Acyclic edge coloring of triangle-free toroidal graphs*

Yian Xu[†]
School of Mathematical Sciences, Nanjing Normal University
1 Wenyuan Road, Nanjing, 210046, China

Abstract

A proper edge coloring c of a graph G is said to be acyclic if G has no bicolored cycle with respect to c. It is proved that every triangle-free toroidal graph G admits an acyclic edge coloring with $(\Delta(G) + 5)$ colors. This generalizes a theorem from [8].

Key Words: acyclic edge coloring; maximum degree; toroidal graphs.
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1 Introduction

We only consider finite and simple graphs. Undefined signs are all from [6].

Let G be a graph. If G can be drawn on the torus such that any two edges intersect only at their common endvertex, then G is called a *toroidal graph.* Let G be a toroidal graph. For convenience, we use G to denote an embedding of G on the torus, and use $V(G)$, $E(G)$ and $F(G)$ to denote the sets of vertices, edges and faces of G, respectively. For a face f of G, we use $d_G(f)$ ($d(f)$ for short in case without confusion) to denote the degree of f that is defined to be the length of the closed walk bounding f.

Let G be a graph, k be a positive integer, and let c be a proper k-edge coloring of G. If G contains no bicolored cycle with respect to c, then c is called an *acyclic* k-edge coloring of G. G is said to be *acyclically k-edge colorable* if it admits an acyclic k-edge coloring. The *acyclic chromatic index* of G, denoted by $\alpha'(G)$, is defined to be the least integer k such that G is acyclically k-edge colorable.

The concept of acyclic edge coloring was first introduced by Fiamcik [9] in 1978. In 1991, Alon et al. [1] proved that $\alpha'(G) \leq 64\Delta(G)$ for

*Supported by the Innovation Fund. of NJNU.
[†]Email: yian1990@gmail.com.

all graphs. Molloy and Reed [12] improved this upper bound to $\alpha'(G) \le 16\Delta(G)$ in 1998. In 2005, Muthu, Narayanan and Subramanian [13] proved that $\alpha'(G) \le 4.52\Delta(G)$ for graphs with girth at least 220, where the *girth* of a graph is the length of a shortest cycle in it. Subramanian [18] presented a simple greedy heuristic algorithm for acyclically $(5\Delta(\log\Delta + 2))$-edge coloring a graph of maximum degree Δ.

In 2001, Alon, Sudakov and Zaks [2] proved that $\alpha'(G) \le \Delta(G) + 2$ for almost all regular graphs, and for all graphs whose girth are at least $c\Delta(G)\log\Delta(G)$ for some constant c. In 2005, Nešetřil and Wormald [16] improved the former result to $\alpha'(G) \le \Delta(G) + 1$ for almost all regular graphs. The long-standing conjecture [2, 9] that $\alpha'(G) \le \Delta(G) + 2$ for all graphs G is still open. Some known results verify this conjecture on restricted families of graphs including connected graphs with $\Delta(G) \le 4$ and $|E(G)| \le 2|V(G)| - 1$ [4], complete bipartite graphs $K_{p,p}$ with p being an odd prime [5], grid-like graphs [14], outerplanar graphs [15], and subcubic graphs [17]. It is also proved that each non-regular subcubic graph admits an acyclic 4-edge coloring [3].

There are quite a few papers (see [7, 8, 10, 11, 19, 20]) studying the acyclic edge colorability of graphs embedded into some surface. In [8], the authors improved a theorem of [10] and proved that every triangle-free planar graph admits an acyclic $(\Delta(G) + 5)$-edge coloring.

In this paper, we extend the above result to toroidal graphs, and show that every triangle-free toroidal graph admits an acyclic $(\Delta(G) + 5)$-edge coloring.

Theorem 1 *Let G be a triangle-free toroidal graph. Then,* $\alpha'(G) \le \Delta(G) + 5$.

Let G and H be two graphs. The *Cartesian product* of G and H, denoted by $G \diamond H$, is defined to be the graph with vertex set $V(G) \times V(H)$ and edge set consisting of the edges joining (u_1, u_2) and (v_1, v_2) iff either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$. The following theorem will be used in the proof of Theorem 1.

