Hyperbolicity and parameters of graphs

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Abstract

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is *δ-hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we find some relations between the hyperbolicity constant of a graph and its order, girth, cycles and edges. In particular, if g denotes the girth, we prove $\delta(G) \geq g(G)/4$ for every (finite or infinite) graph; if G is a graph of order n and edges with length k (possibly with loops and multiple edges), then $\delta(G) \leq nk/4$. We find a large family of graphs for which the first (non-strict) inequality is in fact an equality; besides, we characterize the set of graphs with $\delta(G) = nk/4$. Furthermore, we characterize the graphs with edges of length k with $\delta(G) < k$.

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1 Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [7, 3, 4, 10, 19, 20, 21, 23, 24, 26, 29, 30, 32, 33].

The theory of Gromov's spaces was used initially for the study of finitely generated groups (see [12, 13] and the references therein), where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [25]), that play an important role in sciences of the computation. Another important application of this spaces is secure transmission of information by internet (see [19, 20]). In particular, the hyperbolicity also plays an important role in the spread of viruses through the network (see [19, 20]). The hyperbolicity is also useful in the study of DNA data (see [7]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [14]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [1, 2, 5, 15, 16, 17, 18, 26, 27, 28, 30, 31, 32]. In particular, in [26, 30, 32, 33] it is proved the equivalence of the hyperbolicity of Riemann surfaces (with their Poincaré metrics) and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [11]. We give now the basic facts about Gromov's spaces. If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), we can define the *length* of γ as

$$L(\gamma) := \sup\Big\{\sum_{i=1}^{n} d(\gamma(t_{i-1}),\gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b\Big\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x,y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the notation $[u,v]$ for the edge of a graph joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify any edge $[u,v] \in E(G)$ with the real interval $[0,l]$ (if $l := L([u,v])$); hence, if we consider $[u,v]$ as a graph with just one edge, then it is isometric to $[0,l]$. Therefore, any point in the interior of the edge $[u,v]$ is a point of G. A connected graph G is naturally equipped with a distance or, more precisely, metric defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Along the paper we just consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges). These conditions guarantee that the graph is a geodesic space. We allow

loops and multiple edges in the graphs, and the edges can have arbitrary lengths.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e. $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$.

A *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

Remark 1. *There are several definitions of Gromov hyperbolicity (see e.g. [6, 11]). These different definitions are equivalent in the sense that if X is δ_A-hyperbolic with respect to the definition A, then it is δ_B-hyperbolic with respect to the definition B, and there exist universal constants c_1, c_2 such that $c_1\delta_A \leq \delta_B \leq c_2\,\delta_A$. However, for a fixed $\delta \geq 0$, the set of δ-hyperbolic graphs with respect to the definition A, is different, in general, from the set of δ-hyperbolic graphs with respect to the definition B. We have chosen this definition since it has a deep geometric meaning (see e.g. [11]).*

Remark 2. *Some authors (see e.g. [7]) study Gromov hyperbolicity for graphs G such that every edge has length 1; in this context, they define $\delta(G)$ as*

$$\sup\{\delta(T) : T \text{ is a geodesic triangle in } G \text{ with vertices in } V(G)\}.$$

This definition is equivalent (in the sense of the previous Remark) to our definition if every edge in G has length 1. However, if we want to deal with graphs with edges of arbitrary lengths, we must consider geodesic triangles with vertices in G. Furthermore, our definition of hyperbolic graph is exactly the classical one (see e.g. [11]) when the geodesic metric space is a graph, and we think that it is more natural from a geometric viewpoint.

The following are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so that $\mathbb{R}^2$ with the Euclidean metric is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed

vector space E is hyperbolic if and only if dim $E = 1$. Every arbitrary length metric tree is 0-hyperbolic: in fact, all point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle. Every bounded metric space X is (diam X)-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -k^2$, for some positive constant k, is hyperbolic. We refer to [6, 11] for more background and further results.

If D is a closed subset of X, such that D contains a rectifiable path joining each $z, w \in D$, we always consider in D the *inner metric* obtained by the restriction of the metric in X, that is $d_D(z,w) := \inf\{L_X(\gamma) : \gamma \subset D$ is a continuous curve joining z and $w\} \geq d_X(z,w)$.

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$. We always have $d_D(z,w) < \infty$ for every $z, w \in D$.

