On resolvable packing RMP$(3, 3, v)$ and covering RMC$(3, 3, v)$

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Abstract

Let $v \equiv k-1, 0$ or 1 $(mod\ k)$. An RMP(k, λ, v) (resp. RMC(k, λ, v)) is a resolvable packing (resp. covering) with maximum (resp. minimum) possible number $m(v)$ of parallel classes which are mutually distinct, each parallel class consists of $\lfloor (v-k+1)/k \rfloor$ blocks of size k and one block of size $v - k\lfloor (v-k+1)/k \rfloor$, and its leave (resp. excess) is a simple graph. Such designs were first introduced by Fang and Yin. They have proved that these designs can be used to construct certain uniform designs which have been widely applied in industry, system engineering, pharmaceutics, and natural science. In this paper, direct and recursive constructions are discussed for such designs. The existence of an RMP$(3, 3, v)$ and an RMC$(3, 3, v)$ is proved for any admissible v.

Key words: uniform design; resolvable; packing; covering; frame

1 Introduction

Let v and λ be positive integers. A *packing* (resp. *covering*) $P(K, \lambda, v)$ (resp. $C(K, \lambda, v)$) is an ordered pair $(V, \mathcal{B})$ where V is a v-set of points, and $\mathcal{B}$ is a collection of subsets of V with sizes from K, called blocks, such

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that each pair of points of V occurs at most (resp. at least) λ times in the blocks.

For any pair $e = \{x, y\}$ of distinct points , let $m(e)$ be the number of blocks containing e. The *leave* (resp. *excess*) of a packing (resp. covering) $P(K, \lambda, v)$ (resp. $C(K, \lambda, v)$) is the multigraph spanned by all pairs e of distinct points with multiplicity $\lambda - m(e)$ (resp. $m(e) - \lambda$).

A packing (resp. covering) is called *resolvable* if its block set admits a partition into parallel classes, each *parallel class* being a partition of the point set V. Denote by $RP(K, \lambda; v, m)$ (resp. $RC(K, \lambda; v, m)$) a resolvable packing (resp. covering) $P(K, \lambda, v)$ (resp. $C(K, \lambda, v)$) with m parallel classes.

Let $v \equiv k-1, 0$ or 1 $(mod\ k)$. An $RMP(k, \lambda, v)$ (resp. $RMC(k, \lambda, v)$) is a resolvable packing (resp. covering) with maximum (resp. minimum) possible number $m(v)$ of parallel classes which are mutually distinct, each parallel class consists of $\lfloor (v-k+1)/k \rfloor$ blocks of size k and one block of size $v - k\lfloor (v-k+1)/k \rfloor$, and its leave (resp. excess) is a simple graph.

Some simple computation shows:

Lemma 1.1 *If there exists an $RMP(k, \lambda, v)$, then $m(v) \le n(v)$ where*

$$n(v) = \begin{cases} \lfloor \frac{\lambda(v-1)}{k-1} \rfloor & v \equiv 0 \ (mod\ k) \\ \lfloor \frac{\lambda v(v-1)}{(k-1)v+k+1} \rfloor & v \equiv 1 \ (mod\ k) \\ \lfloor \frac{\lambda v}{k-1} \rfloor & v \equiv k-1 \ (mod\ k) \end{cases}$$

Lemma 1.2 *If there exists an $RMC(k, \lambda, v)$, then $m(v) \ge n(v)$ where*

$$n(v) = \begin{cases} \lceil \frac{\lambda(v-1)}{k-1} \rceil & v \equiv 0 \ (mod\ k) \\ \lceil \frac{\lambda v(v-1)}{(k-1)v+k+1} \rceil & v \equiv 1 \ (mod\ k) \\ \lceil \frac{\lambda v}{k-1} \rceil & v \equiv k-1 \ (mod\ k) \end{cases}$$

RMP and RMC were first studied by Fang etc in [6, 7]. They have proved that these designs can be used to construct certain uniform designs in statistics which have been widely applied in industry, system engineering, pharmaceutics and natural science.

Theorem 1.3 ([7]) *Suppose* n, k, λ *and* m *are positive integers and* $n \equiv r \pmod{k}$ *where* $r \in \{0, 1, k-1\}$. *Then the factorial design derived from an RMP*(k, λ, n) *(resp. RMC*(k, λ, n)*) with* m *parallel classes is a uniform design* $U_n(q^m)$, *where* $q = \lfloor (n-k+1)/k \rfloor + 1$.

When $k = 3$ and $\lambda \in \{1, 2\}$, the existence of an RMP$(3, \lambda, v)$ or RMC$(3, \lambda, v)$ has been solved for every positive integer v with five possible exceptions [1, 2, 3, 4, 7, 14, 16, 18, 19, 21]. There are also some known results on RMP$(4, \lambda, v)$ and RMC$(4, \lambda, v)$ for $\lambda \in \{1, 2\}$ [2, 7, 8, 10, 12, 15].

In this paper, we shall deal with the existence of an RMP$(3, 3, v)$ and an RMC$(3, 3, v)$ for every positive integer v. Direct and recursive constructions are discussed for these designs. The existence of an RMP$(3, 3, v)$ and an RMC$(3, 3, v)$ will be proved for any integer $v \geq 5$, except for RMP$(3, 3, 6)$.

Theorem 1.4 *There exists an RMP*$(3, 3, v)$ *and an RMC*$(3, 3, v)$ *for each* $v \geq 5$ *except for an RMP*$(3, 3, 6)$.

2 Preliminaries

In this section we shall define some of the auxiliary designs and establish some of the fundamental results which will be used later. The reader is referred to [5] for more information on designs, and, in particular, group divisible designs and frames.

Let K be a set of positive integers. A *group-divisible design* (K, λ)-GDD is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{X}$ is a finite set of points,
2. $\mathcal{G}$ is a partition of $\mathcal{X}$ into subsets called groups,

3. $\mathcal{B}$ is a collection of subsets of $\mathcal{X}$ with sizes from K, called blocks, such that every pair of points from distinct groups occurs in exactly λ blocks, and

4. No pair of points belonging to a group occurs in any block.

A (K, λ)-GDD $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ is *resolvable* if the blocks of $\mathcal{B}$ can be partitioned into parallel classes. When $K = \{k\}$, we write (K, λ)-GDD as (k, λ)-GDD. Further, we denote $(K, 1)$-GDD as K-GDD and $(k, 1)$-GDD as k-GDD.

The *type* of the GDD $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: type $1^i 2^j 3^k \cdots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on. An RB(v, k, λ) is a resolvable (k, λ)-GDD of type 1^v. A *transversal design* $TD(k, n)$ is a k-GDD of type n^k. It is well known that a $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares of order n.

A (K, λ)-*frame* is a GDD $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ in which the collection of blocks $\mathcal{B}$ can be partitioned into holey parallel classes, each *holey parallel class* being a partition of $\mathcal{X} \setminus G_j$ for some $G_j \in \mathcal{G}$. The groups in a (K, λ)-frame are often referred to as holes. A *uniform* frame is a frame in which all groups are of the same size. A $(3, \lambda)$-frame is also called a Kirkman frame with index λ. In a $(3, \lambda)$-frame, it is not difficult to prove that to each group G_j there are exactly $\lambda |G_j|/2$ holey parallel classes that partition $\mathcal{X} \setminus G_j$.

A design is called *simple* if all its blocks are distinct. From [22], we have the following results for simple $(3, 2)$-frames and simple $(3, 3)$-frames.

Theorem 2.1 (1) *There exists a simple* $(3, 2)$*-frame of type* t^u *if and only if* $u \geq 4$ *and* $t(u - 1) \equiv 0 \ (mod\ 3)$. (2) *There exists a simple* $(3, 3)$*-frame of type* t^u *if and only if* $u \geq 4$, t *is even and* $t(u - 1) \equiv 0 \ (mod\ 3)$.

The main technique that we will be using throughout the remainder of the article is a variant of Stinson's 'Filling in Holes' construction. To apply that construction, we will require simple $(3,3)$-frames in which the groups are not necessarily all of the same size. To get these, we shall use the following recursive construction.

Lemma 2.2 ([20]) *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_m^{t_m}$ and that for each $k \in K$ there is a simple $(3,3)$-frame of type h^k. Then there is a simple $(3,3)$-frame of type $(hg_1)^{t_1}(hg_2)^{t_2} \cdots (hg_m)^{t_m}$.*

In order to use the 'Filling in Holes' construction, we need the notion of an incomplete RMP (IRMP) (resp. RMC (IRMC)).

Let $a = v - 2n(v)/3$ and $v \equiv h \pmod 2$, $h \geq 3$. For $h \geq a$, an *IRMP*$(3,3;v,h)$ (resp. *IRMC*$(3,3;v,h)$) is defined to be a triple $(V,H,\mathcal{B})$ which satisfies the following properties:

1. V is a v-set of points, H is an h-subset of V (called "*hole*") and $\mathcal{B}$ is a collection of subsets of V (called *blocks*), each block having size 3 or $v - 3\lfloor (v-2)/3 \rfloor$;
2. $|H \cap B| \leq 1$ for all $B \in \mathcal{B}$;
3. any two points of V appear either in H or in t blocks of $\mathcal{B}$, $2 \leq t \leq 3$ (resp. $3 \leq t \leq 4$);
4. $\mathcal{B}$ admits a partition into $3(v-h)/2$ distinct parallel classes, each consists of $\lfloor (v-2)/3 \rfloor$ blocks of size 3 and one block of size $v - 3\lfloor (v-2)/3 \rfloor$ on V, and $3(h-a)/2$ ($3(h-a)/2 - 1$ for IRMP$(3,3;16,4)$) auxiliary parallel classes, each consists of $(v-h)/3$ triples on $V \backslash H$.

For later use we will construct some IRMPs and IRMCs. Instead of listing all the blocks of the parallel classes of the desired design, we only list the blocks of some initial parallel classes. We write (a,i) as a_i, IRMP$(3,3;v,h)$ (IRMC$(3,3;v,h)$) as IRMP(v,h) (IRMC(v,h)) and RMP$(3,3,v)$ (RMC$(3,3,v)$) as RMP(v) (RMC(v)) for brevity.