Theorem 2 ([*14*]) *Let G and H be two connected graphs of order at least 2 such that* $\max\{\alpha'(G), \alpha'(H)\} \ge 2$. *Then,*

$$\alpha'(G \diamond H) \le \alpha'(G) + \alpha'(H).$$

A *k-vertex* is a vertex of degree k, a *k^--vertex* is a vertex of degree at most k, and a *k^+-vertex* is defined similarly.

2 Proof of Theorem 1

To prove Theorem 1, we first need to extend a technical lemma from [10]. In [10], the authors presented some configurations that may appear in a graph G with $\delta(G) \geq 2$ and $|E(G)| \leq 2|V(G)| - 1$.

Lemma 1 ([*10*]) *Let G be a graph such that $|E(G)| \leq 2|V(G)| - 1$ and $\delta(G) \geq 2$. Then, G contains one of the following configurations.*

(C_1) *a 2-vertex adjacent to a 5^--vertex;*

(C_2) *a 3-vertex adjacent to at least two 5^--vertices;*

(C_3) *a 6-vertex adjacent to at least five 3^--vertices;*

(C_4) *a 7-vertex adjacent to seven 3^--vertices;*

(C_5) *a vertex x such that at least $d(x) - 3$ of its neighbors are 3^--vertices, and moreover one of them is a 2-vertex.*

Before stating our lemma, we need to define a configuration, called $\mathcal{B}$-figure. A $\mathcal{B}$-figure in a toroidal graphs consists of two adjacent 4-faces f_1 and f_2 such that (1) f_1 and f_2 share an edge joining to 3-vertices, and (2) all the other four vertices are 7-vertices (see Figure 1).

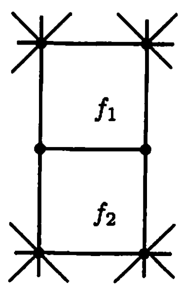

Figure 1: A $\mathcal{B}$-figure

With the same method, we show that for a triangle-free toroidal graph G with $\delta(G) \geq 2$, if G is not 4-regular, and contains non of the configurations $(C_2) \sim (C_5)$, then it has either a 2-vertex adjacent to a 6^--vertex, or a 3-vertex adjacent to a 6-vertex and a 4^--vertex, or a 7^+-vertex v adjacent to exact $(d(v) - 4)$ 2-vertices etc. To be precisely, we have

Lemma 2 *Let G be a triangle-free toroidal graph with $\delta(G) \geq 2$. If G is not 4-regular, and does not contain any of the configurations $(C_2) \sim (C_5)$ as listed in Lemma 1, then G contains one of the following configurations.*

(C'_1) *a 2-vertex adjacent to a* 6^-*-vertex;*

(C'_2) *a 3-vertex adjacent to a 6-vertex and a* 4^-*-vertex;*

(C'_3) *a $\mathcal{B}$-figure;*

(C'_4) *a* 7^+*-vertex* v *adjacent to exact* $(d(v) - 4)$ *2-vertices;*

(C'_5) *a 7-vertex adjacent to six 3-vertices and a 4-vertex;*

(C'_6) *an 8-vertex adjacent to eight 3-vertices.*

Proof. Assume to the contrary that G does not contain any one of the configurations $(C_2) \sim (C_5)$ and $(C'_1) \sim (C'_6)$, and is not 4-regular.

By the Euler's Formula on toroidal graphs, $|V(G)|+|F(G)|-|E(G)| \geq 0$. For element $x \in V(G) \cup F(G)$, let $\omega(x) = d(x) - 4$. We see that

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = 4(|E(G)| - |V(G)| - |F(G)| \leq 0,$$

and hence $\delta(G) \leq 3$ as G is triangle-free but not 4-regular by our assumption.

We will redistribute the weight ω between vertices following a discharging rule below, and denote the resulting new weight as ω^*. Let x be a 6^+-vertex, and let y be a neighbor of x.

- x transfers 1 to y if $d(y) = 2$, and transfers $\frac{1}{2}$ to y if $d(y) = 3$.

Now, let us calculate the new weight ω^*. It is certain that $\omega^*(f) \geq 0$ for each face f, and

$$\omega^*(f) > 0 \text{ if } d(f) \geq 5. \tag{1}$$

Let v be a k-vertex.

If $k = 2$, then v is adjacent to two 7^+-vertices of which each sends 1 to v, and thus $\omega^*(v) = -2 + 2 = 0$.

If $k = 3$, then v is adjacent to at least two 6^+-vertices, as G contains no (C_2), of which each sends $\frac{1}{2}$ to v, and so $\omega^*(v) = -1 + 2 \cdot \frac{1}{2} = 0$.

