On a well-spread halving of directed multigraphs

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Abstract

Let G be an even degree multigraph and let $\deg(v)$ and $\mu(uv, G)$ denote the degree of vertex v in G and the multiplicity of edge (u, v) respectively in G. A decomposition of G into multigraphs G_1 and G_2 is said to be a *well-spread halving* of G into two *halves* G_1 and G_2, if for each vertex v, $\deg(v, G_1) = \deg(v, G_2) = \frac{1}{2}\deg(v, G)$, and $|\mu(uv, G_1) - \mu(uv, G_2)| \leq 1$ for each edge $(u, v) \in E(G)$. A sufficient condition was given in [7] under which there exists a well-spread halving of G if we allow the addition/removal of a Hamilton cycle to/from G. Analogous to [7], in this paper we define a well-spread halving of a directed multigraph D and give a sufficient condition under which there exists a well-spread halving of D if we allow the addition/removal of a particular type of Hamilton cycle to/from D.

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1 Introduction

Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. We denote the degree of a vertex v in $V(G)$ by $\deg(v, G)$ and the maximum degree of G by $\Delta(G)$. The *multiplicity* of an edge $(u, v) \in E(G)$ is the number of edges joining u and v and is denoted by $\mu(uv, G)$. The maximum edge-multiplicity of G is denoted by $\mu(G)$. G is said to be *simple* if $\mu(G) = 1$. We will say that multigraph G is an *even degree* multigraph if $\deg(v, G)$ is even for all $v \in V(G)$. Let $\mathrm{red}(G)$ denote the simple graph obtained from G by deleting all edges of even multiplicity and replacing all edges of odd multiplicity by single edges. We refer the reader to ([2,3]) for all terminology and notation that is not defined in this paper.

Let G be an even degree multigraph. A decomposition of G into multigraphs G_1 and G_2 is said to be a *well-spread halving* of G into two *halves* G_1 and G_2, if for each vertex v, $\deg(v, G_1) = \deg(v, G_2) = \frac{1}{2}\deg(v, G)$, and, $|\mu(uv, G_1) - \mu(uv, G_2)| \leq 1$ for each edge $(u, v) \in E(G)$. The following theorem from [7] gives a simple necessary and sufficient condition for the existence of a well-spread halving of an even degree multigraph.

Theorem 1 *Let G be an even degree multigraph. There exists a well-spread halving of G if and only if* $\mathrm{red}(G)$ *has no components with an odd number of edges.*

If G is an even degree multigraph and if $\mathrm{red}(G)$ has some component with an odd number of edges then by Theorem 1, there does not exist a well-spread halving of G. However, the following theorem in [7] shows that if G is a Δ-regular multigraph on n vertices with n and Δ being even, then, generally there exists a well-spread halving of G if we allow the addition/removal of a Hamilton cycle to/from G. We denote by $K_n^{(r)}$ the complete multigraph on n vertices with r parallel edges between each pair of vertices.

Theorem 2 *Let G be a Δ-regular multigraph of even order n and maximum multiplicity $\mu(G) \leq \rho$ with ρ and Δ being even.*

(i) If $\Delta \geq \rho(\frac{n}{2} + 1)$, then G contains a Hamilton cycle H such that there exists a well-spread halving of $G - E(H)$.

(ii) If $\Delta \leq \rho(\frac{n}{2} - 2)$, then the complement of G relative to $K_n^{(\rho)}$ contains a Hamilton cycle H such that there exists a well-spread halving of $G \cup E(H)$.

Let D be a directed multigraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, the *indegree* (respectively, *outdegree*) of v in D denoted by $\mathrm{indegree}(v, D)$ (respectively, $\mathrm{outdegree}(v, D)$) is the number of arcs of D directed into v (respectively, directed out of v), and we denote by $\deg(v, D)$ the sum $\text{indegree}(v, D) + \text{outdegree}(v, D)$. Note that $\sum_{v \in V(D)} \text{indegree}(v, D) = \sum_{v \in V(D)} \text{outdegree}(v, D) = |A(D)|$. The *multiplicity* of an arc $(u, v) \in A(D)$ is the number of arcs directed from u to v and is denoted by $\mu(uv, D)$. The maximum arc-multiplicity of D is denoted by $\mu(D)$. D is said to be *simple* if $\mu(D) = 1$. The *simple directed graph* underlying a directed multigraph D, denoted by D_{simp}, is the directed graph obtained from D be replacing all arcs of multiplicity greater than one by single arcs. We will say that a directed multigraph D is an *even degree* directed multigraph if $\deg(v, D)$ is even for all $v \in V(D)$. An *anti-directed walk* (correspondingly, *path/circuit/Euler circuit/Hamilton cycle*) in a directed multigraph D is a walk (correspondingly, path/circuit/Euler circuit/Hamilton cycle) in the graph underlying D such that no pair of consecutive arcs in the walk (correspondingly, path/circuit/Euler circuit/Hamilton cycle) form a directed path in D in either direction. Note that an anti-directed circuit in D must have an even number of arcs. The *reduced directed graph* of D denoted by $\text{red}(D)$ is the simple directed graph obtained from D by deleting all arcs of even multiplicity and replacing all arcs of odd multiplicity by single arcs.