Lemma 2.3 *There exists an IRMP*$(13,3)$.

Proof: Take the point set $V = (Z_5 \times Z_2) \cup \{\infty_1, \infty_2, \infty_3\}$. The required 15 parallel classes will be generated from the following three initial parallel classes by $(+1\ mod\ 5,\ -)$.

P_1 :	$0_01_02_00_1$	$\infty_13_03_1$	$\infty_24_04_1$	$\infty_31_12_1$
P_2 :	$0_02_01_13_1$	$\infty_14_00_1$	$\infty_22_14_1$	$\infty_31_03_0$
P_3 :	$0_02_13_14_1$	$\infty_14_01_1$	$\infty_21_02_0$	$\infty_33_00_1$

It is easy to check that the leave of this IRMP(13, 3) consists of $3K_3s$ based on the point set $\{\infty_1, \infty_2, \infty_3\}$. □

Lemma 2.4 *There exists an IRMP*(16, 4).

Proof: Take the point set $V = Z_{12} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The auxiliary parallel class will be generated from an initial block $\{0, 4, 8\}$ by $(+1\ mod\ 12)$. The required parallel classes will be generated from the following six initial parallel classes by $(+4\ mod\ 12)$.

P_1 :	0 1 2 3	∞_1 4 5	∞_2 6 7	∞_3 8 9	∞_4 10 11
P_2 :	0 2 5 7	∞_1 1 3	∞_2 4 8	∞_3 6 11	∞_4 9 10
P_3 :	0 2 5 11	∞_1 1 6	∞_2 4 10	∞_3 3 9	∞_4 7 8
P_4 :	0 3 5 10	∞_1 2 6	∞_2 1 9	∞_3 7 8	∞_4 4 11
P_5 :	0 3 6 9	∞_1 4 8	∞_2 7 11	∞_3 2 10	∞_4 1 5
P_6 :	0 7 9 10	∞_1 3 11	∞_2 1 6	∞_3 5 8	∞_4 2 4

Lemma 2.5 *There exists an IRMP*(28, 4).

Proof: Take the point set $V = Z_{24} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The auxiliary parallel class will be generated from an initial block $\{0, 8, 16\}$ by $(+1\ mod\ 24)$. The required parallel classes will be generated from the following two initial parallel classes by $(+2\ mod\ 24)$ and $(+1\ mod\ 24)$ respectively.

P_1 :	0 1 2 3	4 5 6	7 8 10	9 11 14	12 15 17	$\infty_1$13 18	$\infty_2$16 21
	$\infty_3$19 22	∞_4 20 23					
P_2 :	0 4 8 12	1 6 15	2 9 19	3 14 20	7 16 22	$\infty_1$5 18	$\infty_2$10 21
	$\infty_3$11 17	∞_4 13 23					

Lemma 2.6 *There exists an IRMP*$(v, 4)$ *for each* $v \in \{22, 34\}$.

Proof: Take the point set $V = (Z_u \times Z_2) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, where $u = (v-4)/2$. The required parallel classes will be generated from the following two initial parallel classes by $(+1\ mod\ u,\ -)$ and $(+1\ mod\ u,\ +1\ mod\ 2)$ respectively.

$v = 22$	$0_0 2_0 1_1 3_1$	$4_0 6_0 5_1$	$7_0 0_1 2_1$	$\infty_1 1_0 7_1$	$\infty_2 3_0 8_1$	$\infty_3 5_0 4_1$	$\infty_4 8_0 6_1$
	$0_0 2_0 8_0 5_1$	$1_0 6_0 1_1$	$2_1 3_1 6_1$	$\infty_1 3_0 4_0$	$\infty_2 5_0 7_1$	$\infty_3 7_0 0_1$	$\infty_4 4_1 8_1$

$v = 34$	$0_0 2_0 1_1 3_1$	$3_0 12_0 0_1$	$4_0 6_0 5_1$	$7_0 11_0 2_1$	$8_0 10_0 7_1$	$13_0 4_1 10_1$
	$9_1 11_1 13_1$	$\infty_1 1_0 6_1$	$\infty_2 5_0 14_1$	$\infty_3 9_0 12_1$	$\infty_4 14_0 8_1$	
	$0_0 4_0 8_0 13_1$	$2_0 7_0 14_0$	$6_0 12_0 1_1$	$3_0 7_1 10_1$	$5_0 9_1 12_1$	$11_0 4_1 11_1$
	$2_1 3_1 8_1$	$\infty_1 9_0 10_0$	$\infty_2 1_0 14_1$	$\infty_3 13_0 0_1$	$\infty_4 5_1 6_1$	

For each v, the required auxiliary parallel class will be generated from an initial block B by $(+1\ mod\ u,\ +1\ mod\ 2)$ respectively, where $B = \{0_0, 3_0, 6_0\}$ for $v = 22$ and $B = \{0_0, 5_0, 10_0\}$ for $v = 34$. $\square$

Lemma 2.7 *There exists an IRMP(25, 7).*

Proof: Take the point set $V = (Z_9 \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_7\}$. The required 27 parallel classes will be generated from the following three initial parallel classes by $(+1\ mod\ 9,\ -)$.

P_1 :	$0_0 1_0 6_0 3_1$	$\infty_1 2_0 3_0$	$\infty_2 7_0 8_0$	$\infty_3 4_0 4_1$	$\infty_4 5_0 0_1$	$\infty_5 1_1 7_1$
	$\infty_6 2_1 8_1$	$\infty_7 5_1 6_1$				
P_2 :	$0_0 2_0 1_1 3_1$	$\infty_1 5_1 7_1$	$\infty_2 4_1 8_1$	$\infty_3 8_0 2_1$	$\infty_4 1_0 5_0$	$\infty_5 4_0 6_0$
	$\infty_6 7_0 6_1$	$\infty_7 3_0 0_1$				
P_3 :	$0_0 0_1 5_1 6_1$	$\infty_1 3_0 3_1$	$\infty_2 6_0 4_1$	$\infty_3 4_0 2_1$	$\infty_4 7_1 8_1$	$\infty_5 8_0 1_1$
	$\infty_6 1_0 7_0$	$\infty_7 2_0 5_0$				

The six auxiliary parallel classes will be generated from the following two auxiliary parallel classes by $(+1\ mod\ 9,\ -)$.

Q_1 :	$0_0 1_1 5_1$	$4_0 8_0 3_1$	$6_0 2_1 7_1$	$1_0 5_0 0_1$	$3_0 4_1 8_1$	$2_0 7_0 6_1$
Q_2 :	$0_0 2_0 7_1$	$1_0 3_1 5_1$	$6_0 8_0 4_1$	$7_0 0_1 2_1$	$3_0 5_0 1_1$	$4_0 6_1 8_1$

Lemma 2.8 *There exists an IRMP(31, 7).*

Proof: Take the point set $V = (Z_{12} \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_7\}$. The rth parallel class will consists of two parts Q_r and F_r, $0 \le r \le 35$. The main part Q_r will be generated from the following three initial classes Q_0, Q_1 and Q_2 by $(+1\ mod\ 12,\ -)$. Here, $Q_{12i+j} = Q_i + j$, $0 \le i \le 2$, $0 \le j \le 11$. Let $F_{12i+j} = F_j$. We assume that the ith 3-frame of type 2^4 is based on the set $\{i, 3+i, 6+i, 9+i\} \times Z_2$. Suppose $j \equiv k (mod\ 3)$ $0 \leq k \leq 2$, then take F_j to be the holey parallel class with the hole $\{j\} \times Z_2$ in the kth 3-frame. The blocks in Q_0, Q_1 and Q_2 are listed below.

Q_0 :	$0_0 1_0 2_0 0_1$	$\infty_1 7_1 8_1$	$\infty_2 11_0 10_1$	$\infty_3 4_0 2_1$	$\infty_4 5_0 1_1$	$\infty_5 7_0 5_1$
	$\infty_6 8_0 4_1$	$\infty_7 10_0 11_1$				
Q_1 :	$0_0 0_1 2_1 4_1$	$\infty_1 2_0 7_1$	$\infty_2 1_0 5_0$	$\infty_3 4_0 11_1$	$\infty_4 7_0 8_1$	$\infty_5 8_0 1_1$
	$\infty_6 10_0 5_1$	$\infty_7 11_0 10_1$				
Q_2 :	$0_0 4_0 4_1 8_1$	$\infty_1 5_0 7_0$	$\infty_2 0_1 11_1$	$\infty_3 1_0 2_1$	$\infty_4 2_0 7_1$	$\infty_5 8_0 10_1$
	$\infty_6 11_0 1_1$	$\infty_7 10_0 5_1$				

The six auxiliary parallel classes will be generated from the following two auxiliary parallel classes by $(+1\ mod\ 12,\ -)$.

$0_0 1_0 5_0$	$3_0 4_0 8_0$	$6_0 7_0 11_0$	$9_0 10_0 2_0$	$0_1 1_1 5_1$	$3_1 4_1 8_1$	$6_1 7_1 11_1$	$9_1 10_1 2_1$
$0_0 2_0 7_0$	$3_0 5_0 10_0$	$6_0 8_0 1_0$	$9_0 11_0 4_0$	$0_1 2_1 7_1$	$3_1 5_1 10_1$	$6_1 8_1 1_1$	$9_1 11_1 4_1$

Lemma 2.9 *There exists an IRMP$(v, 7)$ for each $v \in \{37, 49, 61\}$.*

Proof: Take the point set $V = (Z_u \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_7\}$, where $u = (v-7)/2$. The required parallel classes will be generated from the following two initial parallel classes by $(+1\ mod\ u,\ -)$ and $(+1\ mod\ u,\ +1\ mod\ 2)$ respectively.