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

In this paper we find some relations between the hyperbolicity constant of a graph and its order, girth, cycles and edges. In particular, if g denotes the girth, we prove $\delta(G) \geq g(G)/4$ for every (finite or infinite) graph in Section 3 (see Theorem 17); in Section 4 we prove that if G is a graph of order n and edges with length k (possibly with loops and multiple edges), then $\delta(G) \leq nk/4$ (see Theorem 30). We find a large family of graphs for which the first inequality is attained (see Theorems 23, 24 and 25); besides, we characterize the set of graphs with $\delta(G) = nk/4$ (see Theorem 30 and Proposition 29). Furthermore, we characterize the graphs with edges of length k with $\delta(G) < k$ in Section 2 (see Theorem 11).

2 Relations between the hyperbolicity constant and cycles of graphs

As usual, by *cycle* we mean a simple closed curve, i.e. a path with different vertices, unless the last one, which is equal to the first vertex.

A subgraph Γ of G is said *isometric* if $d_\Gamma(x,y) = d_G(x,y)$ for every

$x, y \in \Gamma$.

The following results appear in [29, Lemma 4 and Theorem 10].

Lemma 3. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \leq \delta(G)$.*

Let us fix a positive constant k. We denote by C_n the cycle graph with n vertices and n edges with length k which is a cycle of length kn. By K_n we denote the complete graph with n vertices and $n(n-1)/2$ edges with length k. We denote by $K_{m,n}$ the complete bipartite graph with $m+n$ vertices and mn edges with length k, i.e. the bipartite graph obtained from a set M of m vertices and a set N of n vertices, by connecting with an edge each vertex of M with each vertex of N. By W_n we denote the wheel graph with n vertices and $2n-2$ edges with length k, obtained from C_{n-1} and another vertex v, by connecting with an edge each vertex of C_{n-1} with v.

Theorem 4. *The following graphs with edges of length k have these precise values of δ:*

- *The cycle graphs verify $\delta(C_n) = nk/4$ for every $n \geq 3$.*
- *The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3k/4$, $\delta(K_n) = k$ for every $n \geq 4$.*
- *The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = k$ for every $m, n \geq 2$.*
- *The wheel graph with n vertices W_n verifies $\delta(W_4) = \delta(W_5) = k$, $\delta(W_n) = 3k/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5k/4$ for $n = 6$ and for every $n \geq 11$.*

Proposition 5. *Let G be a graph with edges of length k. If there exists a cycle g in G with length $L(g) = 3k$, then $\delta(G) \geq \frac{3k}{4}$.*

Proof. Since every edge has length k and $L(g) = 3k$, then g is an isometric subgraph of G. Therefore, Lemma 3 gives $\delta(G) \geq \delta(g)$ and Theorem 4 implies $\delta(g) = \frac{3k}{4}$. □

Given a graph G, we define $A(G)$ as the graph G without its loops, and $B(G)$ as the graph G without its multiple edges, obtained by replacing each multiple edge by a single edge with the minimum length of the edges corresponding to that multiple edge.

We will need the following lemma (see [29, Theorem 7]):

Lemma 6. *In any graph G the inequality $\delta(G) \leq \frac{1}{2} \operatorname{diam} G$ holds.*

Theorem 7. *Let G be a graph with edges of length k. If there exists a cycle g in G with length $L(g) \geq 4k$, then*

$$\delta(G) \geq \frac{1}{4} \min \left\{ L(\sigma) : \sigma \text{ is a cycle in } G \text{ with } L(\sigma) \geq 4k \right\}.$$

Note that Theorem 7 improves Proposition 5: for instance, if there are cycles of lengths $3k$ and $7k$ in a graph G and there are not cycles of lengths $4k$, $5k$ and $6k$, then Proposition 5 gives $\delta(G) \geq \frac{3k}{4}$ and Theorem 7 gives $\delta(G) \geq \frac{7k}{4}$.

Proof. Let us consider a cycle g_0 in G with length $L(g_0) \geq 4k$ and such that

$$L(g_0) = \min \left\{ L(\sigma) : \sigma \text{ is a cycle in } G \text{ with } L(\sigma) \geq 4k \right\}.$$

Assume first that $L(g_0) = 4k$. Let us consider the subgraph Γ of G with the four vertices of g_0 and whose edges are the edges in G connecting these four vertices. Then Γ is isomorphic either to the cycle graph C_4, to the complete graph K_4 or to the graph H obtained by deleting an edge to K_4, since G has not loops nor multiple edges. By Lemma 3 we have that $\delta(G) \geq \delta(\Gamma)$, since Γ is an isometric subgraph of G.

