Linear transformations preserving log-convexity *

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Abstract

In this paper we study linear transformations preserving log-convexity, when the triangular array satisfies some ordinary convolution. As applications, we show that the Stirling transformations of two kinds, the Lah transformation, the generalized Stirling transformation of the second kind and the Dowling transformations of two kinds preserve the log-convexity.

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1 Introduction

Let $x_0, x_1, x_2, \ldots$ be a sequence of nonnegative numbers and with no internal zeros. By the latter we mean that there are no three indices $i < j < k$ such that $x_i, x_k \neq 0$ and $x_j = 0$. We say that the sequence is log-convex (LCX) if $x_i^2 \leq x_{i-1}x_{i+1}$ for all $i > 0$ and is log-concave (LC) if $x_i^2 \geq x_{i-1}x_{i+1}$ for all $i > 0$. Although the log-convexity of a sequence of positive numbers is formally equivalent to the log-concavity of its reciprocal sequence, we would have a hard time proving the log-convexity by the log-concavity of its reciprocal. One possible reason for this is that the sequence satisfies nice recurrence relations since its strong background in combinatorics, but the reciprocal sequence does not. Many famous sequences in combinatorics, including the Bell numbers, the Catalan numbers and

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the Motzkin numbers, are log-convex respectively [14], but log-convexity has not received nearly as much attention as log-concavity. For the log-concavity problems, there have been quite a few attempts (see the survey articles [5, 16] and some recent developments [18, 19, 20, 21]). However, there is only a little study of the log-convexity of sequences [14].

Let $\{a(n,k)\}_{0\le k\le n}$ be a triangular array of nonnegative numbers. Define two linear transformations by

$$z_n = \sum_{k=0}^{n} a(n,k)x_k, \quad n = 0,1,2,\ldots \tag{1.1}$$

and

$$z_n = \sum_{k=0}^{n} a(n,k)x_k y_{n-k}, \quad n = 0,1,2,\ldots \tag{1.2}$$

respectively. We say that the linear transformation (1.1) has the PLCX (resp. PLC) property if it preserves the log-convexity (resp. log-concavity) of sequences, i.e., the log-convexity (resp. log-concavity) of $\{x_n\}_{n\ge 0}$ implies that of $\{z_n\}_{n\ge 0}$. We say that the linear transformation (1.2) has double PLCX (resp. PLC) property if the log-convexity (resp. log-concavity) of $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ implies that of $\{z_n\}_{n\ge 0}$. The corresponding triangle $\{a(n,k)\}_{0\le k\le n}$ is also called PLCX and double PLCX (resp. PLC). Clearly, the double PLCX property implies the PLCX property (resp. PLC).

Given two triangles $\{a(n,k)\}_{0\le k\le n}$ and $\{b(n,k)\}_{0\le k\le n}$, define their ordinary convolution $\{T(n,k)\}_{0\le k\le n}$ by

$$T(n,k) = \sum_{j=k}^{n} a(n,j)b(j,k), \quad n = 0,1,2,\ldots.$$

The paper is devoted to the study of the PLCX property of linear transformations whose triangular array satisfies some ordinary convolution. There are many triangles satisfying this recurrence in combinatorics. For example, it is well known that the Stirling number of the second kind $S(n,k)$ satisfies the recurrence

$$S(n,k) = \sum_{j=k-1}^{n-1} \binom{n-1}{j} S(j,k-1)$$

(see [9] for instance). So far there have been found some important transformations that are PLC (see [4, 18, 19, 21] for instance). But linear transformations which preserve the log-convexity of sequences have not been attached importance to since H. Davenport and G. Pölya showed the following result.

Proposition 1.1. *[10] The binomial convolution*

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \quad n = 0, 1, 2, \ldots,$$

has the double PLCX property.

Only till recently, Liu and Wang [14] showed that linear transformations given by the triangular arrays of binomial coefficients and Stirling numbers of two kinds preserve the log-convexity respectively. And using a result of [14], Chen *et. al.* [6, 7, 8] obtained that the Narayana transformation, the Bessel transformation and the Narayana transformation of type B preserve the log-convexity.