$v = 37$	$0_0 2_0 1_1 3_1$	$4_0 6_0 9_1$	$7_0 10_0 13_1$	$1_0 8_1 11_1$	$8_0 5_1 7_1$	$\infty_1 3_0 0_1$
	$\infty_2 5_0 12_1$	$\infty_3 9_0 2_1$	$\infty_4 11_0 4_1$	$\infty_5 12_0 6_1$	$\infty_6 13_0 10_1$	$\infty_7 14_0 14_1$
	$0_0 4_0 8_0 14_0$	$1_0 11_0 5_1$	$2_0 10_0 4_1$	$7_0 13_0 11_1$	$0_1 1_1 7_1$	$\infty_1 9_0 12_0$
	$\infty_2 3_0 14_1$	$\infty_3 5_0 3_1$	$\infty_4 6_0 6_1$	$\infty_5 2_1 12_1$	$\infty_6 8_1 9_1$	$\infty_7 10_1 13_1$
$v = 49$	$0_0 2_0 1_1 3_1$	$3_0 20_0 15_1$	$4_0 6_0 9_1$	$7_0 17_0 17_1$	$13_0 19_0 4_1$	$8_0 5_1 7_1$
	$14_0 18_0 11_1$	$12_0 2_1 19_1$	$8_1 12_1 18_1$	$\infty_1 1_0 16_1$	$\infty_2 5_0 14_1$	$\infty_3 9_0 6_1$
	$\infty_4 10_0 13_1$	$\infty_5 11_0 20_1$	$\infty_6 15_0 0_1$	$\infty_7 16_0 10_1$		
	$0_0 6_0 12_0 19_1$	$2_0 10_0 20_0$	$3_0 16_0 19_0$	$5_0 1_1 15_1$	$11_0 2_1 3_1$	$14_0 6_1 16_1$
	$15_0 4_1 11_1$	$0_1 7_1 12_1$	$9_1 14_1 17_1$	$\infty_1 8_0 9_0$	$\infty_5 7_0 5_1$	$\infty_3 1_0 18_1$
	$\infty_4 4_0 10_1$	$\infty_2 17_0 18_0$	$\infty_6 13_0 13_1$	$\infty_7 8_1 20_1$		
$v = 61$	$0_0 2_0 1_1 3_1$	$1_0 7_0 22_0$	$3_0 13_0 21_0$	$4_0 6_0 9_1$	$9_0 26_0 26_1$	
	$10_0 16_0 13_1$	$14_0 18_0 11_1$	$20_0 24_0 15_1$	$8_0 5_1 7_1$	$12_0 19_1 23_1$	
	$0_1 17_1 21_1$	$4_1 10_1 25_1$	$6_1 16_1 24_1$	$\infty_1 5_0 14_1$	$\infty_2 11_0 20_1$	
	$\infty_3 15_0 22_1$	$\infty_4 17_0 8_1$	$\infty_5 19_0 2_1$	$\infty_6 23_0 12_1$	$\infty_7 25_0 18_1$	
	$0_0 8_0 16_0 25_1$	$5_0 12_0 21_0$	$3_0 4_0 17_1$	$6_0 20_0 8_1$	$10_0 23_0 2_1$	
	$15_0 18_0 26_1$	$24_0 25_0 14_1$	$2_0 16_1 21_1$	$7_0 5_1 19_1$	$11_0 7_1 23_1$	
	$14_0 10_1 20_1$	$0_1 15_1 22_1$	$6_1 9_1 18_1$	$\infty_1 17_0 22_0$	$\infty_2 19_0 26_0$	
	$\infty_3 1_0 24_1$	$\infty_4 9_0 3_1$	$\infty_5 13_0 13_1$	$\infty_6 1_1 4_1$	$\infty_7 11_1 12_1$	

The six auxiliary parallel classes will be generated from six initial blocks $B \in \{0_0 1_1 5_1,\ 2_0 6_0 1_1,\ 2_0 3_1 7_1,\ 0_0 2_0 7_1,\ 8_0 1_1 3_1,\ 2_0 4_0 9_1\}$ by $(+3\ mod\ u, +1\ mod\ 2)$. □

Note that the leave of the IRMP$(v,7)$ for $v \in \{25,31,37,49,61\}$ constructed in Lemmas 2.7- 2.9 consists of $3K_7s$ based on the point set $\{\infty_1, \infty_2, \cdots, \infty_7\}$. This is important for the constructions of the corresponding RMPs and RMCs.

Lemma 2.10 *There exists an IRMP*(34, 10).

Proof: Take the point set $V = Z_{24} \cup \{\infty_1, \infty_2, \cdots, \infty_{10}\}$. The required 36 parallel classes will be generated from the following three initial parallel classes by $(+2\ mod\ 24)$.

P_1 :	0 1 2 3	∞_1 4 6	∞_2 5 7	∞_3 8 11	∞_4 9 10	∞_5 12 15
	∞_6 13 16	∞_7 14 20	∞_8 17 21	∞_9 18 22	∞_{10} 19 23	
P_2 :	0 4 9 12	∞_1 1 6	∞_2 2 7	∞_3 3 8	∞_4 5 13	∞_5 10 19
	∞_6 14 20	∞_7 15 21	∞_8 11 18	∞_9 17 23	∞_{10} 16 22	
P_3 :	0 7 13 21	∞_1 1 11	∞_2 2 12	∞_3 3 16	∞_4 10 20	∞_5 4 19
	∞_6 5 17	∞_7 6 15	∞_8 14 22	∞_9 8 23	∞_{10} 9 18	

For each block $B \in \{0\,2\,7, 1\,3\,8, 2\,4\,9, 0\,1\,11, 1\,2\,12, 2\,3\,13, 0\,7\,11, 1\,8\,12, 2\,9\,13\}$, we can generate an auxiliary parallel class from B by $(+3\ mod\ 24)$. Thus we obtain 9 auxiliary parallel classes. The last auxiliary parallel class will be generated from the block $\{0,8,16\}$ by $(+1\ mod\ 24)$. □

Lemma 2.11 *There exists an IRMP*$(v,10)$ *for each* $v \in \{40,46\}$.

Proof: Take the point set $V = Z_u \cup \{\infty_1, \infty_2, \cdots, \infty_{10}\}$, where $u = v - 10$. The required parallel classes will be generated from the following two initial parallel classes by $(+2\ mod\ u)$ and $(+1\ mod\ u)$ respectively.

$v = 40$	0 1 2 3	4 6 9	5 7 26	∞_1 8 23	∞_2 10 25
	∞_3 11 16	∞_4 12 29	∞_5 13 22	∞_6 14 27	∞_7 15 18
	∞_8 17 28	∞_9 19 20	∞_{10} 21 24		
	0 3 7 12	1 5 11	2 10 19	∞_1 4 16	∞_2 6 18
	∞_3 8 24	∞_4 9 22	∞_5 13 29	∞_6 14 28	∞_7 15 23
	∞_8 17 25	∞_9 20 26	∞_{10} 21 27		

$v = 46$	0 1 2 3	4 6 9	5 7 8	10 13 16	11 17 34
	∞_1 12 29	∞_2 14 33	∞_3 15 20	∞_4 18 35	∞_5 19 32
	∞_6 21 26	∞_7 22 27	∞_8 23 30	∞_9 24 31	∞_{10} 25 28
	0 4 8 14	1 7 15	2 10 19	3 12 23	11 24 33
	∞_1 5 25	∞_2 6 26	∞_3 9 27	∞_4 13 28	∞_5 16 29
	∞_6 17 32	∞_7 18 30	∞_8 20 35	∞_9 21 31	∞_{10} 22 34

For each block $B \in \{0\,2\,7, 1\,3\,8, 2\,4\,9, 0\,1\,11, 1\,2\,12, 2\,3\,13, 0\,7\,11, 1\,8\,12, 2\,9\,13\}$, we can generate an auxiliary parallel class from B by $(+3 \ mod\ u)$. Thus we obtain 9 auxiliary parallel classes. The last auxiliary parallel class will be generated from the block $\{0, u/3, 2u/3\}$ by $(+1\ mod\ u)$. □

Lemma 2.12 *There exists an IRMP(v, 13) for each $v \in \{55, 67\}$.*

Proof: Take the point set $V = (Z_u \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_{13}\}$, where $u = (v-13)/2$. The required parallel classes will be generated from the following two initial parallel classes by $(+1\ mod\ u,\ -)$ and $(+1\ mod\ u,\ +1\ mod\ 2)$ respectively.

$v = 55$	$0_0 2_0 0_1 1_1$	$10_0 16_0 19_0$	$13_0 14_0 6_1$	$8_0 17_1 19_1$	$9_1 15_1 18_1$
	$\infty_1 1_0 11_1$	$\infty_2 3_0 10_1$	$\infty_3 4_0 7_1$	$\infty_4 5_0 20_1$	$\infty_5 6_0 8_1$
	$\infty_6 7_0 3_1$	$\infty_7 9_0 14_1$	$\infty_8 11_0 2_1$	$\infty_9 12_0 16_1$	$\infty_{10} 15_0 12_1$
	$\infty_{11} 17_0 4_1$	$\infty_{12} 18_0 13_1$	$\infty_{13} 20_0 5_1$		
	$0_0 2_0 4_0 1_1$	$8_0 13_0 0_1$	$11_0 17_0 17_1$	$18_0 7_1 8_1$	$20_0 5_1 11_1$
	$\infty_1 3_0 6_0$	$\infty_2 5_0 14_0$	$\infty_3 9_0 16_0$	$\infty_4 1_0 3_1$	$\infty_5 7_0 16_1$
	$\infty_6 10_0 6_1$	$\infty_7 12_0 14_1$	$\infty_8 15_0 18_1$	$\infty_9 19_0 2_1$	$\infty_{10} 4_1 9_1$
	$\infty_{11} 10_1 19_1$	$\infty_{12} 12_1 15_1$	$\infty_{13} 13_1 20_1$		
$v = 67$	$0_0 2_0 0_1 1_1$	$3_0 12_0 20_0$	$10_0 15_0 16_0$	$7_0 14_0 2_1$	$1_0 13_1 19_1$
	$6_0 8_1 25_1$	$22_0 4_1 9_1$	$23_0 7_1 26_1$	$3_1 10_1 12_1$	$\infty_1 4_0 20_1$
	$\infty_2 5_0 18_1$	$\infty_3 8_0 15_1$	$\infty_4 9_0 14_1$	$\infty_5 11_0 21_1$	$\infty_6 13_0 6_1$
	$\infty_7 17_0 11_1$	$\infty_8 18_0 24_1$	$\infty_9 19_0 23_1$	$\infty_{10} 21_0 17_1$	$\infty_{11} 24_0 5_1$
	$\infty_{12} 25_0 22_1$	$\infty_{13} 26_0 16_1$			
	$0_0 2_0 4_0 1_1$	$1_0 22_0 25_0$	$3_0 18_0 7_1$	$10_0 24_0 12_1$	$8_0 6_1 21_1$
	$14_0 4_1 22_1$	$20_0 5_1 11_1$	$2_1 13_1 18_1$	$8_1 9_1 23_1$	$\infty_1 5_0 23_0$
	$\infty_2 6_0 26_0$	$\infty_3 12_0 15_0$	$\infty_4 16_0 21_0$	$\infty_5 7_0 24_1$	$\infty_6 9_0 3_1$
	$\infty_7 11_0 20_1$	$\infty_8 13_0 10_1$	$\infty_9 17_0 17_1$	$\infty_{10} 19_0 25_1$	$\infty_{11} 0_1 16_1$
	$\infty_{12} 14_1 15_1$	$\infty_{13} 19_1 26_1$			