By Theorem 4 we know that $\delta(C_4) = \delta(K_4) = k$. Let us consider the two vertices x, y in H with degree 2 (note that $d(x,y) = 2k$) and the bigon B with vertices $\{x, y\}$ and sides γ_1, γ_2 such that $\gamma_1 \cup \gamma_2 = g_0$. The midpoint p of γ_1 satisfies $\delta(H) \geq \delta(B) \geq d(p, \gamma_2) = k$. Moreover, we know that $\delta(H) \leq \frac{1}{2} \operatorname{diam} H = k$ by Lemma 6, and hence $\delta(H) = k$. Therefore, in any case $\delta(G) \geq \delta(\Gamma) = k$.

Assume now that $L(g_0) > 4k$. Fix a vertex v in g_0, and consider the point $u \in g_0$ such that the two subcurves $g_1, g_2 \subset g_0$ with $g_1 \cup g_2 = g_0$ which join u and v verify $L(g_1) = L(g_2) = L(g_0)/2$. We are going to show that g_1 and g_2 are geodesics. Seeking for a contradiction assume that there exists a curve η joining u and v with $L(\eta) < L(g_0)/2$; then $g_1 \cup \eta$ is a closed curve in G with length $L(g_1 \cup \eta) < L(g_0)$; therefore there exists a cycle $\sigma \subseteq g_1 \cup \eta$ with length $L(\sigma) \leq 3k$, since $L(g_0) > 4k$ is the minimum length of the cycles in G with length greater or equal than $4k$. Since G has not loops nor multiple edges, we deduce that $L(\sigma) = 3k$; then $L(g_1 \cap \sigma) = 2k$ and $L(\eta \cap \sigma) = k$; therefore, if we replace in $g_1 \cap \eta$ the subcurve $g_1 \cap \sigma$ by $\eta \cap \sigma$, then we obtain a cycle with length $L(g_0) - k$. This is the contradiction we are looking for, and then $d(u,v) = L(g_1) = L(g_2) = L(g_0)/2$.

Finally, let us consider the bigon $B = \{\gamma_1, \gamma_2\}$ and the midpoint p of γ_1 (note that B is a bigon just because $d(u,v) = L(g_0)/2$). We have $\delta(B) \geq d(p, \gamma_2) = L(g_0)/4$ (this can be proved by assuming that there exists a shorter curve joining p and γ_2 and finding a contradiction with a similar

argument to the previous one). Hence, $\delta(G) \geq \delta(\Gamma) \geq \delta(B) \geq L(g_0)/4$, and this finishes the proof. $\square$

Given a graph G and a family of subgraphs $\{G_n\}_{n\in\Lambda}$ of G verifying $\cup_n G_n = G$ and that $G_n \cap G_m$ is either a vertex or the empty set for each $n \neq m$, we define a graph R as follows: for each index $n \in \Lambda$, let us consider a point v_n (v_n is an abstract point which is not contained in G_n) and we define the set of vertices of R as $V(R) = \{v_n\}_{n\in\Lambda}$; two vertices of R are neighbors, i.e. $[v_n, v_m] \in E(R)$ if and only if $G_n \cap G_m \neq \varnothing$. We say that the family of subgraphs $\{G_n\}_n$ of G is a *tree-decomposition* of G if the graph R is a tree.

A tree-decomposition of G always exists, as we will show by introducing the canonical tree-decomposition of G, before Theorem 23.

We will need the following result (see [3, Theorem 5]):

Lemma 8. *Let G be a graph and $\{G_n\}_n$ be a tree-decomposition of G. Then $\delta(G) = \sup_n \delta(G_n)$.*

In [23], the authors characterize the bridged graphs with edges of length 1 which have hyperbolicity constant 1, for a different (but equivalent in the sense of Remark 1) definition of hyperbolicity constant.

An interesting question is how to characterize the graphs G with edges of length k and $\delta(G) = k$, but it seems very difficult to give a description of such graphs in a simple way. Theorem 4 and Theorem 28 show a large variety of graphs with $\delta(G) = k$ (e.g. K_n for every $n \geq 4$, $K_{m,n}$ for every $m, n \geq 2$, and W_n for $n = 4, 5$).

