Super-Mixed-Connected Line Digraphs *

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Abstract

A digraph D is said to be super-mixed-connected if every minimum general cut of D is a local cut. In this paper, we characterize non-super-mixed-connected line digraphs. As a consequence, if D is a super-arc-connected digraph with $\delta(D) \geq 3$, then the n-th iterated line digraph of D is super-mixed-connected for any positive integer n. In particular, Kautz network $K(d, n)$ is super-mixed-connected for $d \neq 2$, and de Bruijn network $B(d, n)$ is always super-mixed-connected.

Keywords: Super-mixed-connected; Line digraph; General cut; Local cut; de Bruijn network; Kautz network

1 Introduction

A network can be modelled as a graph or a digraph. In this paper, we consider digraphs in which loops or digons (arcs uv and vu form a digon) are permitted but multiple arcs (arcs with the same head and tail) are not present.

Besides the classical connectivity κ and edge-connectivity λ, some other concepts were proposed to measure the fault-tolerance of networks. A strongly connected digraph D is said to be *super-connected*, if every mini-

*This work is supported by NSFC (60603003) and XJEDU.
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mum vertex cut of D is either the out-neighbor set or the in-neighbor set of a vertex. It is *super-arc-connected* if every minimum arc-cut of D is either the out-going arcs or the in-going arcs of a vertex. In 2004, Ramras proposed the concept of super-mixed-connectedness for (undirected) graphs [2]. We generalize this concept to digraphs. A *general cut* of a strongly connected digraph D is a set S of vertices and/or arcs such that $D - S$ is no longer strongly connected. Denote by $\kappa_g(D)$ the cardinality of a minimum general cut. For $u \in V(D)$, a *local cut at* u is a general cut consisting of, for each neighbor v of u, either the vertex v or one arc of $\{uv, vu\}$, but not both. Call a graph D *super-mixed-connected* if every minimum general cut of D is a local cut.

In [2], Ramras showed that any hyper-connected graph with girth at least 4 is super-mixed-connected (a graph is *hyper-connected* if the deletion of every minimum vertex-cut results in exactly two connected components, one of which is an isolated vertex [3].) As a corollary, the hypercube Q_k ($k \geq 3$) is super-mixed-connected.

An intuitive guess may be that D is super-mixed-connected if D is both super-connected and super-arc-connected. But this is generally not true, as can be seen from the following digraph.

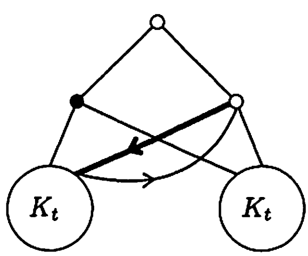


Figure 1. The lines without arrows are all bidirectional arcs.
A minimum general cut which is not a local cut is blackened.

Then, how about special classes of digraphs, such as line digraphs? In the design of communication networks, line digraphs are often used as the topology, for they meet many requirements such as small delays and high reliability [4]. The *line digraph* $L(D)$ of a digraph $D = (V(D), A(D))$ has $A(D)$ as its vertex set and a vertex xy is adjacent to a vertex wz in $L(D)$ if and only if $y = w$. For an integer n, the *n-th iterated line digraph* of D is recursively defined as $L^n(D) = L(L^{n-1}(D))$ with $L^0(D) = D$. The well known de Bruijn networks and Kautz networks can be defined as iterated line digraphs. For any integers $d \geq 2$ and $n \geq 1$, the de Bruijn network

$B(d,1) = K_d^+$ and $B(d,n) = L^{n-1}(K_d^+)$, where K_d^+ is the digraph obtained from a complete digraph K_d by adding a loop at each vertex; the *Kautz* network $K(d,1) = K_{d+1}$ and $K(d,n) = L^{n-1}(K_{d+1})$.

In this paper, we characterize non-super-mixed-connected line digraphs. As a consequence, if D is super-arc-connected with minimum degree $\delta(D) \geq 3$, then $L^n(D)$ is super-mixed-connected for any positive integer n. In particular, $K(d,n)$ is super-mixed-connected for $d \neq 2$, and $B(d,n)$ is always super-mixed-connected.