The fifteen auxiliary parallel classes will be generated from the following five initial blocks by $(+1\ mod\ u,\ +1\ mod\ 2)$.

$v = 55:$	$0_0 4_0 8_0$	$0_0 10_0 5_1$	$0_0 14_0 7_1$	$0_0 8_0 16_0$	$0_0 10_0 20_0$
$v = 67:$	$0_0 4_0 8_0$	$0_0 10_0 5_1$	$0_0 14_0 7_1$	$0_0 8_0 19_1$	$0_0 10_0 23_1$

Note that the leave of the IRMP$(v, 13)$ constructed above consists of $3K_{13}s$ based on the point set $\{\infty_1, \infty_2, \cdots, \infty_{13}\}$. □

Lemma 2.13 *There exists an IRMC*$(16, 4)$.

Proof: Take the point set $V = Z_{12} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Take a auxiliary parallel class generated from the block $\{0, 4, 8\}$ by $(+1\ mod\ 12)$ and copy it. This gives the required two auxiliary parallel classes. The required parallel classes will be generated from the following six initial parallel classes by $(+4\ mod\ 12)$.

P_1 :	0 1 2 3	∞_1 4 5	∞_2 6 7	∞_3 8 9	∞_4 10 11
P_2 :	0 2 5 7	∞_1 1 3	∞_2 4 6	∞_3 8 11	∞_4 9 10
P_3 :	0 3 5 10	∞_1 2 6	∞_2 4 11	∞_3 1 9	∞_4 7 8
P_4 :	0 5 6 11	∞_1 3 8	∞_2 2 9	∞_3 7 10	∞_4 1 4
P_5 :	∞_1 0 6 8	3 7 9	∞_2 1 11	∞_3 2 4	∞_4 5 10
P_6 :	∞_1 1 7 10	0 6 9	∞_2 5 8	∞_3 2 11	∞_4 3 4

Lemma 2.14 *There exists an IRMC*$(28, 4)$.

Proof: Take the point set $V = Z_{24} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Cycling the block $\{0, 8, 16\}$ twice gives the required two auxiliary parallel classes. The required parallel classes will be generated from the following two initial parallel classes by $(+2\ mod\ 24)$ and $(+1\ mod\ 24)$ respectively.

P_1 :	0 1 2 3	4 5 6	7 8 10	9 11 14	12 15 17	∞_1 13 18
	∞_2 16 21	∞_3 19 22	∞_4 20 23			
P_2 :	0 4 8 13	1 5 14	2 9 19	3 15 21	10 16 23	∞_1 6 18
	∞_2 7 17	∞_3 11 20	∞_4 12 22			

Lemma 2.15 *There exists an IRMC*$(v, 4)$ *for each* $v \in \{22, 34\}$.

Proof: Take the point set $V = (Z_u \times Z_2) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, where $u = (v-4)/2$. Cycle the block $\{0_0, (u/3)_0, (2u/3)_0\}$ twice by $(+1\ mod\ u,\ +1\ mod\ 2)$. This gives the required two auxiliary parallel classes. The required parallel classes will be generated from the following two initial parallel classes by $(+1\ mod\ u,\ -)$ and $(+1\ mod\ u,\ +1\ mod\ 2)$ respectively.

$v = 22$	$0_0 2_0 1_1 3_1$	$4_0 6_0 5_1$	$7_0 0_1 2_1$	$\infty_1 1_0 7_1$	$\infty_2 3_0 8_1$	
	$\infty_3 5_0 4_1$	$\infty_4 8_0 6_1$				
	$0_0 2_0 8_0 5_1$	$1_0 5_0 1_1$	$4_0 2_1 6_1$	$\infty_1 6_0 7_0$	$\infty_2 3_0 3_1$	
	$\infty_3 0_1 4_1$	$\infty_4 7_1 8_1$				
$v = 34$	$0_0 2_0 1_1 3_1$	$3_0 12_0 0_1$	$4_0 6_0 5_1$	$7_0 11_0 2_1$	$8_0 10_0 7_1$	$13_0 4_1 10_1$
	$9_1 11_1 13_1$	$\infty_1 1_0 6_1$	$\infty_2 5_0 14_1$	$\infty_3 9_0 12_1$	$\infty_4 14_0 8_1$	
	$0_0 4_0 8_0 13_1$	$5_0 12_0 12_1$	$6_0 14_0 1_1$	$2_0 0_1 6_1$	$3_0 7_1 10_1$	$7_0 11_1 14_1$
	$2_1 3_1 8_1$	$\infty_1 1_0 13_0$	$\infty_2 10_0 11_0$	$\infty_3 9_0 9_1$	$\infty_4 4_1 5_1$	

Lemma 2.16 *There exists an IRMC*(34, 10).

Proof: Take the point set $V = Z_{24} \cup \{\infty_1, \infty_2, \cdots, \infty_{10}\}$. The required parallel classes will be generated from the following three initial parallel classes by (+2 *mod* 24).

P_1 :	0 1 2 3	∞_1 4 6	∞_2 5 7	∞_3 8 11	∞_4 9 10	∞_5 12 15
	∞_6 13 16	∞_7 14 20	∞_8 17 21	∞_9 18 22	∞_{10} 19 23	
P_2 :	0 4 9 12	∞_1 1 6	∞_2 2 7	∞_3 3 8	∞_4 5 15	∞_5 10 19
	∞_6 13 21	∞_7 11 18	∞_8 14 20	∞_9 17 23	∞_{10} 16 22	
P_3 :	0 9 15 21	∞_1 1 11	∞_2 2 12	∞_3 3 14	∞_4 4 16	∞_5 5 18
	∞_6 6 20	∞_7 7 19	∞_8 8 23	∞_9 10 17	∞_{10} 13 22	

For each block $B \in \{0\,2\,7, 1\,3\,8, 2\,4\,9, 0\,1\,11, 1\,2\,12, 2\,3\,13, 0\,7\,11, 1\,8\,12, 2\,9\,13\}$, we can generate an auxiliary parallel class from B by (+3 *mod* 24). Thus we obtain 9 auxiliary parallel classes. The last two auxiliary parallel classes will be generated from the block $\{0, 8, 16\}$ by (+1 *mod* 24). □

Lemma 2.17 *There exists an IRMC*(82, 16).

Proof: Take the point set $V = (Z_{33} \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_{16}\}$. The required parallel classes will be generated from the following two initial parallel classes by (+1 *mod* 33, −) and (+1 *mod* 33, +1 *mod* 2) respectively.

P_1 :	$2_0 6_0 1_1 3_1$	$3_0 8_0 20_0$	$7_0 14_0 17_0$	$10_0 30_0 24_1$	$26_0 32_0 6_1$	$27_0 29_0 17_1$
	$16_0 21_1 28_1$	$22_0 8_1 14_1$	$23_0 27_1 31_1$	$5_1 10_1 26_1$	$9_1 12_1 22_1$	$\infty_1 0_0 11_1$
	$\infty_2 1_0 23_1$	$\infty_3 4_0 30_1$	$\infty_4 5_0 29_1$	$\infty_5 9_0 25_1$	$\infty_6 11_0 7_1$	$\infty_7 12_0 18_1$
	$\infty_8 13_0 16_1$	$\infty_9 15_0 32_1$	$\infty_{10} 18_0 20_1$	$\infty_{11} 19_0 4_1$	$\infty_{12} 21_0 19_1$	$\infty_{13} 24_0 0_1$
	$\infty_{14} 25_0 2_1$	$\infty_{15} 28_0 15_1$	$\infty_{16} 31_0 13_1$			
P_2 :	$2_0 1_1 3_1 5_1$	$8_0 20_0 32_0$	$1_0 16_0 22_1$	$5_0 25_0 17_1$	$10_0 24_0 6_1$	$12_0 13_0 21_1$
	$19_0 28_0 28_1$	$11_0 11_1 27_1$	$14_0 0_1 18_1$	$2_1 15_1 20_1$	$4_1 9_1 23_1$	$\infty_1 0_0 27_0$

$\infty_2 7_0 18_0$	$\infty_3 21_0 30_0$	$\infty_4 22_0 23_0$	$\infty_5 26_0 29_0$	$\infty_6 3_0 13_1$	$\infty_7 4_0 10_1$
$\infty_8 6_0 8_1$	$\infty_9 9_0 7_1$	$\infty_{10} 15_0 25_1$	$\infty_{11} 17_0 14_1$	$\infty_{12} 31_0 16_1$	$\infty_{13} 12_1 19_1$
$\infty_{14} 24_1 30_1$	$\infty_{15} 26_1 29_1$	$\infty_{16} 31_1 32_1$			

Cycle the block $\{0_0, 11_0, 22_0\}$ twice by $(+1\ mod\ 33,\ +1\ mod\ 2)$. This gives two auxiliary parallel classes. The other 18 auxiliary parallel classes will be generated from 6 initial blocks $B \in \{0_0 4_0 11_1, 0_0 10_0 5_1, 0_0 7_1 17_1, 0_0 8_0 16_0, 0_0 8_0 19_1, 0_0 26_0 13_1\}$ by $(+1\ mod\ 33,\ +1\ mod\ 2)$. □

3 Direct constructions for small orders

In this section, we construct RMPs and RMCs with small orders, some of which will be used as input designs in recursive constructions of Section 4.