However, the following theorem allows to characterize the graphs with $\delta(G) < k$.

We also have some results which guarantee that many graphs satisfy $\delta(G) > k$ (see Theorem 12).

We will need the following lemmas (see [3, Theorems 8 and 10]):

Lemma 9. *If G is a graph with some loop and every edge has length k, then*

$$\delta(G) = \max\left\{\delta(A(G)), \frac{k}{4}\right\}.$$

Lemma 10. *If G is a graph with some multiple edge and every edge has length k, then*

$$\delta(G) = \max\left\{\delta(B(G)), \frac{k}{2}\right\} = \max\left\{\delta(B(A(G))), \frac{k}{2}\right\}.$$

Given any graph G we define, as usual, its *girth* $g(G)$ as the infimum of the lengths of the cycles in G.

Theorem 11. *Let G be a graph with edges of length k.*

- $\delta(G) < k/4$ *if and only if* G *is a tree.*
- $\delta(G) < k/2$ *if and only if* $A(G)$ *is a tree.*
- $\delta(G) < 3k/4$ *if and only if* $B(A(G))$ *is a tree.*
- $\delta(G) < k$ *if and only if every cycle* g *in* G *has length* $L(g) \leq 3k$.

Furthermore, if $\delta(G) < k$, *then* $\delta(G) \in \{0, k/4, k/2, 3k/4\}$.

Proof. If G is a tree, we know that $\delta(G) = 0 < k/4$.

Conversely, assume that $\delta(G) < k/4$. Seeking for a contradiction let us assume that there exists a cycle in G. By Theorem 17 we have $k \leq g(G) \leq 4\delta(G) < k$, which is the contradiction we are looking for; therefore, G is a tree.

If $A(G)$ is a tree, we know that $\delta(A(G)) = 0$. If G has not loops, then $A(G) = G$ and $\delta(A(G)) = 0 < k/2$. If G has some loop, then by Lemma 9, we have that

$$\delta(G) = \max\left\{\delta(A(G)), \frac{k}{4}\right\} = \max\left\{0, \frac{k}{4}\right\} = \frac{k}{4} < \frac{k}{2}.$$

Conversely, assume that $\delta(G) < k/2$. If G is a tree, then $A(G) = G$ is a tree. Assume that G has some cycle. There are not multiple edges in G, since in other case Lemma 10 gives $\delta(G) \geq k/2$. There are not cycles in G with length $3k$, since otherwise $\delta(G) \geq 3k/4$, by Proposition 5. There are not cycles in G with length greater than $3k$, since otherwise Theorem 7 gives $\delta(G) \geq k$. Consequently, every cycle of G is a loop, and $A(G)$ is a tree.

If $B(A(G))$ is a tree, we have $\delta(B(A(G))) = 0$. If G has not multiple edges, then $B(A(G)) = A(G)$ is a tree and then $\delta(G) < k/2 < 3k/4$. If G has some multiple edge, then by Lemma 10, we have that

$$\delta(G) = \max\left\{\delta(B(A(G))), \frac{k}{2}\right\} = \max\left\{0, \frac{k}{2}\right\} = \frac{k}{2} < \frac{3k}{4}.$$

Conversely, assume that $\delta(G) < 3k/4$. If G is a tree, then $B(A(G)) = G$ is a tree. Assume that G has some cycle. There are not cycles in G with length $3k$, since otherwise $\delta(G) \geq 3k/4$, by Proposition 5. There are not cycles in G with length greater than $3k$, since in other case Theorem 7 gives $\delta(G) \geq k$. Consequently, every cycle of G is a loop or a double edge, and $B(A(G))$ is a tree.