In this paper, we present that the linear transformation

$$z_n = \sum_{k=0}^{n} T(n,k) x_k$$

has the PLCX property when the triangular array $\{T(n,k)\}_{0 \le k \le n}$ satisfies some ordinary convolution. As applications, we show that the Stirling transformations of two kinds, the Lah transformation, the generalized Stirling transformation of the second kind and the Dowling transformations of two kinds preserve the log-convexity.

2 Main Results

In this section, we give the main result of the PLCX property first.

Theorem 2.1. *Suppose that $\{a(n,k)\}_{0 \le k \le n}$ and $\{b(n,k)\}_{0 \le k \le n}$ are two triangles of nonnegative numbers. If linear transformations $t_n = \sum_{k=0}^{n} a(n, k) x_k$ and $s_n = \sum_{k=0}^{n} b(n,k) x_k$ preserve log-convexity, then so does*

$$z_n = \sum_{k=0}^{n} T(n,k) x_k, \quad n = 0, 1, 2, \ldots,$$

where $T(n,k) = \sum_{j=k}^{n} a(n,j) b(j,k)$.

Proof. Note that

$$z_n = \sum_{k=0}^{n} T(n,k) x_k = \sum_{k=0}^{n} x_k \sum_{j=k}^{n} a(n,j) b(j,k).$$

After changing the order of the summation, we have

$$z_n = \sum_{j=0}^{n} a(n,j) \left[\sum_{k=0}^{j} b(j,k) x_k \right].$$

Let $y_j = \sum_{k=0}^{j} b(j,k)x_k$ for $0 \le j \le n$. Then the sequence $y_0, y_1, \ldots, y_n$ is log-convex by the condition that the triangle $\{b(n,k)\}_{0\le k\le n}$ preserves the log-convexity, so is the sequence $z_0, z_1, \ldots, z_n$ by the PLCX property of the triangle $\{a(n,k)\}_{0\le k\le n}$. This completes the proof. □

When the triangle $\{T(n,k)\}_{0\le k\le n}$ is the ordinary convolution of itself and another triangle $\{a(n,k)\}_{0\le k\le n}$, Theorem 2.1 is particularly interesting and useful as we shall see in the next section.

Corollary 2.1. *Suppose that $\{a(n,k)\}_{0\le k\le n}$ is a triangle of nonnegative numbers with $a(n,n) = 1$. If the linear transformation $t_n = \sum_{k=0}^{n} a(n,k)x_k$ preserves log-convexity, then so does*

$$z_n = \sum_{k=0}^{n} T(n,k)x_k, \quad n = 0, 1, 2, \ldots,$$

where $T(n,k) = \sum_{j=k-1}^{n-1} a(n-1,j)T(j,k-1)$ and $T(n,0) = T(0,k) = 0$ unless $T(0,0) = 1$.

Proof. Let $\{x_k\}_{k\ge 0}$ be a log-convex sequence. We need to show that the sequence $\{z_n\}_{n\ge 0}$ is log-convex. We proceed by induction on n. Since

$$\begin{aligned} z_0 &= T(0,0)x_0 = x_0, \\ z_1 &= T(1,0)x_0 + T(1,1)x_1 = a(0,0)x_1 = x_1, \\ z_2 &= T(2,0)x_0 + T(2,1)x_1 + T(2,2)x_2 = a(1,0)x_1 + x_2, \end{aligned}$$

we find $z_1^2 \le z_0 z_2$ by the log-convexity of $\{x_k\}_{k\ge 0}$. Now assume that $n \ge 3$ and $z_0, z_1, \ldots, z_{n-1}$ is log-convex, i.e., $z_j = \sum_{k=0}^{j} T(j,k)x_k$ preserves the log-convexity for $0 \le j \le n-1$. Note that

$$T(n,k) = \sum_{j=k-1}^{n-1} a(n-1,j)T(j,k-1).$$

So we have

$$\begin{aligned} z_n &= \sum_{k=0}^{n} T(n,k)x_k = \sum_{k=0}^{n} x_k \sum_{j=k-1}^{n-1} a(n-1,j)T(j,k-1) \\ &= \sum_{j=0}^{n-1} a(n-1,j)\left[\sum_{k=0}^{j} T(j,k)x_{k+1}\right]. \end{aligned}$$