If $k = 4$ or 5, then $\omega^*(v) = \omega(v) \geq 0$, and

$$\omega^*(v) > 0 \text{ whenever } k = 5. \tag{2}$$

If $k = 6$, then v is not adjacent to 2-vertices as G contains no (C'_1), and is adjacent to at most four 3-vertices as G contains no (C_3). By the discharging rule, v totally sends out at most $4 \cdot \frac{1}{2}$ to its neighbors, and thus $\omega^*(v) \geq \omega(v) - 4 \cdot \frac{1}{2} = 0$.

Suppose that $k \geq 7$. If v is not adjacent to 2-vertex, then v is adjacent to at most $(k-1)$ 3^--vertices whenever $k \leq 8$ as G contains neither (C_4) nor (C_6'). Hence, $\omega^*(v) \geq \omega(v) - 6 \cdot \frac{1}{2} = 0$ whenever $k = 7$,

$$\omega^*(v) \geq \omega(v) - 7 \cdot \frac{1}{2} > 0 \text{ whenenver } k = 8, \tag{3}$$

and

$$\omega^*(v) \geq \omega(v) - k \cdot \frac{1}{2} = \frac{k-8}{2} > 0 \text{ whenenver } k \geq 9, \tag{4}$$

If $k \geq 7$ and v is adjacent to a 2-vertex, then v is adjacent to at most $(d(v)-4)$ 3^--vertices as G contains no (C_5), and furthermore if v is adjacent to exact $(d(v)-4)$ 3^--vertices then at least one of the 3^--vertices is a 3-vertex as G contains no (C_4'). By the discharging rule, v totally transfers out at most $(d(v) - \frac{9}{2})$ to its neighbors, and thus

$$\omega^*(v) \geq \omega(v) - (d(v)-5) - \frac{1}{2} > 0 \text{ if } k \geq 7 \text{ and } v \text{ is adjacent to a 2-vertex.} \tag{5}$$

We have shown that $\omega^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. Note that the discharging procedure does not change the total sum of the weights.

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(v) \leq 0.$$

It follows that

$$\omega^*(x) = 0 \text{ for each element } x \in V(G) \cup F(G). \tag{6}$$

Therefore, G has no 5^+-face by (1), has no 5-vertex as $\omega^*(u) > 0$ for each 5-vertex u by (2), $\Delta(G) \leq 7$ by (3), (4) and (5), and each 7-vertex is adjacent to six 3-vertices.

If $\delta(G) = 2$, let v be a 2-vertex, and let u be a neighbor of v, then $d(u) \geq 7$ and thus $\omega^*(u) > 0$ by (5). Therefore, $\delta(G) = 3$.

Let v be a 6-vertex. Since $\omega^*(v) = 0$ by (6), v must be adjacent to four 3-vertices. Let x be a 3-vertex adjacent to v. Then, x is adjacent to three 6^+-vertex as G contains neither 5-vertex nor configuration (C_2'), and thus $\omega^*(v) = -1 + \frac{3}{2} > 0$, a contradiction to (6). Therefore, G contains no 6-vertex, and thus every vertex in G has degree 3, 4, or 7, and each 7-vertex is adjacent to six 3-vertices and a 7-vertex as G contains neither (C_4) nor (C_5').

By (1), each face of G is a 4-face. Let v be a 7-vertex, let u be a 7-vertex adjacent to v, and let f be the 4-face bounded by $uvwxu$. If $d(w) = 7$, then v is adjacent to two 7-vertices implying $\omega^*(v) > 0$, contradicting (6). Therefore, both w and x are 3-vertices by symmetry. Let f' be the other face incident with wx bounded by $wxyzw$. Since G contains no (C_2), each of y and z is a 7-vertex, and hence (C_3') occurs in G. This contradiction completes the proof of our lemma. ∎

Now we are ready to prove Theorem 1. The approach is the same as that of [8] and [10]. Assume to the contrary that the theorem does not hold. We choose G to be a counterexample with minimum number of vertices. Suppose that $k = \Delta(G) + 5$. Let H be a proper subgraph of G. By the minimality of G, H is acyclic k-edge colorable. Let c be an acyclic k-edge coloring of H. We use $C(v)$ to denote the set of the colors appearing on the edges incident with vertex v.