Assume now that every cycle g in G has length $L(g) \leq 3k$. Then we have either that $B(A(G))$ is a tree (and then we have seen that $\delta(G) < 3k/4$) or that every cycle g in $B(A(G))$ has length $L(g) = 3k$. In this last

case, let us consider the set of cycles $\{g_n\}_n$ in $B(A(G))$ with length $3k$, and the closures $\{H_m\}_m$ of the connected components of $B(A(G)) \setminus \cup_n\{g_n\}$. It is clear that the union of $\{g_n\}_n$ and $\{H_m\}_m$ are a tree-decomposition of $B(A(G))$, and then Lemma 8 gives that

$$\delta(B(A(G))) = \max\left\{\sup_n \delta(g_n),\ \sup_m \delta(H_m)\right\}.$$

Since each g_n is isometric to C_3, Theorem 4 gives that $\delta(g_n) = \delta(C_3) = 3k/4$. Since every H_m is a tree, we have $\delta(H_m) = 0$. Hence, $\delta(B(A(G))) = 3k/4$. By Lemma 10, we have that

$$\delta(G) = \max\left\{\delta(B(A(G))),\ \frac{k}{2}\right\} = \max\left\{\frac{3}{4}, \frac{k}{2}\right\} = \frac{3k}{4} < k.$$

Conversely, assume that $\delta(G) < k$. Then Theorem 7 gives that every cycle g in G has length $L(g) \leq 3k$. We have seen in the proof that if $\delta(G) < k$, then $\delta(G) \in \{0, k/4, k/2, 3k/4\}$.

□

We give now some results which guarantee that many graphs satisfy $\delta(G) > k$.

Theorem 12. *Let G be a graph with edges of length k. If there exists a cycle g in G with length $L(g) = 5k$ and a vertex $w \in g$ such that $[w, w'] \in E(G)$ for just two vertices $w' \in g$, then $\delta(G) \geq 5k/4$.*

Remark 13. *Note that the hypothesis about the vertex w is necessary in Theorem 12, as shows Proposition 27 below (with $n = 5$).*

Proof. Let us consider the subgraph Γ of G with the vertices of g and whose edges are the edges in G connecting these vertices.

We are going to show that Γ is an isometric subgraph of G: consider two vertices x, y in Γ; if $d_G(x, y) = k$, then $[x, y] \in E(G)$, and consequently, $[x, y] \in E(\Gamma)$ and $d_\Gamma(x, y) = k = d_G(x, y)$; if $d_G(x, y) > k$, then $d_\Gamma(x, y) > k$ and therefore $d_\Gamma(x, y) = 2k$ (since Γ has five vertices and contains the subgraph g isomorphic to C_5), and hence, $d_G(x, y) = 2k = d_\Gamma(x, y)$. Consequently, Γ is an isometric subgraph of G and $\delta(G) \geq \delta(\Gamma)$, by Lemma 3.

Therefore, it suffices to prove that $\delta(\Gamma) \geq 5k/4$. Since $[w, w'] \in E(G)$ for just two vertices $w' \in g$, there exists a point $z \in g$ with $d(w, z) = 5k/2$. Let us consider the bigon $\{w, z\}$ with sides γ_1, γ_2 whose union is g. The midpoint p of γ_1 satisfies $d(p, \gamma_2) = 5k/4$, and then $\delta(G) \geq \delta(\Gamma) \geq 5k/4$. □

As a consequence of Theorem 12 we obtain the following result.

Corollary 14. *Let G be a graph with edges of length k. If there exists a cycle g in G with length $L(g) = 5k$ and a vertex $w \in g$ with degree two, then $\delta(G) \geq 5k/4$.*

Now, as a consequence of Proposition 5 and Theorems 7, 11 and 12, we have for every graph G with edges of length k:

If G has some loop, then $\delta(G) \geq k/4$.
If G has some multiple edge, then $\delta(G) \geq k/2$.
If G has some cycle with three edges, then $\delta(G) \geq 3k/4$.
If G has some cycle with four edges, then $\delta(G) \geq k$.
If G has some cycle g with five edges and a vertex $w \in g$ such that $[w, w'] \in E(G)$ for just two vertices $w' \in g$, then $\delta(G) \geq 5k/4$.

3 Hyperbolicity constant and girth of a graph

Theorem 17 relates the hyperbolicity constant with an interesting parameter of a graph G, as is its girth $g(G)$. We need two previous results.

Lemma 15. *For any cycle graph C we have $\delta(C) = \frac{L(C)}{4}$.*

Proof. Since $\operatorname{diam} C = L(C)/2$, Lemma 6 gives that $\delta(C) \leq L(C)/4$. Let us consider a bigon B with vertices $\{x, y\}$ at distance $L(C)/2$, with sides γ_1, γ_2 such that $\gamma_1 \cup \gamma_2 = C$. The midpoint p of γ_1 satisfies $\delta(C) \geq \delta(B) \geq d(p, \gamma_2) = d(p, \{x, y\}) = L(C)/4$. Consequently, $\delta(C) = L(C)/4$. □

The following result is the main tool in the proof of Theorem 17. Furthermore, it is interesting by itself. Note that it holds trivially for finite graphs.