Let D be an even degree directed multigraph. A decomposition of D into directed multigraphs D_1 and D_2 is said to be a *well-spread halving* of D into two *halves* D_1 and D_2, if

(1) $\text{indegree}(v, D_1) = \text{indegree}(v, D_2)$, and, $\text{outdegree}(v, D_1) = \text{outdegree}$ (v, D_2), for each $v \in V(D)$, and,

(2) $|\mu(uv, D_1) - \mu(uv, D_2)| \le 1$, for each arc $uv \in D$.

In Section 2 we prove the following characterization of directed multigraphs that have a well-spread halving. This characterization of directed multigraphs that have a well-spread halving is simpler than the characterization (Theorem 1) of multigraphs that have a well-spread halving.

Theorem 3 *Let D be a directed multigraph. There exists a well-spread halving of D if and only if* $\text{indegree}(v)$ *and* $\text{outdegree}(v)$ *are even for each* $v \in V(D)$.

Let D be a directed multigraph, and let $V(D) = P \cup Q$ be a partition of $V(D)$. A (P, Q)-*directed Hamilton cycle* in D is a Hamilton cycle H in D such that for each $v \in P$, consecutive arcs of H incident on v do not form a directed path in D, and, for each $v \in Q$, consecutive arcs of H incident on v form a directed path in D. Hence, a usual directed Hamilton cycle in D is a $(\emptyset, V(D))$-directed Hamilton cycle. We note that if D is an even degree directed multigraph, then $V(D)$ partitions into $V(D) = P \cup Q$, where for each $v \in P$, indegree(v) is even and outdegree(v) is even, and, for each $v \in Q$, indegree(v) is odd and outdegree(v) is odd.

If D is an even degree directed multigraph and if the indegree/outdegree of some vertex is odd, then by Theorem 3, there does not exist a well-spread halving of D. However, analogous to Theorem 2 in [7], we prove the following theorem in Section 2 that shows that if D is an even degree directed multigraph of even order n, then, often there exists a well-spread halving of D if we allow the addition/removal of a particular (P, Q)-directed Hamilton cycle to/from D. We denote by $\vec{K}_n^{(r)}$ the complete directed multigraph on n vertices with r parallel arcs directed from each vertex to every other vertex.

Theorem 4 *Let D be an even degree directed multigraph of even order n and maximum multiplicity $\mu(D) \leq \rho$. Let $V(D) = P \cup Q$ be the partition of $V(D)$, where for each $v \in P$,* indegree(v) *is even and* outdegree(v) *is even, and, for each $v \in Q$,* indegree(v) *is odd and* outdegree(v) *is odd, and, suppose that $|P| = 2j$ for some integer $j \geq 0$.*

- *(i) If* indegree$(v, D) \geq \rho(\frac{n}{2}+j)$ *and* outdegree$(v, D) \geq \rho(\frac{n}{2}+j)$ *for each $v \in V(D)$, then D contains a (P, Q)-directed Hamilton cycle H such that there exists a well-spread halving of $D - A(H)$.*
- *(ii) If* indegree$(v, D) \leq \rho(\frac{n}{2}-j-1)$ *and* outdegree$(v, D) \leq \rho(\frac{n}{2}-j-1)$ *for each $v \in V(D)$, then the complement of D relative to $\vec{K}_n^{(\rho)}$ contains a (P, Q)-directed Hamilton cycle H such that there exists a well-spread halving of $D \cup A(H)$.*

We note that the degree conditions in Theorem 4 that we have been able to prove to be sufficient for the existence of a (P, Q)-directed Hamilton cycle H in a directed multigraph D that guarantees the existence of a well-spread halving of $G - A(H)$ depend on $|P|$ and are not as strong as those in Theorem 2. Our proof of Theorem 4 is based on a sufficient condition for the existence of a (P, Q)-directed Hamilton cycle in a directed graph D that we prove in Theorem 7 in Section 2. Note that in the special case when $V = P$, Theorem 4 follows trivially from Theorem 3. We note here that a $(V, \emptyset)$-directed cycle in a directed graph D is an anti-directed Hamilton cycle in D. Grant [5] and Haggkvist and Thomason [6] have given sufficient conditions for the existence of an anti-directed Hamilton cycle in a directed graph $D = (V, A)$ that are interesting in their own right.