It is certain that G is connected, and $\Delta(G) \geq 3$. If G has a cut edge, say uv, let G_1 and G_2 be the two components of $G - uv$. Then, both G_1 and G_2 are acyclically k-edge colorable. By rearranging the colors appeared at u and v, we can always choose a color for uv and get an acyclic k-edge coloring of G. Therefore, G is 2-edge connected, and hence $\delta(G) \geq 2$.

By the Euler's Formula on toroidal graphs, $\sum_{x \in V(G) \cup F(G)} (d(x) - 4) \leq 0$.

If G is 4-regular, then each face is of degree 4 as G is triangle-free, and hence G is isomorphic to $C_r \diamond C_s$ for some integers $r, s \geq 4$. By Theorem 2, $\alpha'(G) \leq 6 < k$ as each cycle has acyclic chromatic index 3. Therefore, G is not 4-regular.

If G contains one of the configurations $(C_2) \sim (C_5)$ listed in Lemma 1, then $\alpha'(G) \leq k$ as proved in [8]. So, we assume that G does not contain any configuration of $(C_2) \sim (C_5)$. By Lemma 2, G contains one of the configurations $(C_1') \sim (C_6')$. Our proof is divided into six cases according to the configuration contained in G. Let $L = \{1, 2, \cdots, k\}$ be the color set.

Case 1. G contains (C_1').

Let v be a 2-vertex with neighbors u and w. Suppose that $d(u) \leq 6$. By the minimality of G, $H = G - vw$ admits an acyclic k-edge coloring c. If $c(vu) \notin C(w)$, we can always choose a color in $L \setminus (C(w) \cup \{c(vu)\})$ for vw to get an acyclic k-edge coloring of G. Otherwise, we suppose that $c(vu) \in C(w)$, then $|C(u) \cup C(w)| \leq \Delta(G) - 1 + 5 = k - 1$, and we can also choose a color in $L \setminus (C(u) \cup C(w))$ for vw to get an acyclic k-edge coloring of G.

Case 2. G contains (C_2').

Let v be a 3-vertex with neighbors u_0, u_1 and u_2. Suppose that $d(u_0) \leq 4$ and $d(u_2) = 6$. Let $H = G - vu_0$, and let c be an acyclic k-edge coloring of H with minimum $|C(v) \cap C(u_0)|$. Suppose that $c(vu_i) = i$ for $i = 1, 2$.

If $C(v) \cap C(u_0) = \emptyset$, then we can choose a color in $L \setminus (C(u_0) \cup \{1, 2\})$ for vu_0 to obtain an acyclic k-edge coloring of G. Therefore, we assume that $C(v) \cap C(u_0) \neq \emptyset$.

First assume that $|C(v) \cap C(u_0)| = 1$. If $i \in C(u_0)$, then $|C(u_i) \cup C(u_0)| \leq \Delta(G) + 2 < k - 1$, and we can choose a color in $L \setminus (C(u_1) \cup$

$C(u_2) \cup C(u_0))$ for vu_0 to get an acyclic k-edge coloring of G, contradicting the choice of G.

Now, we assume that $|C(v) \cap C(u_0)| = 2$. Then, $\{1,2\} = C(v) \subseteq C(u_0)$.

If $i \notin C(u_{3-i})$ for some $i \in \{1,2\}$, note that $|C(u_0) \cup C(u_{3-i})| \leq \Delta + 2$, we may choose a color in $L \setminus (C(u_0) \cup C(u_{3-i}))$ to recolor vu_{3-i} and get an acyclic k-edge coloring c' of H with $|C'(v) \cap C'(u_0)| = 1$, contradicting the choice of c.

So, we suppose that $1 \in C(u_2)$ and $2 \in C(u_1)$, and hence $C(v) = \{1,2\} \subseteq C(u_0) \cap C(u_1) \cap C(u_2)$. If $C(u_1) \cap C(u_2) \neq \{1,2\}$, then $|C(u_0) \cup C(u_1) \cup C(u_2)| \leq \Delta + (6-3) + 1 = k - 1$, we can choose a color in $L \setminus (C(u_0) \cup C(u_1) \cup C(u_2))$ for vu_0 and get an acyclic k-edge coloring of G. The similar occurs whenever $C(u_0) \cap C(u_1) \neq \{1,2\}$ or $C(u_0) \cap C(u_2) \neq \{1,2\}$. Therefore,

$$\{1,2\} = C(u_0) \cap C(u_1) = C(u_0) \cap C(u_2) = C(u_1) \cap C(u_2).$$

Without loss of generality, we may assume that $C(u_0) = \{1,2,3\}$, $C(u_2) = \{1,2,4,5,6,7\}$ and $C(u_1) = \{1,2,8,\ldots,k\}$. Then, we recolor vu_1 with color 3, recolor vu_2 with color 8, and get an acyclic k-edge coloring c'' of H with $|C''(v) \cap C''(u_0)| = 1$, contradicting the choice of c. This completes the proof of Case 2.