Theorem 16. *Let C be any cycle in any graph G. There exists an isometric cycle in G which contains at least an edge of C.*

Proof. If there is not a simple curve γ joining $x_0, y_0 \in V(C)$ with γ not contained in C, then γ is an isometric cycle in G, and the claim holds.

In other case, there is a simple curve γ joining $x_0, y_0 \in V(C)$ with γ not contained in C. Let us define

$$C^* := \big\{(u, v) \in V(C) \times V(C) : \text{there is a simple curve } \eta \text{ joining } u, v \text{ with } \eta \text{ not contained in } C\big\}.$$

Let $(x, y) \in C^*$ with $d_C(x, y) \leq d_C(u, v)$ for every $(u, v) \in C^*$.

Let us define

$$C' := \big\{\eta : \eta \text{ is a simple curve joining } x, y \text{ with } \eta \text{ not contained in } C\big\}.$$

Consider $\sigma \in C'$ with $L(\sigma) \leq L(\eta)$ for every $\eta \in C'$.

We can write $C = C^1 \cup C^2$, where C^1, C^2 are curves in C joining x and y. Without loss of generality we can assume that $L(C^1) \leq L(C^2)$, and we define a new cycle $C_0 := C^1 \cup \sigma$.

Now we prove that C_0 is an isometric cycle in G. Seeking for a contradiction assume that there exist $x_1, y_1 \in V(C_0)$ with $d(x_1, y_1) < d_{C_0}(x_1, y_1)$.

First, we show that it is not possible to have $x_1, y_1 \in C^1$. If $\{x_1, y_1\} \neq \{x, y\}$, then $d_C(x_1, y_1) < d_C(x, y)$, and $[x_1 y_1]$ is not contained in C; hence, $(x_1, y_1) \in C^*$ and this fact contradicts the definition of x and y. If $\{x_1, y_1\} = \{x, y\}$, then $L([xy]) = d(x, y) < d_{C_0}(x, y) \leq L(\sigma)$ and $[xy]$ is not contained in C since $L([xy]) = d(x, y) < d_{C_0}(x, y) \leq L(C^1)$; hence, $[xy] \in C'$ and this fact contradicts the definition of σ.

We check now that it is not possible to have $x_1, y_1 \in \sigma$. If $\{x_1, y_1\} = \{x, y\}$, we have seen that this fact contradicts the definition of σ. If $\{x_1, y_1\} \neq \{x, y\}$ and we denote by σ_0 the subset of σ joining x and y, then $\eta := [x_1, y_1] \cup (\sigma \setminus \sigma_0)$ is not contained in C and $L(\eta) < L(\sigma)$; hence, $\eta \in C'$ and this fact contradicts the definition of σ.

If $x_1 \in C^1 \setminus \{x, y\}, y_1 \in \sigma \setminus \{x, y\}$, then consider a geodesic γ joining x_1 and y_1; hence, the union γ' of γ and a subcurve of σ is a simple curve joining x_1 and y, and γ' is not contained in C. Therefore, $(x_1, y) \in C^*$ and $d_C(x_1, y) < d_C(x, y)$, and these facts contradict the definition of x and y. By symmetry, it is not possible to have $x_1 \in \sigma \setminus \{x, y\}, y_1 \in C^1 \setminus \{x, y\}$.

These are the contradictions we were looking for. Hence, C_0 is an isometric cycle in G. It is clear that C_0 contains at least an edge of C. □

Theorem 17. *For any graph G we have $\delta(G) \geq \frac{g(G)}{4}$ and the inequality is optimal.*

Proof. The inequality in the statement of this Theorem is, in fact, an equality for every cycle graph, by Lemma 15.

If G does not contain cycles, then G is a tree and $g(G) = 0 = \delta(G)$.