Lemma 3.1 *There exists an RMP(7) and an RMC(7).*

Proof: Take the point set $V = Z_7$. The required 7 parallel classes will be generated from an initial parallel class $\{0,1,2,5\}$, $\{3,4,6\}$ by $(+1\ mod\ 7)$. Note that the leave of this RMP(7) is an empty set. So, it is also an RMC(7). □

Lemma 3.2 *There exists an RMP(v) for each $v \in \{10, 28\}$.*

Proof: Take the point set $V = Z_u \cup \{\infty\}$, where $u = v - 1$. Some of the required parallel classes will be generated from the following initial parallel classes by $(+3\ mod\ u)$.

$v = 10$	1 2 4 8	3 5 6	∞ 0 7			
	1 3 6 7	0 4 5	∞ 2 8			
	0 2 3 8	1 5 7	∞ 4 6			
$v = 28$	0 3 7 10	1 5 9	2 4 8	6 11 18	12 13 20	14 23 24
	15 17 25	16 21 26	∞ 19 22			
	0 6 13 18	1 8 14	2 7 12	3 11 22	4 16 26	5 17 19
	9 20 23	10 21 25	∞ 15 24			
	0 1 2 5	3 4 6	7 8 9	10 11 16	12 23 24	14 22 25
	15 18 26	17 19 21	∞ 13 20			
	0 4 13 23	1 7 15	2 17 20	3 5 19	6 12 22	8 14 21
	10 16 25	11 18 24	∞ 9 26			

Other parallel classes for $v = 10$ and 28 are listed below.

$v = 10$:	∞ 0 1 2	3 4 5	6 7 8			
	∞ 3 4 8	0 1 5	2 6 7			
$v = 28$:	∞ 2 11 20	0 9 18	1 10 19	3 12 21	4 13 22	5 14 23
	6 15 24	7 16 25	8 17 26			

Lemma 3.3 *There exists an RMP*(12).

Proof: Take the point set $V = Z_{12}$. The following 5 initial parallel classes $P_1, P_2, \cdots, P_5$ will generate 15 parallel classes by $(+4 \ \ mod\ 12)$. The last parallel class contains the following 4 blocks: 0 4 8, 1 5 9, 2 6 10, 3 7 11.

P_1 :	0 1 2	3 4 5	6 7 8	9 10 11
P_2 :	0 1 2	3 4 7	5 8 10	6 9 11
P_3 :	0 3 6	1 7 9	2 4 8	5 10 11
P_4 :	0 3 7	1 5 10	2 6 8	4 9 11
P_5 :	0 4 9	1 7 10	2 6 11	3 5 8

Lemma 3.4 *There exists an RMP*(13).

Proof: Take the point set $V = Z_{12} \cup \{\infty\}$. The required 15 parallel classes will be generated from the following five initial parallel classes by $(+4\ mod\ 12)$.

P_1 :	0 1 2	3 4 5	6 7 8	∞ 9 10 11
P_2 :	0 1 2	3 4 7	5 8 10	∞ 6 9 11
P_3 :	0 3 5 8	1 6 9	2 7 11	∞ 4 10
P_4 :	0 3 5 9	1 6 10	2 7 11	∞ 4 8
P_5 :	0 6 7 10	1 3 9	2 4 8	∞ 5 11

Lemma 3.5 *There exists an RMP(v) for each $v \in \{16, 22\}$.*

Proof: Take the point set $V = (Z_u \times Z_3) \cup \{\infty\}$, where $u = (v-1)/3$. The required $4u$ parallel classes will be generated from the following four initial parallel classes by $(+1\ mod\ u,\ -)$.

$v = 16$	$1_0 3_0 4_1$	$4_0 0_1 3_1$	$0_0 1_1 2_2$	$2_1 0_2 3_2$	$\infty\ 2_0 1_2 4_2$
	$0_0 1_0 3_0 2_2$	$4_0 1_1 4_1$	$2_0 0_1 0_2$	$2_1 1_2 4_2$	$\infty\ 3_1 3_2$
	$0_0 3_1 4_1 3_2$	$1_0 2_0 2_2$	$3_0 0_1 1_2$	$2_1 0_2 4_2$	$\infty\ 4_0 1_1$
	$0_1 1_1 4_1 3_2$	$3_0 4_0 3_1$	$0_0 0_2 4_2$	$1_0 1_2 2_2$	$\infty\ 2_0 2_1$

$v = 22$	$1_0 2_0 5_0$	$4_0 3_1 6_1$	$0_0 1_1 2_2$	$3_0 4_1 5_2$	$6_0 0_1 1_2$	$2_1 5_1 4_2$	$\infty\ 0_2 3_2 6_2$
	$0_0 2_0 3_0 0_1$	$6_0 2_1 6_2$	$4_0 0_2 2_2$	$5_0 4_2 5_2$	$1_1 3_1 4_1$	$5_1 1_2 3_2$	$\infty\ 1_0 6_1$
	$0_0 6_0 4_1 4_2$	$1_0 3_0 0_1$	$2_0 5_1 3_2$	$5_0 2_2 6_2$	$1_1 3_1 5_2$	$2_1 0_2 1_2$	$\infty\ 4_0 6_1$
	$1_1 2_1 3_1 5_2$	$4_0 6_0 2_2$	$1_0 0_1 0_2$	$2_0 4_1 3_2$	$5_0 5_1 1_2$	$0_0 4_2 6_2$	$\infty\ 3_0 6_1$

Lemma 3.6 *There exists an RMP*(19).

Proof: Take the point set $V = Z_{18} \cup \{\infty\}$. The required 24 parallel classes will be generated from the following two initial parallel classes P and Q by (+1 *mod* 18) and (+3 *mod* 18) respectively.

P:	1 2 4 7	0 6 14	3 9 13	5 12 15	8 10 17	∞ 11 16
Q:	∞ 1 2 3	0 4 5	6 13 17	7 12 14	8 9 10	11 15 16

Lemma 3.7 *There exists an RMC(v) for each* $v \in \{10, 13, 16, 22\}$.

Proof: By Lemma 3.5, there exists an RMP(16) whose leave is an empty set, then it is also an RMC(16). For the other three values, we can obtain an RMC(v) by adding a new parallel class P to the RMP(v) constructed in Lemma 3.2, Lemma 3.4 and Lemma 3.5. The blocks in P are listed below.

$v = 10$:	∞ 5 6 7	0 1 2	3 4 8				
$v = 13$:	∞ 2 6 10	0 5 7	4 9 11	1 3 8			
$v = 22$:	$\infty\ 0_0 0_1 0_2$	$1_0 1_1 1_2$	$2_0 2_1 2_2$	$3_0 3_1 3_2$	$4_0 4_1 4_2$	$5_0 5_1 5_2$	$6_0 6_1 6_2$

Lemma 3.8 *There exists an RMC*(12).

Proof: Take the point set $V = Z_{12}$. The following 4 initial parallel classes P_1, P_2, P_3, P_4 will generate 12 parallel classes by (+4 *mod* 12). Q_1 will generate 4 parallel classes by (+3 *mod* 12). The last parallel class will be generated from the block $\{0, 4, 8\}$ by (+3 *mod* 12).

P_1 :	0 1 2	3 4 7	5 8 10	6 9 11
P_2 :	0 1 6	2 7 8	3 5 9	4 10 11
P_3 :	0 3 5	1 7 10	2 6 8	4 9 11
P_4 :	0 4 9	1 2 7	3 6 8	5 10 11
Q_1 :	0 9 10	1 2 4	3 7 11	5 6 8

Lemma 3.9 *There exists an RMC*(19).

Proof: Take the point set $V = Z_{18} \cup \{\infty\}$. Cycle the block $\{0, 6, 12\}$ by $(+1 \ mod \ 18)$ and add ∞ to the block $\{0, 6, 12\}$. This gives a parallel class. The other 24 parallel classes will be generated from the following 4 initial parallel classes by $(+3 \ mod \ 18)$.

P_1 :	0 1 2	3 4 5	6 7 8	9 12 15	10 13 16	∞ 11 14 17
P_2 :	0 3 7 9	1 4 8	2 5 10	6 14 16	11 12 17	∞ 13 15
P_3 :	0 4 8 13	1 11 12	2 7 9	3 10 17	5 14 15	∞ 6 16
P_4 :	0 4 11 13	1 6 17	2 10 16	3 8 12	5 9 14	∞ 7 15

Lemma 3.10 *There exists an RMC*(28).

Proof: Take the point set $V = Z_{27} \cup \{\infty\}$. The required 38 parallel classes will be generated as follows. For each block $B \in \{0\ 1\ 2,\ 3\ 7\ 11\}$, cycle the block B by $(+3 \ mod \ 27)$ and add ∞ to B to form a new block of size 4. Thus we get two parallel classes. The other 36 parallel classes will be generated from the following two initial parallel classes P_1 and P_2 by $(+3 \ mod \ 27)$ and $(+1 \ mod \ 27)$ respectively.

P_1 :	0 3 9	1 4 15	2 8 17	5 6 18	10 16 25	11 14 19	13 21 26
	20 22 24	∞ 7 12 23					
P_2 :	0 1 2 4	3 6 12	5 13 20	7 17 21	8 18 24	9 14 22	10 19 26
	11 16 23	∞ 15 25					

Lemma 3.11 *There exists an RMC*(v) *for each* $v \in \{34, 40, 58\}$.