Hence the sequence $z_0, z_1, \ldots, z_{n-1}, z_n$ is log-convex by Theorem 2.1 and the induction hypothesis. This completes the proof. □

For the double PLCX property, we have the following results. Using the result that the log-convexity is preserved by componentwise product, they can be proved by the same technique used in the proof of Theorem 2.1 and Corollary 2.1. So we omit their proof for brevity.

Theorem 2.2. *Suppose that sequences $\{u_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 0}$ are log-convex. If the linear transformation $t_n = \sum_{k=0}^{n} a(n,k)x_k y_{n-k}$ preserves double log-convexity and $s_n = \sum_{k=0}^{n} b(n,k)x_k$ preserves log-convexity, then the linear transformation*

$$z_n = \sum_{k=0}^{n} T(n,k)x_k, \quad n = 0,1,2,\ldots,$$

where $T(n,k) = \sum_{j=k}^{n} a(n,j)b(j,k)u_j v_{n-j}$ and $T(n,0) = T(0,k) = 0$ unless $T(0,0) = 1$, also preserves log-convexity.

Theorem 2.3. *Suppose that the sequence $\{v_k\}_{k\geq 0}$ is log-convex. If the linear transformation $t_n = \sum_{k=0}^{n} a(n,k)x_k y_{n-k}$ preserves double log-convexity with $a(n,n) = 1$, then the linear transformation*

$$z_n = \sum_{k=0}^{n} T(n,k)x_k, \quad n = 0,1,2,\ldots,$$

where $T(n,k) = \sum_{j=k-1}^{n-1} a(n-1,j)T(j,k-1)v_{n-j-1}$ and $T(n,0) = T(0,k) = 0$ unless $T(0,0) = 1$, also preserves log-convexity.

3 Applications

In this section we apply results obtained in the previous section to present that the Stirling transformations of two kinds, the Lah transformation, the generalized Stirling transformation of the second kind and the Dowling transformations of two kinds preserve the log-convexity.

3.1 The Stirling transformations of two kinds

Following Riordan [15], the Stirling number $s(n,k)$ of the first kind and the Stirling number $S(n,k)$ of the second kind are defined by relations

$$\begin{aligned}(x)_n &= \sum_{k=0}^{n} s(n,k)x^k, \\ x^n &= \sum_{k=0}^{n} S(n,k)(x)_k, \quad n = 0,1,2,\ldots,\end{aligned}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial. Let $c(n,k) = (-1)^{n+k}s(n,k)$ be the signless Stirling number of the first kind, i.e., the number of permutations of $[n]$ which contain exactly k permutation cycles. It is known that Stirling numbers of two kinds satisfy recurrences

$$c(n,k) = \sum_{j=k-1}^{n-1} \binom{n-1}{j} c(j,k-1)(n-j-1)!, \tag{3.1}$$

$$S(n,k) = \sum_{j=k-1}^{n-1} \binom{n-1}{j} S(j,k-1) \tag{3.2}$$

respectively, with $c(n,0) = c(0,k) = S(n,0) = S(0,k) = 0$, except $c(0,0) = S(0,0) = 1$ [9].

For the signless Stirling number of the first kind, we have the following result immediately from Theorem 2.3 and the recurrence (3.1).

Proposition 3.1. *[14] The Stirling transformation of the first kind* $z_n = \sum_{k=0}^{n} c(n,k)x_k$ *preserves log-convexity.*

For the Stirling number of the second kind, we have the following result immediately from Corollary 2.1 and the recurrence (3.2).