If there exists a cycle in G, then Theorem 16 gives that there exists an isometric cycle C_0 in G. Then Lemmas 3 and 15 give

$$\delta(G) \geq \delta(C_0) = \frac{L(C_0)}{4} \geq \frac{g(G)}{4}.$$

□

Note that it is not possible to obtain the reverse inequality $\delta(G) \leq c\,g(G)$ for some positive constant c: let us consider the graph G_n obtained by attaching a loop to C_n (with edges of length 1); it is clear that $g(G_n) = 1$ and $\delta(G_n) = n/4$.

In [3, Corollary 4] and [29, Theorem 11] we found the following results, respectively.

Lemma 18. *In any graph G,*

$$\delta(G) = \sup\left\{\delta(T) : T \text{ is a geodesic triangle which is a cycle}\right\}.$$

Lemma 19. *Let $C_{a,b,c}$ be the graph with two vertices and three edges joining them with lengths $a \le b \le c$. Then $\delta(C_{a,b,c}) = \frac{c+\min\{b,3a\}}{4}$.*

Proposition 20. *Denote by $C_{a_1,a_2,\ldots,a_k}$ the graph with two vertices and k edges joining them with lengths $a_1 \le a_2 \le \cdots \le a_k$. Then*

(i) $\delta(C_{a_1,a_2,\ldots,a_k}) = \frac{a_k+\min\{a_{k-1},3a_1\}}{4}$.

(ii) $\delta(C_{a_1,a_2,\ldots,a_k}) = \frac{1}{2}\operatorname{diam} C_{a_1,a_2,\ldots,a_k}$ *if and only if* $a_{k-1} \le 3a_1$.

Proof. Let us denote by x_1, x_2 the vertices of $C_{a_1,a_2,\ldots,a_k}$, and by $A_1, A_2, ..A_k$ the edges with lengths $a_1, a_2, \ldots, a_k$, respectively.

Let us consider a geodesic triangle T; in order to compute $\delta(C_{a_1,a_2,\ldots,a_k})$ without loss of generality we can assume that T is a cycle, by Lemma 18. Then the closed curve given by T is $A_i \cup A_j$ with $1 \le i < j \le k$.

If $i = 1$, then $A_1 \cup A_j$ is an isometric subgraph of $C_{a_1,a_2,\ldots,a_k}$. If $i > 1$, then $A_1 \cup A_i \cup A_j$ is an isometric subgraph of $C_{a_1,a_2,\ldots,a_k}$. Hence, by Lemmas 3 and 19 we have

$$\begin{aligned}
\delta(C_{a_1,a_2,\ldots,a_k}) &= \max\left\{\max_{1<j\le k}\delta(C_{a_1,a_j}),\ \max_{1<i<j\le k}\delta(C_{a_1,a_i,a_j})\right\}\\
&= \max\left\{\max_{1<j\le k}\frac{a_j+a_1}{4},\ \max_{1<i<j\le k}\frac{a_j+\min\{a_i,3a_1\}}{4}\right\}\\
&= \max\left\{\frac{a_k+a_1}{4},\ \frac{a_k+\min\{a_{k-1},3a_1\}}{4}\right\}\\
&= \frac{a_k+\min\{a_{k-1},3a_1\}}{4}.
\end{aligned}$$

□

We have the following family of extremal graphs for Theorem 17.

Proposition 21. *$C_{a_1,a_2,\ldots,a_k}$ verifies $\delta(C_{a_1,a_2,\ldots,a_k}) = g(C_{a_1,a_2,\ldots,a_k})/4$ if and only if $a_1 = a_2 = \cdots = a_k$.*

Proof. Note that $g(C_{a_1,a_2,\ldots,a_k}) = a_1 + a_2$.

If $a_1 = a_2 = \cdots = a_k$, then Proposition 20 gives that

$$4\,\delta(C_{a_1,a_2,\ldots,a_k}) = a_k + \min\{a_{k-1},3a_1\} = a_1 + a_2 = g(C_{a_1,a_2,\ldots,a_k}).$$

Assume now that $\delta(C_{a_1,a_2,\ldots,a_k}) = g(C_{a_1,a_2,\ldots,a_k})/4$. Proposition 20 gives that

$$a_1 + a_2 = g(C_{a_1,a_2,\ldots,a_k}) = 4\,\delta(C_{a_1,a_2,\ldots,a_k}) = a_k + \min\{a_{k-1},3a_1\}.$$

We prove first that $a_{k-1} < 3a_1$. Seeking for a contradiction, assume that $\min\{a_{k-1}, 3a_1\} = 3a_1$. Then $a_1 + a_2 = a_k + 3a_1$ and we conclude that $a_k \geq a_2 = a_k + 2a_1 > a_k$, which is the contradiction we were looking for. Hence, we have $a_{k-1} < 3a_1$, and we deduce $a_1 + a_2 = a_k + a_{k-1}$; this implies $a_1 = a_2 = \cdots = a_k$. □

We say that a vertex v of a graph G is a *tree-vertex* if $G \setminus \{v\}$ is not connected. Note that any vertex with degree at least two in a tree is a tree-vertex.