Proof: Take the point set $V = (Z_u \times Z_3) \cup \{\infty\}$, where $u = (v-1)/3$. Similar to the construction in Lemma 3.10, all these required parallel classes consist of three parts. For each block $B \in Q$, cycle the block B by $(+1 \ mod \ u, \ -)$. Thus we can get $n(v) - 4u$ parallel classes by adding ∞ to B to form a new block of size 4. The other $4u$ parallel classes will be generated from the following two initial parallel classes P_1 and P_2 by $(+1 \ mod \ u, \ -)$ and $(+1 \ mod \ u, \ +1 \ mod \ 3)$ respectively. The blocks in P_1, P_2 and Q are listed below.

P_1 :

$v = 34$	$0_0 3_0 9_0$	$4_0 5_0 4_2$	$6_0 1_1 4_1$	$7_0 6_1 7_1$	$8_0 0_1 2_1$	$1_0 2_2 7_2$
	$10_0 3_2 6_2$	$3_1 8_1 0_2$	$5_1 1_2 10_2$	$9_1 8_2 9_2$	$\infty\ 2_0 10_1 5_2$	

$v = 40$	$6_0 10_0 11_0$	$3_0 9_0 1_1$	$1_0 4_0 7_2$	$2_0 11_1 12_1$	$8_0 2_1 10_1$
	$5_0 8_1 6_2$	$0_0 2_2 8_2$	$12_0 9_2 10_2$	$0_1 4_1 7_1$	$3_1 0_2 5_2$
	$5_1 1_2 4_2$	$9_1 3_2 12_2$	$\infty\ 7_0 6_1 11_2$		
$v = 58$	$1_0 14_1 8_2$	$2_0 1_1 2_2$	$3_0 17_1 7_2$	$4_0 12_1 12_2$	$5_0 15_1 18_2$
	$6_0 10_1 17_2$	$7_0 8_1 16_2$	$8_0 4_1 6_2$	$9_0 16_1 11_2$	$10_0 2_1 1_2$
	$11_0 13_1 4_2$	$12_0 9_1 13_2$	$13_0 11_1 9_2$	$14_0 7_1 0_2$	$15_0 5_1 10_2$
	$16_0 0_1 15_2$	$17_0 3_1 14_2$	$18_0 18_1 5_2$	$\infty 0_0 6_1 3_2$	
P_2 :					
$v = 34$	$0_0 1_1 4_1 2_2$	$1_0 7_0 9_1$	$3_0 6_0 10_1$	$5_0 9_0 0_1$	$8_0 10_0 3_2$
	$2_0 2_1 5_2$	$4_0 9_2 10_2$	$3_1 7_1 6_2$	$6_1 8_1 8_2$	$0_2 1_2 7_2$
	$\infty\ 5_1 4_2$				
$v = 40$	$0_0 1_1 4_1 2_2$	$1_0 2_0 8_1$	$5_0 10_0 5_1$	$4_0 8_0 5_2$	$6_0 2_1 11_1$
	$12_0 6_1 7_1$	$3_0 9_1 11_2$	$7_0 4_2 10_2$	$9_0 7_2 9_2$	$0_1 3_1 12_2$
	$10_1 12_1 3_2$	$0_2 6_2 8_2$	$\infty\ 11_0 1_2$		
$v = 58$	$3_0 1_1 2_2 5_2$	$7_0 10_0 11_0$	$4_0 5_0 17_1$	$2_0 16_0 0_2$	$6_0 14_0 10_2$
	$8_0 12_0 1_2$	$9_0 3_1 16_1$	$13_0 6_1 18_1$	$15_0 11_1 14_1$	$18_0 5_1 9_1$
	$17_0 7_2 17_2$	$2_1 8_1 10_1$	$0_1 13_1 3_2$	$7_1 15_1 12_2$	$4_1 13_2 18_2$
	$12_1 4_2 14_2$	$6_2 8_2 15_2$	$9_2 11_2 16_2$	$\infty\ 0_0 1_0$	
Q :					
$v = 34$	$0_0 1_1 2_2$	$3_0 7_1 0_2$	$9_0 5_1 3_2$		
$v = 40$	$0_0 1_1 5_2$	$12_0 7_1 8_2$	$2_0 6_1 1_2$	$3_0 3_1 3_2$	
$v = 58$	$0_0 6_1 5_2$	$6_0 1_1 7_2$	$8_0 7_1 2_2$	$9_0 12_1 1_2$	$2_0 10_1 13_2$
	$14_0 3_1 11_2$	$4_0 4_1 4_2$			

Lemma 3.12 *There exists an $RMC(46)$.*

Proof: Take the point set $V = Z_{45} \cup \{\infty\}$. For each block $B \in \{0\ 13\ 29,\ 1\ 14\ 30,\ 2\ 15\ 31\}$, cycle the block B by $(+3\ mod\ 45)$ and add ∞ to B to form a new block of size 4. Thus we obtain 3 parallel classes. Further, cycle the block $\{0, 15, 30\}$ to get 15 blocks. Add ∞ to the block $\{3, 18, 33\}$ or $\{6, 21, 36\}$ respectively. This gives 2 parallel classes. The other 60 parallel classes can be generated from the following two parallel classes P_1 and P_2 by $(+3\ mod\ 45)$ and $(+1\ mod\ 45)$ respectively.

P_1 :	0 1 3	2 4 5	9 10 11	12 14 15	13 16 18	17 19 21
	20 22 25	23 26 36	24 29 30	27 34 40	28 32 38	31 37 44
	33 39 43	35 41 42	∞ 6 7 8			
P_2 :	0 3 7 11	18 23 43	1 15 29	10 22 34	6 16 32	19 37 42
	17 28 38	4 13 24	5 20 41	8 25 31	2 30 39	9 27 35
	12 26 44	21 33 40	∞ 14 36			

4 Main results

In this section, we shall prove Theorem 1.4. For our purpose we need the following constructions.

Construction 4.1 *If there exists an IRMP(v,h) (resp. IRMC(v,h)) and an RMP(h) (resp. RMC(h)), then an RMP(v) (resp. RMC(v)) exists.*

Lemma 4.2 *There exists an RMP(v) for each $v \in \{25,31,34,37,40,46,49,55,61,67\}$.*

Proof: From Lemma 2.7 to Lemma 2.12, we know that there exists an IRMP$(v,7)$ for each $v \in \{25,31,37,49,61\}$, an IRMP$(v,10)$ for each $v \in \{34,40,46\}$ and an IRMP$(v,13)$ for each $v \in \{55,67\}$. Using Construction 4.1, we obtain the required RMP. The input designs RMP(7), RMP(10), RMP(13) exist by Lemma 3.1, Lemma 3.2 and Lemma 3.4. □

Lemma 4.3 *There exists an RMC(82).*

Proof: By Lemma 2.17 and Lemma 3.7, there exists an IRMC(16) and an RMC(16). Then there exists an RMC(82) by Construction 4.1. □

Construction 4.4 *If there exist*

1. *a $(3,3)$-frame of type $g_1, g_2 \cdots g_u$, $g_i \equiv g_j (mod\ 6)$, $1 \le i < j \le u$,*
2. *IRMP$(3,3;g_i+h,h)$s (resp. IRMC$(3,3;g_i+h,h)$s) for $1 \le i \le u-1$,*
3. *an RMP$(3,3,g_u+h)$ (resp. RMC$(3,3,g_u+h)$).*

Then an RMP$(3,3,\sum_{i=1}^{u} g_i + h)$ (resp. RMC$(3,3,\sum_{i=1}^{u} g_i + h)$) exists.

Proof: For $1 \le i < u$, there are $3g_i/2$ holey parallel classes missing the group of size g_i, and the same number of parallel classes in the IRMP$(3,3;g_i+h,h)$; match them up arbitrarily, placing the g_i points of the IRMP on the ith group of the frame and the h points in its hole on h new points.

Next, each IRMP contains $3(h-a)/2$ auxiliary parallel classes of triples. From unions of these with $3(h-a)/2$ parallel classes of the RMP$(3,3,g_u+h)$, to form $3(h-a)/2$ additional parallel classes. There remain $3g_u/2$ parallel classes of the RMP$(3,3,g_u+h)$, which can be matched arbitrarily with the $3g_u/2$ holey parallel classes of the uth group to complete the construction.

It is easy to check that this construction gives an RMP$(3,3,\sum_{i=1}^{u} g_i+h)$. The proof of RMC is similar to RMP. □

We can also use some known results about RMP$(3,1,v)$s (resp. RMC $(3,1,v)$s) and simple (3,2)-frames to construct RMP$(3,3,v)$s (resp. RMC $(3,3,v)$s).

Construction 4.5 *If there exists an RMP$(3,1,3t)$ (resp. RMC $(3,1,3t)$) with $\lfloor\frac{3t-1}{2}\rfloor$ (resp. $\lfloor\frac{3t}{2}\rfloor$) parallel classes and a simple $(3,2)$-frame of type 3^t, then an RMP$(3,3,3t)$ (resp. RMC $(3,3,3t)$) exists.*

Proof: Start from an RMP$(3,1,3t)$ with $\lfloor\frac{3t-1}{2}\rfloor$ parallel classes. Take a parallel class P from them arbitrarily and construct a simple $(3,2)$-frame of type 3^t whose groups are these t blocks in P. For every group, there are 3 holey parallel classes. Match them up with the corresponding hole. Thus we have 3t new parallel classes. Now we get $3t-1+\lfloor\frac{3t-1}{2}\rfloor$ parallel classes. Clearly, each pair of points occurs at most 3 times in the blocks, and the leave is the same as that in the beginning RMP$(3,1,3t)$, so it is a simple graph. It is easy to check that these parallel classes are mutually distinct. So this construction gives an RMP$(3,3,3t)$.

Similarly, we can get an RMC$(3,3,3t)$ by using an RMC$(3,1,3t)$ and a simple (3,2)-frame of type 3^t. □

A resolvable 3-GDD of type 1^v is called a *Kirkman triple system* and denoted by $KTS(v)$. Two $KTS(v)$s on the same set X are said to be *disjoint* if there is no common block. A set of $v-2$ pairwise disjoint $KTS(v)$s is called a *large set* of Kirkman triple systems, briefly an LKTS(v). LKTS(v) can also be used to construct some kinds of RMP and RMC.