We recall that the motivation behind studying well-spread halvings of multigraphs in [7] was to split a given set of games between teams into two halves such that multiple encounters between teams are well-spread over the two halves of the season. The model corresponds to a multigraph whose vertices are the teams and whose edges are the games between teams that need to be scheduled. If we wish to incorporate the notion of 'home' games and 'away' games we are naturally led to consider directed multigraphs where an arc uv might correspond to a game that team u plays with team v at the home field of team v. A well-spread halving of the directed multigraph in this model would correspond to a split of the given set of games between teams into two halves such that multiple 'home' encounters and multiple 'away' encounters are well-spread over the two halves of the season.

2 Proofs of Theorems 3 and 4

We first prove Theorem 3 stated in the Introduction.

Proof of Theorem 3. Clearly if D has a well-spread halving then indegree(v) and outdegree(v) must be even for each $v \in V(D)$. Now suppose that indegree(v) and outdegree(v) are even for each $v \in V(D)$. For each arc $uv \in A(D)$, we begin by including $\lfloor \frac{(\mu(uv,D))}{2} \rfloor$ parallel arcs directed from vertex u to vertex v in each of D_1 and D_2 and deleting these arcs from D. This leaves us with red(G). This pre-processing of D will guarantee that the decomposition of D into D_1 and D_2 that we produce satisfies condition (2) in the definition of a well-spread halving of D. Since indegree(v, D) and outdegree(v, D) are even for each vertex of $v \in V(D)$, we have that indegree(v, red(D)) and outdegree(v, red(D)) are also even for each vertex of $v \in V$(red(D)). Hence $|E(\text{red}(D))|$ must be even, and red(D) contains an anti-directed circuit C. We place arcs of C alternately in D_1 and D_2. Now, all vertices in red(D) $- E(C)$ have even indegrees and outdegrees. We now find an anti-directed circuit in red(D) $- E(C)$ and place its arcs alternately in D_1 and D_2. Continuing in this fashion we obtain a well spread halving of D into halves D_1 and D_2. ■

Note that if D is a connected directed graph with indegree(v) and outdegree(v) being even for each $v \in V(D)$, then D contains a directed Euler circuit. In contrast, we point out here that if D is a connected directed graph with indegree(v) and outdegree(v) being even for each $v \in V(D)$, then D need not contain an anti-directed Euler circuit. A simple example to illustrate this is the following D. Let $D = (V, A)$ with $V = \{1,2,3,4\}$ and $A = \{12,21,23,32,34,43,14,41\}$. Then, the indegree and outdegree of each vertex of D is 2 but D does not contain an anti-directed Euler circuit. The following theorem [1] gives a necessary and sufficient condition for a directed graph to have an antidirected Euler circuit. We state this theorem without proof and purely as a matter of side interest.

Theorem 5 [1] *A directed graph D has an anti-directed Euler circuit if and only if* indegree(v) *and* outdegree(v) *are even for each $v \in V(D)$, any* *two vertices in $V(D)$ are joined by an anti-directed path in D, and, there exists an anti-directed walk of odd length from v to itself for each $v \in V(D)$ with* indegree$(v) > 0$ *and* outdegree$(v) > 0$.

In order to prove Theorem 4 in the Introduction, we will use the following theorem that gives a sufficient condition for the existence of a directed Hamilton cycle in a directed graph. This theorem is a direct corollary of a theorem by Ghouila-Houri [4].

Theorem 6 [4] *If D is a directed graph of order n with* indegree$(v) \geq \frac{n}{2}$ *and* outdegree$(v) \geq \frac{n}{2}$ *for each $v \in V(D)$, then D contains a directed Hamilton cycle.*

The proof of Theorem 4 in the Introduction will follow easily from the following theorem.

Theorem 7 *Let D be a directed graph of even order n and let $V(D) = P \cup Q$ be a partition of $V(D)$. If $|P| = 2j$ for some integer $j \geq 0$, and* indeg$(v) \geq \frac{n}{2} + j$ *and* outdeg$(v) \geq \frac{n}{2} + j$ *for each $v \in V(D)$, then D contains a (P, Q)-directed Hamilton cycle.*

Proof. We proceed by induction on j. If $j = 0$, then $P = \emptyset$ and indeg$(v) \geq \frac{n}{2}$ and outdeg$(v) \geq \frac{n}{2}$ for each $v \in V(D)$. Theorem 6 implies that D contains a directed Hamilton cycle which is a $(\emptyset, V(D))$-directed Hamilton cycle in D. Now suppose the theorem is true for all integers $j \leq k - 1$ and suppose that $j = k \geq 1$.