Proposition 3.2. *[14] The Stirling transformation of the second kind* $z_n = \sum_{k=0}^{n} S(n,k)x_k$ *preserves log-convexity.*

A combinatorial interpretation of the Stirling number $S(n,k)$ of the second kind is the number of partitions of the set $[n]$ having exactly k blocks. The number closely related to $S(n,k)$ is the Bell number, defined as the total partition number of $[n]$, i.e.,

$$B_n = \sum_{k=0}^{n} S(n,k).$$

The log-convexity of the Bell numbers follows immediately from Proposition 3.2, which is originally due to Engel [12]. On the other hand, the Bell numbers satisfy the recurrence

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

(see [17] for instance). So we can give another interpretation of the log-convexity of the Bell numbers using Proposition 1.1.

A more general number related to $S(n,k)$ is

$$F_n = \sum_{k=0}^{n} k!S(n,k),$$

which can be given a combinatorial interpretation analogous to the Bell numbers B_n: if X is an n-set, then F_n is the number of distinct ordered partition of X [17]. Since the factorial sequence $\{k!\}_{k\geq 0}$ is log-convex, we can get the following more general result.

Corollary 3.1. *The linear transformation* $z_n = \sum_{k=0}^{n} k!S(n,k)x_k$ *preserves log-convexity.*

Corollary 3.1 or Proposition 3.2 gives a natural interpretation to the log-convexity of the sequence $\{F_n\}_{n\geq 0}$.

3.2 The Lah transformation

The Lah numbers $L_{n,k}$ are coefficients expressing rising factorials in terms of falling factorials, i.e.,

$$(-x)_n = (-1)^n \langle x\rangle_n = \sum_{k=0}^{n} L_{n,k}(x)_k,$$

where $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$ is the rising factorial and $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial (see [15] for instance). Let $\ell_{n,k} = (-1)^n L_{n,k}$ be the unsigned Lah numbers, i.e., the number of partitions of the set $[n]$ into k nonempty linearly ordered blocks. It is known that the Lah numbers satisfy the recurrence

$$L_{n,k} = \sum_{j=k}^{n} (-1)^j s(n,j)S(j,k)$$

(see[9] for instance). So the unsigned Lah numbers satisfy

$$\ell_{n,k} = \sum_{j=k}^{n} c(n,j)S(j,k). \tag{3.3}$$

Thus the following corollary is an immediate consequence of Theorem 2.1 and Propositions 3.1, 3.2.

Corollary 3.2. *The Lah transformation* $z_n = \sum_{k=0}^{n} \ell_{n,k}x_k$ *preserves log-convexity.*

3.3 The generalized Stirling transformation of the second kind

Given a finite Coxeter group W, define the Eulerian polynomial of W by

$$P(W,x) = \sum_{\pi\in W} x^{d_W(\pi)},$$

where $d_W(\pi)$ is the number of W-descents of π. For Coxeter group of type B_n, Brenti [5] defined a q-analogue of $P(B_n, x)$, which reduces to the Eulerian polynomial $A_n(x)$ when $q = 0$ and to $P(B_n, x)$ when $q = 1$, by

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_B(\sigma)},$$

where $N(\pi) = |i \in [n] : \pi(i) < 0|$. Liu and Wang [13] showed that $B_n(x; q)$ has only real zeros for $q \geq 0$. Now we consider coefficients of $B_n(x; q)$ when expressed in powers of $(x - 1)$. Brenti showed that

$$B_n(x; q) = \sum_{k=0}^{n} k! S_B(n, k; q)(x - 1)^{n-k}.$$

And he also showed that the sequence $\{S_B(n, k; q)\}_{n \geq 0}$ satisfies the relation

$$S_B(n, k; q) = \sum_{i=k}^{n} \binom{n}{i} S(i, k) q^{n-i} (1 + q)^i. \tag{3.4}$$

In particular, if $q = 0$, then $S_B(n, k; q) = S(n, k)$, where $S(n, k)$ is the Stirling number of the second kind.

Using Propositions 1.1, 3.2 and Theorem 2.2, we can obtain the following result, which reduces to Proposition 3.2 when $q = 0$.