Case 3. G contains (C_3').

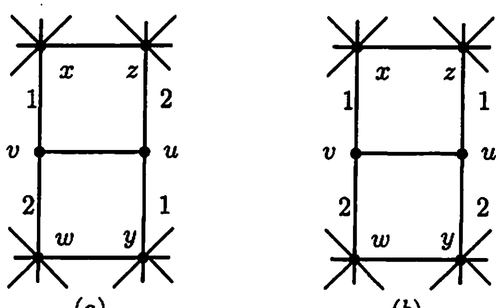

Figure 2: $\mathcal{B}$-figure

Let u and v be two adjacent 3-vertices in the $\mathcal{B}$-figure. Suppose that $N(v) = \{u, w, x\}$, and $N(u) = \{v, y, z\}$ (see Figure 2).

Let $H = G - vu$, and let c be an acyclic k-edge coloring of H with minimum $|C(v) \cap C(u)|$. Suppose that $c(vx) = 1$ and $c(vw) = 2$. The situations that $|C(v) \cap C(u)| \leq 1$ can be dealt with the same arguments on v as that used in Case 2. So, we suppose that $|C(v) \cap C(u)| = 2$, i.e., $C(u) = C(v) = \{1,2\}$.

If $1 \notin C(w)$, we choose a color in $L \setminus (\{1\} \cup C(w))$ to recolor vw to produce an acyclic k-edge coloring c' of H with $|C'(v) \cap C'(u)| = 1$. If

$2 \notin C(x)$, we choose a color in $L \setminus (\{2\} \cup C(x))$ to recolor vx to produce an acyclic k-edge coloring c'' of H with $|C''(v) \cap C''(u)| = 1$. Both contradict the choice of c.

Suppose that $1 \in C(w)$ and $2 \in C(x)$, and hence $C(u) = C(v) = \{1,2\} \subseteq C(w) \cap C(x)$. If $C(w) \cap C(x) \neq \{1,2\}$, then $|C(u) \cup C(w) \cup C(x)| \leq \Delta + (7-3) = k-1$, we choose a color in $L \setminus C(u) \cup C(w) \cup C(x)$ for vu and get an acyclic k-edge coloring of G. Therefore,

$$\{1,2\} = C(u) \cap C(w) = C(u) \cap C(x) = C(w) \cap C(x).$$

By symmetry, we may also suppose that

$$\{1,2\} = C(v) \cap C(y) = C(u) \cap C(z) = C(y) \cap C(z).$$

Without loss of generality, we may assume that $C(w) = \{1,2,3,4,5,6,7\}$ and $C(x) = \{1,2,8,\ldots,12\}$.

First, we consider the situation that $c(uy) = 1$ and $c(uz) = 2$ (see Figure 2(a)). Then, at least one color, say $\beta \in \{8,9,10,11,12\}$ does not appear in $C(y)$. By coloring uv with β, we get an acyclic k-coloring of G.

So, we suppose that $c(uy) = 2$ and $c(uz) = 1$ (see Figure 2(b)). If $C(w) \setminus C(y) \neq \emptyset$, we choose a color in $C(w) \setminus C(y)$ for uv and get an acyclic k-edge coloring of G. So, we suppose, by symmetry, that $C(w) = C(y)$ and $C(x) = C(z)$. Since c is an acyclic edge coloring of H, either $c(xz) \neq 2$ or $c(wy) \neq 1$ as otherwise $vxzuywv$ would be a cycle with color 1 and 2. By symmetry, we suppose that $c(xz) \neq 2$. Then, by recoloring vw with $c(xz)$, we get an acyclic k-edge coloring c''' of H with $|C'''(v) \cap C'''(u)| = 1$, contradicting the choice of c. This completes the proof of Case 3.

Case 4. G contains (C_4').