Construction 4.6 *Suppose $v \equiv 3 \pmod 6$, if there exists an $LKTS(v)$, then an RMP$(3,\lambda,v)$ and an RMC$(3,\lambda,v)$ exists for any $1 \le \lambda \le v-2$.*

For our main results, we also need some frames with different group sizes. To construct these frames, we need the notation of PBD. A *pairwise balanced design* (v, K)-PBD is a K-GDD of type 1^v. From [5], we have the following known results.

Lemma 4.7 *There exists a* $(v, \{4,5,6\})$*-PBD for each integer* $v \geq 4$ *except for* $v \in \{7-12, 14, 15, 18, 19, 23, 47\}$.

Lemma 4.8 *For each* $t \geq 13$, $t \notin \{14, 15, 18, 19, 23, 47\}$, *there exists a simple* $(3,3)$*-frame of type* $18^a 24^b 30^c$, *where* $v = 6(t-1) = 18a + 24b + 30c$, $a, b, c \geq 0$.

Proof: By Lemma 4.7, we have a $(t, \{4,5,6\})$-PBD based on $\mathcal{X}$ for every positive integer $t \geq 13$ except for $t \in \{14, 15, 18, 19, 23, 47\}$. Deleting a point from $\mathcal{X}$, then we get a $\{4,5,6\}$-GDD of type $3^a 4^b 5^c$ for certain $a, b, c \geq 0$. Applying Lemma 2.2 with $h = 6$, we get the required simple $(3,3)$-frame, the input simple $(3,3)$-frames of type 6^u exist by Theorem 2.1. □

To get more frames we also need the following results on 4-GDD of type $g^4 m^1$. From Rees [11] and [17] we have:

Theorem 4.9 *There exists a 4-GDD of type* $g^4 m^1$ *with* $m > 0$ *if and only if* $g \equiv m \equiv 0 \pmod 3$ *and* $0 < m \leq 3g/2$.

Lemma 4.10 *If* $g \equiv m \equiv 0 \pmod 6$ *and* $0 < m \leq 3g/2$, *then there exists a simple (3,3)-frame of type* $g^4 m^1$.

Proof: Since $g \equiv m \equiv 0 \pmod 6$ and $0 < m \leq 3g/2$, then $g/2 \equiv m/2 \equiv 0 \pmod 3$ and $0 < m/2 \leq 3g/4$. From Theorem 4.9, there exists a 4-GDD of type $(\frac{g}{2})^4 (\frac{m}{2})^1$. Applying Theorem 2.1 and Lemma 2.2 with $h = 2$, we get the required simple (3,3)-frame of type $g^4 m^1$. □

Theorem 4.11 *There exists an RMP(v) and an RMC(v) with n(v) parallel classes for each* $v \equiv 0 \pmod 3$, $v \geq 9$.

Proof: By [14, 18, 19], there exists an RMP$(3,1,3t)$ with $\lfloor\frac{3t-1}{2}\rfloor$ parallel classes when $t \neq 2,4$. By Theorem 2.1, there is a simple (3,2)-frame of type 3^t when $t \geq 4$. Applying Construction 4.5, we obtain an RMP(v) for each $v \equiv 0 \pmod 3$, $v \geq 15$. From Lemma 3.3, we get an RMP(12). From [13], we have an LKTS(9). Applying Construction 4.6 we get an RMP(9). So there is an RMP(v) for $v \equiv 0 \pmod 3$, $v \geq 9$. The proof of RMC is similar to RMP. □

Deleting a point from these designs constructed above, we get the following lemma.

Theorem 4.12 *There exists an RMP(v) and an RMC(v) with $n(v)$ parallel classes for each $v \equiv 2 \pmod 3$ and $v \geq 8$.*

Lemma 4.13 *There exists an RMP(v) and an RMC(v) for each $v \equiv 1 \pmod 6$, $v \geq 79$ and $v \notin \{85,91,109,115,139,283\}$.*

Proof: Let $v_1 = v - 7$. By Lemma 4.8, there exists a simple (3,3)-frame of type $18^a 24^b 30^c$, where $v_1 = 18a + 24b + 30c \notin \{78,84,102,108,132,276\}$. Apply Construction 4.4 with $h = 7$. The input designs IRMP$(25,7)$, IRMP$(31,7)$, IRMP$(37,7)$ and RMPs exist by Lemmas 2.7- 2.9 and 4.2. Thus we obtain the required RMP(v).

It is easy to see that the leave of the RMP(v) consists of 9 edges which form a parallel class H of the hole. Furthermore, we partition these v_1 points into $v_1/3$ triples which form an auxiliary parallel class Q. Let $P = H \cup Q$. It is a parallel class on v points. Adding P to the RMP constructed above, we obtain an RMC(v). □

Lemma 4.14 *There exists an RMP(v) for each $v \equiv 1 \pmod 6$ and $v \in \{7-73,85,91,109,115,139,283\}$.*

Proof: The order $v = 7,13,19$ comes from Lemmas 3.1, 3.4, 3.6, respectively. By Lemma 4.2, there exists an RMP(v) for each $v \in \{25,31,37,49,$

$55, 61, 67\}$. From Theorem 2.1, we have simple (3,3)-frames of type 10^u, $u \in \{4, 7, 28\}$. Apply Construction 4.4 with $h = 3$ to fill in holes using IRMP$(13, 3)$ and RMP(13) from Lemma 2.3 and Lemma 3.4. Thus we get an RMP(v) for every $v \in \{43, 73, 283\}$. Similarly, from Lemma 4.10, we have simple (3,3)-frames of type a^4b^1, $a \in \{18, 24, 30\}$ and $b \in \{6, 12\}$. Applying Construction 4.4 with $h = 7$, we obtain an RMP(v) for $v \in \{85, 91, 109, 115, 139\}$. Here the input designs IRMP$(a+7, 7)$ and RMP$(b+7)$ come from Lemmas 2.7- 2.9. This completes the proof. □

Lemma 4.15 *There exists an RMP(v) for each $v \equiv 4 \pmod 6$, $v \geq 76$ and $v \notin \{82, 88, 106, 112, 136, 280\}$.*

Proof: Let $v_1 = v - 4$. By Lemma 4.8, there exists a simple (3,3)-frame of type $18^a24^b30^c$, where $v_1 = 18a + 24b + 30c \notin \{78, 84, 102, 108, 132, 276\}$. Apply Construction 4.4 with $h = 4$. The input designs IRMP$(22, 4)$, IRMP$(28, 4)$, IRMP$(34, 4)$ and RMPs exist by Lemmas 2.5, 2.6, 3.2, 3.5 and 4.2. Thus we obtain the required RMP(v). □

Lemma 4.16 *There exists an RMP(v) for each $v \equiv 4 \pmod 6$ and $v \in \{10 - 70, 82, 88, 106, 112, 136, 280\}$.*

Proof: The order $v \in \{10, 16, 22, 28, 34, 40, 46\}$ comes from Lemmas 3.2, 3.5 and 4.2. From Theorem 2.1, we have simple (3,3)-frames of type 12^u, $u \in \{4, 5\}$. Apply Construction 4.4 with $h = 4$ to fill in holes using IRMP$(16, 4)$ and RMP(16) from Lemma 2.4 and Lemma 3.5. Then we get an RMP(v) for each $v \in \{52, 64\}$. Further, we have simple (3,3)-frames of type a^4b^1, $a \in \{12, 18, 24, 30\}$ and $b \in \{6, 12, 18\}$ by Lemma 4.10. Applying Construction 4.4 with $h = 4$, we get an RMP(v) for each $v \in \{58, 70, 82, 88, 106, 112, 136\}$. Here the input designs IRMP$(a + 4, 4)$ and RMP$(b + 4)$ exist by Lemmas 2.4- 2.6, 3.5 and 3.2. Deleting a point from a 5-GDD of type 8^6 ([9]), we get a $\{5, 8\}$-GDD of type $4^{10}7^1$. Delete a point from the group of size 7. This gives a $\{4, 5, 7, 8\}$-GDD of type $4^{10}6^1$.

Applying Theorem 2.1 and Lemma 2.2 with $h = 6$, we get a simple (3,3)-frame of type $24^{10}36^1$. Apply Construction 4.4 with $h = 4$ to fill in holes using IRMP(28, 4) and RMP(40) from Lemma 2.5 and Lemma 4.2. Thus we get an RMP(280). The proof is complete. □

Combining Lemmas 4.13- 4.16, we have the following theorem.

Theorem 4.17 *There exists an RMP(v) with $n(v)$ parallel classes for each $v \equiv 1 \pmod 3$ and $v \geq 7$.*

Lemma 4.18 *There exists an RMC(v) for each $v \equiv 1 \pmod 6$, $v \in \{7-37, 49-67\}$.*

Proof: An RMC(7) exists by Lemma 3.1. The order $v = 13, 19$ comes from Lemma 3.7 and Lemma 3.9 respectively. By Lemmas 2.7- 2.9 and Lemmas 2.12, there exists an IRMP(v, 7) for each $v \in \{25, 31, 37, 49, 61\}$ and an IRMP(v, 13) for $v = 55, 67$. Partition these points which are not in the hole arbitrarily to form an auxiliary parallel class. Add this auxiliary parallel class to the known IRMPs. Thus we get an IRMC(v, 7) for each $v \in \{25, 31, 37, 49, 61\}$ and an IRMC(v, 13) for $v = 55, 67$. Using Construction 4.1 to fill in holes with an RMC(7) or an RMC(13), we obtain the required RMC. □

Lemma 4.19 *There exists an RMC(v) for $v \in \{43, 73, 85, 91, 109, 115, 139, 283\}$.*

Proof: From Theorem 2.1, we have simple (3,3)-frames of type 10^u, $u \in \{4, 7, 28\}$. Apply Construction 4.4 with $h = 3$ to fill in holes using IRMP (13, 3). This gives an IRMP($10u+3$, 13). Similar to the proof of Lemma 4.18, we get an RMC(v) for every $v \in \{43, 73, 283\}$. Similarly, from Lemma 4.10, we have simple (3,3)-frames of type a^4b^1, $a \in \{18, 24, 30\}$ and $b \in \{6, 12\}$. Applying Construction 4.4 with $h = 7$, we obtain an IRMP($v, b+7$) for $v \in \{85, 91, 109, 115, 139\}$. Here the input designs IRMP($a+7$, 7) come from

Lemmas 2.7- 2.9. Then we obtain an RMC(v) for $v \in \{85, 91, 109, 115, 139\}$. This completes the proof. □

The proof of the following lemma is similar to the proof of Lemma 4.15. Here, the input designs IRMC(22, 4), IRMC(28, 4), IRMC(34, 4) and RMCs exist by Lemmas 2.14 and 2.15, Lemmas 3.10 and 3.11 and Lemma 3.7.