Corollary 3.3. *The linear transformation $z_n = \sum_{k=0}^{n} S_B(n, k; q) x_k$ preserves log-convexity for $q \geq 0$.*

3.4 The Dowling transformations of two kinds

The Dowling lattice $Q_n(G)$ is a geometric lattice of rank n over a finite group G of order m and has many remarkable properties (see [1, 2, 3, 11] for instance). When $m = 1$, that is, G is the trivial group, $Q_n(G)$ is the lattice $\prod_{n+1}$ of partitions of an $(n + 1)$-element set. So the Dowling lattices can be viewed as group-theoretic analogs of the partition lattices. Let $w_m(n, k)$ and $W_m(n, k)$ be the Whitney numbers of the first kind and the second kind respectively. Denote by $t_m(n, k) = (-1)^{n+k} w_m(n, k)$. It is known that the Whitney numbers of two kinds satisfy recurrences

$$t_m(n, k) = \sum_{i=k}^{n} c(n, i) \binom{i}{k} m^{n-i} = m^n \sum_{i=k}^{n} c(n, i) \binom{i}{k} m^{-i}, \tag{3.5}$$

$$W_m(n, k) = \sum_{i=k}^{n} \binom{n}{i} S(i, k) m^{i-k} = m^{-k} \sum_{i=k}^{n} \binom{n}{i} S(i, k) m^i \tag{3.6}$$

respectively (see [2] for instance).

Using Theorem 2.1, we can give a more direct approach to the PLCX properties of $t_m(n,k)$ and $W_m(n,k)$.

Corollary 3.4. *Suppose that the sequence $\{u_k\}_{k\geq 0}$ is log-convex. If linear transformations $t_n = \sum_{k=0}^{n} a(n,k)x_k$ and $s_n = \sum_{k=0}^{n} b(n,k)x_k$ preserve log-convexity, then so does*

$$z_n = \sum_{k=0}^{n} T(n,k)x_k, \quad n = 0, 1, 2, \ldots,$$

where $T(n,k) = \sum_{j=k}^{n} a(n,j)b(j,k)u_j$.

We next give another interpretation of the PLCX property of $W_m(n,k)$. It is also known that $W_m(n,k)$ satisfies the recurrence

$$W_m(n,k) = W_m(n-1,k) + \sum_{i=k-1}^{n-1} \binom{n-1}{i} W_m(i,k-1)m^{n-i-1} \tag{3.7}$$

(see [2] for instance). Note that the log-convexity is preserved under componentwise sum.

Lemma 3.1. *[14] If both $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are LCX, then so is the sequence $\{x_n + y_n\}_{n\geq 0}$.*

Proof. By the log-convexity of $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}(x_{n-1}+y_{n-1})(x_{n+1}+y_{n+1}) &= x_{n-1}x_{n+1} + (x_{n-1}y_{n+1} + x_{n+1}y_{n-1}) \\ &\quad + y_{n-1}y_{n+1} \\ &\geq x_n^2 + 2x_ny_n + y_n^2 = (x_n+y_n)^2.\end{aligned}$$

□

Thus we can also obtain the following result by induction and Theorem 2.3.

Proposition 3.3. *The Dowling transformation of the second kind $z_n = \sum_{k=0}^{n} W_m(n,k)x_k$ preserves log-convexity.*

Dowling [11] has given combinatorial interpretations for sequences

$$\{k!W_m(n,k)\}_{0\leq k\leq n} \quad \text{and} \quad \{k!m^kW_m(n,k)\}_{0\leq k\leq n}$$

for $m \geq 0$. From Proposition 3.3 and the log-convexity of sequences $\{k!\}_{k\geq 0}$ and $\{m^k\}_{k\geq 0}$, we have the following.

Corollary 3.5. *Linear transformations $z_n = \sum_{k=0}^{n} k!W_m(n,k)x_k$ and $z_n = \sum_{k=0}^{n} k!m^kW_m(n,k)x_k$ preserve log-convexity.*

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