In the remaining of this section, we introduce some terminologies used in this paper. For $u \in V(D)$, denote by $N_D^+(u)$ the set of out-neighbors of u, $N_D^-(u)$ the set of in-neighbors of u. Write $d_D^+(u) = |N_D^+(u)|$, $d_D^-(u) = |N_D^-(u)|$. When there is no danger of confusion, the subscriptions are omitted in the above notation. A general cut S of D is said to be *associated with a vertex cut C of D*, if S is composed of a subset of vertices $C' \subseteq C$ and a subset of arcs A', such that each arc in A' is incident with exactly one vertex in $C \setminus C'$, and every vertex in $C \setminus C'$ is incident with exactly one arc in A'. We follow [1] and [5] for terminologies and notation not given here.

2 Main Results

First, we show the relation between $\kappa_g(D)$ and $\kappa(D)$.

Lemma 1. *Let D be a strongly connected digraph. Then $\kappa_g(D) = \kappa(D)$.*

Proof. Since every minimum vertex cut is a general cut, we have $\kappa_g(D) \leq \kappa(D)$. On the other hand, let S be a minimum general cut of D. Set $A = S \cap V(D)$, $B = S \cap A(D)$, $D' = D - A$. By the choice of S, B is a minimum arc cut of D'. Let A' be a minimum vertex cut of D'. Then $|A'| = \kappa(D') \leq \lambda(D') = |B|$. Since $A \cup A'$ is a vertex cut of D, we have $\kappa(D) \leq |A| + |A'| \leq |A| + |B| = \kappa_g(D)$. □

Lemma 2. *Let D be a strongly connected digraph with $\kappa(D) = \lambda(D)$, then every minimum general cut of D is associated with some minimum vertex cut of D.*

Proof. If $\kappa(D) = |V(D)| - 1$, then D is a complete digraph, and the result is obvious. So, assume $\kappa(D) < |V(D)| - 1$ in the following. Let S be a minimum general cut of D. Use the notation as in the proof of Lemma 1.

Since B is a minimum arc cut of D', there exists a vertex subset $X \subset V(D')$, such that there is no arc from X to $\bar{X}$ in $D' - B$, where $\bar{X} = V(D) \setminus X$. By the minimality of B, every arc in B has its tail in X and head in $\bar{X}$. Write

$$N_1 = \{x \in X : \ x \text{ is the tail of some arc in B}\},$$
$$N_2 = \{x \in \bar{X} : \ x \text{ is the head of some arc in B}\}.$$

In the case $|N_1| < |X|$, set $T = N_1$. In the case $|N_1| = |X|$ and $|N_2| < |\bar{X}|$, set $T = N_2$. In the case that $N_1 = X$ and $N_2 = \bar{X}$, there must be two vertices $x \in X$ and $y \in \bar{X}$, such that $xy \notin D'$. In fact if this is not true, then $|B| = |X||\bar{X}| = |X|(|V(D')| - |X|) \geq |V(D')| - 1$, and thus $\kappa(D) = \kappa_g(D) = |A|+|B| \geq |V(D)|-1$. Set $T = N_1 \setminus \{x\} \cup \{z \in \bar{X} | xz \in B\}$ in the third case. In any case, T is a vertex cut of D', and $|T| \leq |B|$. Note that $\kappa(D') = \kappa(D) - |A| = \lambda(D) - |A| = |B| = \lambda(D')$, it follows from $\kappa(D') \leq |T| \leq |B| = \kappa(D')$ that $|T| = |B| = \kappa(D')$. So, T is a minimum vertex cut of D', and every vertex in T is incident with exactly one edge of B. Then S is associated with $T \cup A$. □

For simplicity of statement, use N_u to denote $N^+(u)$ or $N^-(u)$. So, $N_u = N_v$ refers to either $N^+(u) = N^+(v)$ or $N^-(u) = N^-(v)$ or $N^+(u) = N^-(v)$ or $N^-(u) = N^+(v)$. The following theorem characterizes non-super-mixed-connected digraphs.