Lemma 4.20 *There exists an RMC(v) for each $v \equiv 4 \pmod 6$, $v \geq 76$ and $v \notin \{82, 88, 106, 112, 136, 280\}$.*

Lemma 4.21 *There exists an RMC(v) for each $v \equiv 4 \pmod 6$, $v \in \{10-70, 82, 88, 106, 112, 136, 280\}$.*

Proof: By Lemmas 3.10- 3.12 and Lemma 3.7, there exists an RMC(v) for every $v \in \{10, 16, 22, 28, 34, 40, 46, 58\}$. From Theorem 2.1, we have simple (3,3)-frames of type 12^u, $u \in \{4, 5, 7, 9, 11, 23\}$. Apply Construction 4.4 with $h = 4$ to fill in holes with an RMC(16) and an IRMC(16, 4) from Lemma 2.13. Then we get an RMC(v) for $v \in \{52, 64, 88, 112, 136, 280\}$. Similarly, start from a simple (3,3)-frame of type 24^4. Applying Construction 4.4 with $h = 10$ to fill in holes with an RMC(34) and an IRMC(34, 10) from Lemma 2.16, we get an RMC(106). An RMC(70) can be obtained from a simple (3,3)-frame of type $12^4 18^1$ from Lemma 4.10. By Lemma 2.17, there exists an IRMC(82, 16). Apply Construction 4.1 with $h = 16$ to fill in hole with an RMC(16). Then we get an RMC(82). This completes the proof. □

Combining Lemma 4.18 to Lemma 4.21, we have the following theorem.

Theorem 4.22 *There exists an RMC(v) with $n(v)$ parallel classes for each $v \equiv 1 \pmod 3$ and $v \geq 7$.*

It is easy to see that there doesn't exist an RMP(3, 3, v) and an RMC(3, 3, v) for $v = 2, 3, 4$. Now we consider the case $v = 5, 6$.

Lemma 4.23 *There doesn't exist an RMP*(6).

Proof: Let the point set $V = Z_6$. It is easy to see that $5 \leq m(6) \leq 7$. We distinguish 3 cases.

Case 1. $m(6) = 5$.

In this case, the RMP(6) is an RB(6, 3, 2) indeed. But by [5], this design doesn't exist, then $m(6) \neq 5$.

Case 2. $m(6) = 6$.

In this case, the leave is a 3-regular graph. Without loss of generality, suppose these three edges $\{0,2\}, \{0,3\}, \{0,4\}$ are in the leave. Now we consider these blocks which contain the point 0. Furthermore, we distinguish 2 cases as follow, the other cases must isomorphic to one of them.

(1) If the blocks containing $\{0,1\}$ are $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,1,4\}$, then the other three blocks containing the point 0 must contain the point 5 also. So, the edge $\{1,5\}$ doesn't appear in any block. That is a contradiction.

(2) If the blocks contain $\{0,1\}$ are $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,1,5\}$, since the edge $\{0,4\}$ must be contained in two of the other three blocks containing the point 0, so there is at most one block containing the edge $\{1,4\}$. This is a contradiction.

Case 3. $m(6) = 7$.

It is easy to prove that the leave in this case is a 1-factor. Suppose it consists of three edges $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$. Now we consider these blocks containing the point 0. similarly to Case 2, we distinguish 2 cases as follow.

(1) If the blocks containing $\{0,1\}$ are $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,1,4\}$, then in the other parallel classes $0,1$ must appear in distinct blocks. Since the edge $\{0,5\}$ must be contained in three blocks, so there is at most one block containing the edge $\{1,5\}$. This is a contradiction.

(2) If the blocks containing $\{0,1\}$ are $\{0,1,2\}$, $\{0,1,4\}$ and $\{0,1,5\}$, then in the other parallel classes $0,1$ must appear in distinct blocks. Since there exactly two blocks contain the edge $\{0,4\}$, so the edge $\{1,4\}$ must be contained in three blocks. But the edge $\{1,4\}$ is in the leave, this is a contradiction.

So there doesn't exist an RMP(6). □

Lemma 4.24 (1) *There doesn't exist an RMP(5) with* $n(5) = 7$ *parallel classes.* (2) *There exists an RMP(5) with 6 parallel classes.*

Proof: Let $V = Z_5$. Suppose there are seven parallel classes in an RMP(5), then the leave consists of two disjoint edges. So there is a point who does not appear in the leave. Suppose this point is 0. It is not difficult to prove that the point 0 must appear in five blocks of size three and two blocks of size two. Without loss of generality, suppose these two blocks of size two are $\{0,1\}$ and $\{0,2\}$. Then each of the two edges $\{0,1\}$ and $\{0,2\}$ must be contained in the other five blocks twice. We distinguish 2 cases. (1) If these blocks are $\{0,1,x\}$, $\{0,1,y\}$, $\{0,2,z\}$ and $\{0,2,w\}$, where $x,y,z,w \in \{3,4\}$. Then there is at most one block containing the edge $\{1,2\}$, this is a contradiction. (2) If these blocks are $\{0,1,2\}$, $\{0,1,x\}$ and $\{0,2,y\}$, where $x,y \in \{3,4\}$. Then there exist two same parallel classes. This is a contradiction. So there doesn't exist an RMP(5) with 7 parallel classes.

Now we construct an RMP(5) with 6 parallel classes. The required parallel classes are listed below.

0 1 2	0 1 3	0 1 4	0 2 3	0 2 4	0 3 4
3 4	2 4	2 3	1 4	1 3	1 2

□

Lemma 4.25 (1) *There doesn't exist an RMC(6) with* $n(6) = 8$ *parallel classes.* (2) *There exists an RMC(6) with 9 parallel classes.*

Proof: Let the point set $V = Z_4 \cup \{a, b\}$. In this case, it is not difficult to show that the excess is a 1-factor. Suppose the excess consists of three edges $\{a, b\}$, $\{0, 1\}$ and $\{2, 3\}$. Since these parallel classes are distinct, we can take four parallel classes as follows:

a b 0	a b 1	a b 2	a b 3
1 2 3	0 2 3	0 1 3	0 1 2

In these parallel classes, the point pairs in Z_4 are all appear twice. Since the two edges $\{0, 1\}$ and $\{2, 3\}$ are in excess, we can fix another two parallel classes:

a 0 1	a 2 3
b 2 3	b 0 1

The last two parallel classes must contain the following 12 edges, $\{a, 0\}$, $\{a, 1\}$, $\{a, 2\}$, $\{a, 3\}$, $\{b, 0\}$, $\{b, 1\}$, $\{b, 2\}$, $\{b, 3\}$, $\{0, 2\}$, $\{0, 3\}$, $\{1, 2\}$, $\{1, 3\}$, and each edge is contained in exactly one block. Then the last two parallel classes is the block set of an 3-RGDD of type 2^3, it is well known that such a design does not exist. This is a contradiction. The proof is completed.

Now we construct an RMC(6) with 9 parallel classes. The required parallel classes are listed below.

0 1 2	0 1 3	0 1 4	0 1 5	0 2 3	0 2 4	0 2 5	0 3 4	0 3 5
3 4 5	2 4 5	2 3 5	2 3 4	1 4 5	1 3 5	1 3 4	1 2 5	1 2 4

□

Lemma 4.26 (1) *There doesn't exist an RMC*(5) *with* $n(5) = 8$ *parallel classes.* (2) *There exists an RMC*(5) *with* 9 *parallel classes.*

Proof: Take the point set $V = Z_5$. Suppose there are 8 parallel classes in an RMC(5), then the excess consists of two disjoint edges. So there is a point who does not appear in the excess. Suppose this point is 0. It is easy to prove that the point 0 must appear in four blocks of size three and four blocks of size two, and these blocks of size two must be $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$ and $\{0, 4\}$. So in the other blocks, each of the four edges must be contained in two blocks. We distinguish 2 cases. (1) If these blocks are $\{0, 1, x\}$, $\{0, 1, y\}$, $\{0, 2, z\}$ and $\{0, 2, w\}$, where $x, y, z, w \in \{3, 4\}$. Then the edge $\{1,2\}$ appears only twice in all the eight parallel classes, this is a contradiction. (2) If these blocks are $\{0,1,2\}$, $\{0,1,x\}$ and $\{0,2,y\}$, where $x,y \in \{3,4\}$. Then the edge $\{1,3\}$ or $\{1,4\}$ appears only twice in all the eight parallel classes, this is a contradiction. So there doesn't exist an RMC(5) with 8 parallel classes.

Now we construct an RMC(5) with 9 parallel classes. The required parallel classes are listed below.

0 1 2	0 2 3	0 1 3	0 1 4	0 2 4	0 3 4	1 2 3	1 2 4	1 3 4
3 4	1 4	2 4	2 3	1 3	1 2	0 4	0 3	0 2

□

Combining Theorems 4.11, 4.12, 4.17, 4.22 and Lemmas 4.23- 4.26, we obtain our main results.

Theorem 1.4 *There exists an RMP*$(3,3,v)$ *and an RMC*$(3,3,v)$ *for each* $v \geq 5$ *except for an RMP*$(3,3,6)$.

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