Let p and p' be distinct vertices in P. We claim that there exists a vertex $q \in Q$ such that $pq \in A(D)$ and $qp' \in A(D)$. Let $B = \{v \in V(D) : pv \in A(D)\}$ and let $C = \{u \in V(D) : up' \in A(D)\}$. The conditions that indeg$(v) \geq \frac{n}{2} + j$ and outdeg$(v) \geq \frac{n}{2} + j$ for each $v \in V(D)$ imply that $|B \cap C| = |B| + |C| - |B \cup C| \geq (\frac{n}{2} + j) + (\frac{n}{2} + j) - n = 2j$. Since $p \notin B$ and $p' \notin C$, we have that the required vertex $q \in Q$ exists with $pq \in A(D)$ and $qp' \in A(D)$.

We now construct a new directed graph D^* from D with $V(D^*) = (V(D) - \{p, q, p'\}) \cup \{q^*\}$ and with $E(D^*)$ obtained from $A(D)$ as follows: Delete arcs $vp \in A(D)$ for each $v \in V(D)$, delete arcs $p'v \in A(D)$ for each $v \in V(D)$, delete all arcs incident on q, replace arc $pv \in A(D)$ by an arc q^*v for each $v \in V(D)$, and, replace arc $vp' \in A(D)$ by an arc vq^* for each $v \in V(D)$. Let $P^* = P - \{p, p'\}$ and $Q^* = (Q - \{q\}) \cup \{q^*\}$. Clearly, if D^* contains a directed (P^*, Q^*)-directed Hamilton cycle then D contains a (P, Q)-directed Hamilton cycle that includes the arcs pq and qp'. Now, $|V(D^*)| = n - 2$ and $|P^*| = 2(k - 1)$. In addition, it is easy to verify that indeg$(v, D^*) \geq \frac{n}{2} + k - 2 = \frac{n-2}{2} + (k - 1)$, and that outdeg$(v, D^*) \geq \frac{n}{2} + k - 2 = \frac{n-2}{2} + (k - 1)$, for each $v \in D^*$. Hence, by the induction hypothesis we have that D^* contains a (P^*, Q^*)-directed

Hamilton cycle and therefore D contains a (P, Q)-directed Hamilton cycle. ■

We are now ready to prove Theorem 4 in the Introduction.

Proof of Theorem 4. Let D_{simp} be the simple directed graph underlying D. Since indegree$(v, D) \geq \rho(\frac{n}{2}+j)$ and outdegree$(v, D) \geq \rho(\frac{n}{2}+j)$ for each $v \in V(D)$, we have that indegree$(v, D_{simp}) \geq (\frac{n}{2}+j)$ and outdegree$(v, D_{simp}) \geq (\frac{n}{2}+j)$ for each $v \in V(D_{simp})$. Hence, Theorem 7 implies that D_{simp} contains a (P, Q)-directed Hamilton cycle H. Now, the indegree and outdegree of each vertex in $D - A(H)$ is even and therefore Theorem 6 implies that $D - A(H)$ has a well-spread halving into halves D_1 and D_2. To prove (ii) in Theorem 4 let $\overline{D}$ denote the complement of D relative to $\vec{K}_n^{(\rho)}$ and note that because indegree$(v, D) \leq \rho(\frac{n}{2}-j-1)$ and outdegree$(v, D) \leq \rho(\frac{n}{2}-j-1)$ for each $v \in V(D)$, we have that indegree$(v, \overline{D}) \geq \rho(\frac{n}{2}+j)$ and outdegree$(v, \overline{D}) \geq \rho(\frac{n}{2}+j)$ for each $v \in V(\overline{D})$. Now applying the result in (i) to $\overline{D}$ yields the result in (ii). ■

References

[1] K. Badheka, Masters Thesis, Department of Mathematics, Illinois State University, August 2007.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London (1976).

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA(1969).

[4] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit Hamiltonien, *C.R. Acad. Sci. Paris* **156** (1960) 495-497.

[5] D.D. Grant, Anti-directed Hamilton cycles in digraphs, *Ars Combinatoria*, Vol. 10, (1980) 205-209.

[6] R. Haggkvist and A. Thomason, Oriented Hamilton cycles in digraphs, *J. of Graph Theory*, Vol. 19, No. 4, (1995) 471-479.

[7] M. Plantholt and S. Tipnis, On a well-spread halving of regular multigraphs, *Ars Combinatoria*, Vol. 83, (2007) 257-265.