Let v be an l-vertex ($l \geq 7$). Suppose that $N(v) = \{u_1, u_2, \ldots, u_l\}$, and suppose that $u_1, u_2, \ldots, u_{l-4}$ are all 2-vertices. Let w_i be the other neighbor of u_i for $i = 1, 2, \ldots, l-4$, and let $H = G - \cup_{i=1}^{l-4}\{vu_i\}$. By the minimality of G, H admits an acyclic k-edge coloring c.

Let $L' = \{c(vu_{l-3}), c(vu_{l-2}), c(vu_{l-1}), c(vu_l)\}$. Let $S = \{c(u_1w_1), c(u_2w_2), \ldots, c(u_{l-4}w_{l-4})\}$, and let $s = |S|$. Since $k = \Delta(G) + 5$, we may assume that $S \cap L' = \emptyset$. Without loss of generality, we suppose that $c(u_iw_i) = i$ for $i \in \{1, 2, \ldots, s\}$, and $L' = \{k-3, k-2, k-1, k\}$. Then, we color vu_i with color $i+1$ for $i \in \{1, 2, \ldots, s-1\}$, color vu_s with $k-4$, and color $vu_{s+1}, \ldots, vu_{l-4}$ sequentially with colors from $\{s+1, s+2, \ldots, \Delta(G)\}$. This yields an acyclic k-edge coloring of G.

Case 5. G contains (C_5').

Let v be a 7-vertex adjacent to a 4-vertex u_1, and six 3-vertices $u_2, u_3, \ldots, u_7$.

Let $H = G - vu_7$, and let c be an acyclic k-edge coloring of H. Suppose, without loss of generality, that $c(vu_i) = i$ for $i = 1, 2, \ldots, 6$.

If $|C(u_7) \cap C(v)| = 0$, we color vu_7 with a color in $L \setminus (C(v) \cup C(u_7)$ (since $|(C(v) \cup C(u_7)| \leq \Delta(G) + 1$, this color always exists), and get an acyclic k-edge coloring of G.

Suppose that $|C(u_7) \cap C(v)| = 1$, and suppose $i \in C(u_7)$. Note that $|C(v) \cup C(u_7) \cup C(u_i)| \leq 10 < k$ as $\Delta(G) \geq 7$. We can always color vu_7 with a color in $L \setminus (C(v) \cup C(u_7) \cup C(u_i))$, and get an acyclic k-edge coloring of G.

So, we suppose that $|C(u_7) \cap C(v)| = 2$, and suppose that $C(u_7) = \{i, j\}$ for some $1 \leq i < j \leq 6$. Since $|C(v) \cup C(u_i) \cup C(u_j) \cup C(u_7)| = |C(v) \cup C(u_i) \cup C(u_j)| \leq 11$, there always exists a color $\beta \in L \setminus C(v) \cup C(u_i) \cup C(u_j) \cup C(u_7)$. By coloring vu_7 with β, we extend c to G. This proves Case 5.

Case 6. G contains (C_6').

This case is almost the same as above Case 5. Let v be an 8-vertex adjacent to eight 3-vertices $u_1, u_2, \ldots, u_8$.

Let $H = G - vu_8$, and let c be an acyclic k-edge coloring of H. Suppose, without loss of generality, that $c(vu_i) = i$ for $i = 1, 2, \ldots, 7$.

If $|C(u_8) \cap C(v)| = 0$, we color vu_8 with a color in $L \setminus (C(v) \cup C(u_8)$, and extend c to G.

If $|C(u_8) \cap C(v)| = 1$, suppose by symmetry that $C(u_8) = \{1, 8\}$. Since $|C(v) \cup C(u_1) \cup C(u_8)| \leq 10 < k$ as $\Delta(G) \geq 8$, we can color vu_8 with a color in $L \setminus (C(v) \cup C(u_1) \cup C(u_8))$, and extend c to G.

Suppose that $|C(u_8) \cap C(v)| = 2$, and suppose by symmetry that $C(u_8) = \{1, 2\}$. Since $|C(v) \cup C(u_1) \cup C(u_2) \cup C(u_8)| = |C(v) \cup C(u_1) \cup C(u_2)| \leq 11$, there always exists a color $\beta \in L \setminus C(v) \cup C(u_1) \cup C(u_2) \cup C(u_8)$. By coloring vu_8 with β, we extend c to G. This proves Case 6. ∎

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