thus inequality (??) holds. Let $m = 1$. Then, Claim ?? implies that G_v contains a copy of $H - v$ and we are done.

Case 2: $v(F) \leq r$.

By definition of F, we know that $v(F) = v(G_v) - rh = d + m - rh$, hence $rh \geq d + m - r$. Recall that it suffices to show that

$$d + m - r \geq \frac{r(m+1)}{r-1}$$

that is

$$(r-1)d - r^2 \geq m. \tag{2.4}$$

Let $q(r-1)+1 \leq n \leq (q+1)(r-1)$ where $q \geq 1$ is an integer. Then $d = n - q - 1$ and $m \leq q$, hence to prove (??) it suffices to show that $(r-1)(n-q-1) - r^2 \geq q$. Since $n \geq q(r-1)+1$ it follows that to prove the latter inequality it is enough to show that $q(r^2-3r+1)-r^2 \geq 0$ which holds for $q \geq 4$ and $r \geq 4$.

Let $q \in \{1,2,3\}$. By Claim ?? we know that G_v contains a copy of $H-v$, so the edges incident with v can be decomposed into at most $1+d+m-r$ edge disjoint copies of H and single edges. If $m \leq 2$ then $1+d+m-r \leq d-1$ and we are done. Suppose $m = 3$ then $q = 3$ and then $G = K_n$. Since $n \geq 3r-2$ we can easily find two copies of H containing v and the proof is complete. □

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References

[1] B. Bollobás. On complete subgraphs of different orders. *Math. Proc. Cambridge Philos. Soc.*, 79(1):19–24, 1976.

[2] B. Bollobás. *Modern Graph Theory.* Springer–Verlag, 2002.

[3] R. Diestel. *Graph Theory.* Springer–Verlag, 2nd edition, 2000.

[4] P. Erdös, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canad. J. Math.*, 18:106–112, 1966.

[5] F. Harary. *Graph theory.* Addison-Wesley, 1972.

[6] T. Sousa. Decompositions of graphs into 5-cycles and other small graphs. *Electron. J. Combin.*, 12:Research Paper 49, 7 pp. (electronic), 2005.