Theorem 1. *Let D be a super-connected digraph with minimum degree δ. If D is not super-mixed-connected, then there exist two vertices u and v of degree δ such that either $N^+(u) = N^-(v)$, or $|N_u \cap N_v| = \delta - 1$ and $u'v' \in A(D)$ where u' is the only vertex in $N_u \setminus N_v$ and v' is the only vertex in $N_v \setminus N_u$.*

Proof. Let S be a minimum general cut of D which is not a local cut. By Lemma 2, S is associated with a minimum vertex cut C of D. Since D is super-connected, C is the out-neighbor set or the in-neighbor set of a vertex u with degree δ. Suppose, without loss of generality, that $C = N^+(u)$. Since D is not super-mixed-connected, there is an arc $u'v'$ or $v'u'$ in S with $u' \in N^+(u)$ and $v' \notin N^+(u) \cup \{u\}$. By the minimality of S, for any arc $xy \in S$, y is not accessible from x in $D - S$.

In the case that $v'u' \in S$ and $N^+(v') = N^+(u)$, in order that S is not a local cut, there must be another arc $uw \in S$. Then $uu' \notin S$ and $v'w \notin S$. By the minimality of S, there is an (u', v')-path P in $D - (S \setminus \{v'u'\})$. Clearly, $v'u' \notin P$, and thus P is also an (u', v')-path in $D - S$. But then $\{uu'\} \cup P \cup \{v'w\}$ is an (u, w)-path in $D - S$, contradicting that w is not accessible from u in $D - S$..

Suppose $v'u' \in S$ and $N^+(v') \neq N^+(u)$. Since $d_D^+(u) = \delta \leq d_D^+(v')$, $uu' \notin S$, and u' is not accessible from v' in $D - S$, we see that there is a vertex $w \in V(D) \setminus (N^+(u) \cup \{u\})$ with $v'w \in A(D)$. Note that neither $w \in S$ nor $vw \in S$, since w and vw are not associated with C. So any w, u' path in $D - (S \setminus \{v'u'\})$ uses the arc $v'u'$, since otherwise a (v', u')-path in $D - (S \setminus \{v'u'\}$ is also a (v', u')-path in $D - S$, and thus u' is accessible from v' in $D - S$. Then $C' = (N^+(u) \cup \{v'\}) \setminus \{u'\}$ is a minimum vertex cut of D. By the super-connectedness of D, there is a vertex v in D such that $N_v = C'$. Therefore, we arrive at the second case of the theorem.

Suppose $u'v' \in S$ and $N^+(u) \neq N^-(v')$. Then there is a vertex $w \in V(D) \setminus (N^+(u) \cup \{u\})$ such that $wv' \in A(D)$. Similar to the above, $w \notin S$, $wv' \notin S$, and thus w is not accessible from u' in $D - S$, since otherwise v' is accessible from u' in $D - S$. Then, $C' = (N^+(u) \cup \{v'\}) \setminus \{u'\}$ is a minimum vertex cut of D, and again we arrive at the second case of the theorem. □

Note that a non-super-connected digraph can not be super-mixed-connected. So the requirement of D being super-connected in Theorem 1 is natural. The following two results were obtained in [5]

Lemma 3. *[5] Let D be a strongly connected digraph. Then $L(D)$ is super-connected if and only if D is super-arc-connected.*

Lemma 4. *[5] Let D be a strongly connected digraph, and u, v be two distinct vertices in $V(L(D))$. If $N^+_{L(D)}(u) \cap N^+_{L(D)}(v) \neq \emptyset$, then $N^+_{L(D)}(u) = N^+_{L(D)}(v)$. If $N^-_{L(D)}(u) \cap N^-_{L(D)}(v) \neq \emptyset$, then $N^-_{L(D)}(u) = N^-_{L(D)}(v)$.*

The following theorem characterizes super-arc-connected digraph whose line digraph is not super-mixed-connected.

Theorem 2. *Let D be a super-arc-connected digraph which does not have a substructure as in Figure 2 (some vertices may coincide, as will be seen from Figure 3), where $\delta(D) = 2$ and $d_D^+(u_2) = 2$, $d_D^-(v_1) = 2$. Then $L(D)$ is super-mixed-connected.*

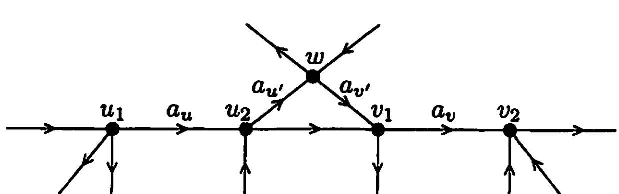


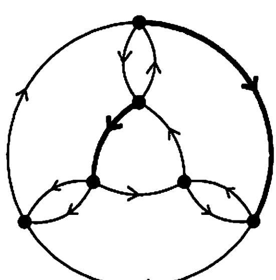
Figure 2. The structure which is excluded by Theorem 2

Proof. When $\delta(L(D)) = 1$, the result is obvious. So, suppose $\delta(L(D)) \geq 2$ in the following. By Lemma 3, $L(G)$ is super-connected. Suppose $L(G)$ is not super-mixed-connected. Then there exist two vertices $u, v \in V(L(D))$ satisfying the conditions of Theorem 1. Suppose they correspond to two arcs $a_u = u_1u_2$ and $a_v = v_1v_2$ in D.

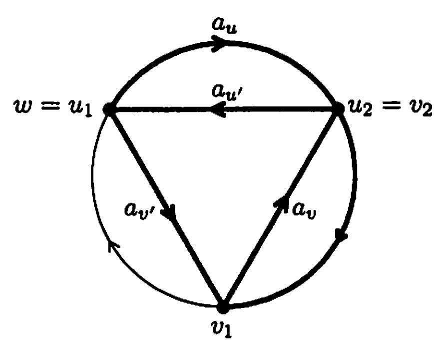
If $N^+_{L(D)}(u) = N^-_{L(D)}(v)$, note that u_2v_1 is the only possible vertex of $L(D)$ common to $N^+_{L(D)}(u)$ and $N^-_{L(D)}(v)$, we have $\delta(L(D)) = 1$, contradicting our assumption that $\delta(L(D)) \geq 2$.

So, suppose that the second case of Theorem 1 occurs, and the two vertices u', v' of $L(D)$ as in Theorem 1 correspond to arcs $a_{u'}, a_{v'}$ in D. By Lemma 4, and the assumption that $\delta(L(D)) \geq 2$, we only need to consider the case that $N_u = N^+_{L(D)}(u)$ and $N_v = N^-_{L(D)}(v)$, or $N_u = N^-_{L(D)}(u)$ and $N_v = N^+_{L(D)}(v)$. By symmetry, assume without loss of generality that $N_u = N^+_{L(D)}(u)$ and $N_v = N^-_{L(D)}(v)$. In this case, $a_{u'} = u_2w$ and $a_{v'} = wv_1$ for some $w \in V(D)$. Note that u_2v_1 is the only possible element common to $N^+_{L(D)}(u)$ and $N^-_{L(D)}(v)$. So $\delta(L(D)) = 2$, and thus $\delta(D) = 2$. Furthermore, $d^+_{L(D)}(u) = d^-_{L(D)}(v) = 2$ (thus $d^+_D(u_2) = d^-_D(v_1) = 2$), and D has a substructure as in Figure 2, a contradiction. □

The following example shows that a digraph having a substructure as in Figure 2 may be non-super-mixed-connected.



(*a*) $K(2,2)$



(*b*) $K(2,1)$

Figure 3. Kautz network $K(2,2) = L(K(2,1))$. In (*a*), a general cut which is not a local cut is indicated by blackened lines. In (*b*), a substructure as in Figure 2 is indicated by blackened lines.

In [5], we have proved

Lemma 5. *Let D be a strongly connected digraph with $\delta(D) \geq 3$. If D is super-arc-connected, then $L^n(D)$ is super-arc-connected and super-connected for any positive integer n.*

Note that $\delta(L^n(D)) = \delta(D)$ and any digraph with minimum degree at least 3 does not have a substructure as in Figure 2. So we have

Corollary 1. *Let D be a super-arc-connected digraph with $\delta(D) \geq 3$. Then $L^n(D)$ is super-mixed-connected for any positive integer n.*

In [5], we also proved that

Lemma 6. *$K(d, n)$ is super-connected and super-arc-connected for any $d \geq 3$. $B(d, n)$ is super-connected and super-arc-connected for any $d \geq 2$.*

Since $K(d, n)$ and $B(d, n)$ are both constructed by line digraph technique, we have

Corollary 2. *$K(d, n)$ is super-mixed-connected if $d \neq 2$. $B(d, n)$ is super-mixed-connected for any d.*

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