We denote by $\{G_n\}_n$ the closures in G of the connected components of the set

$$G \setminus \{v \in V(G) : v \text{ is a tree-vertex of } G\}.$$

It is clear that $\{G_n\}_n$ is a tree-decomposition of G; we call it the *canonical tree-decomposition* of G. Given any positive number N, we say that G belongs to $\mathcal{E}_N$ if G is not a tree and for each n, the graph G_n is either an edge or isometric to $C_{a_1, a_2, \ldots, a_{k_n}}$ for some $k_n \geq 2$, with $a_1 = a_2 = \cdots = a_{k_n} = N/2$.

Remark 22. *Note that every G_n in the canonical tree-decomposition of G is an isometric subgraph of G.*

The next result gives that the inequality in Theorem 17 is attained for the graphs G in the class $\mathcal{E}_N$, with $N = g(G)$.

Theorem 23. *Every graph G which belongs to $\mathcal{E}_{g(G)}$ verifies $\delta(G) = g(G)/4$.*

Proof. Assume that G belongs to $\mathcal{E}_{g(G)}$. Then, for any fixed G_n, Proposition 21 gives that we have either $\delta(G_n) = 0$ or $\delta(G_n) = g(G)/4$. Since G is not a tree, there exists n_0 with $\delta(G_{n_0}) = g(G)/4$ and $g(G_{n_0}) = g(G)$. Hence, Lemma 8 gives $\delta(G) = \sup_n \delta(G_n) = g(G)/4$. □

The following results characterize the classes of graphs with loops or multiple edges which attain the inequality in Theorem 17.

Theorem 24. *Let G be any graph with every edge of length k and with some loop. Then the following conditions are equivalent:*

- $\delta(G) = g(G)/4$.
- *$A(G)$ is a tree.*
- $\delta(G) = k/4$.
- $\delta(G) < k/2$.

Proof. Assume first that $\delta(G) = g(G)/4$. Since G has at least a loop, then $g(G) = k$. Consequently, $\delta(G) = g(G)/4 = k/4$, and Theorem 11 gives that $A(G)$ is a tree.

Assume now that $A(G)$ is a tree. Then every cycle in G is a loop. Since G has at least a loop, then $g(G) = k$. Since G is not a tree, Theorem 11 gives that $\delta(G) = k/4$, and consequently, $\delta(G) = g(G)/4$.

The equivalence of the three last statements is a consequence of Theorem 11, since G has at least a loop. □

Theorem 25. *Let G be any graph with every edge of length k and with some multiple edge. Then the following conditions are equivalent:*

- $\delta(G) = g(G)/4$.
- *$B(G)$ is a tree.*
- *$\delta(G) = k/2$ and G has not loops.*
- *$\delta(G) < 3k/4$ and G has not loops.*

Proof. First of all, assume that $\delta(G) = g(G)/4$. Seeking for a contradiction, assume that G has some loop; hence, Theorem 24 gives that $A(G)$ is a tree, and then G has not multiple edges, which is a contradiction. Consequently, G has not loops and $B(A(G)) = B(G)$. Since G has at least a multiple edge, then $g(G) = 2k$. Consequently, $\delta(G) = g(G)/4 = k/2$, and Theorem 11 gives that $B(A(G)) = B(G)$ is a tree.

Assume now that $B(G)$ is a tree. Then G has not loops and $B(A(G)) = B(G)$ is a tree. Theorem 11 gives that $\delta(G) \in \{0, k/4, k/2\}$. Since G has some multiple edge, $\delta(G) \geq g(G)/4 = k/2$ by Theorem 17. Consequently, $\delta(G) = g(G)/4 = k/2$.

The equivalence of the three last statements is a consequence of Theorem 11, since G has at least a multiple edge and has not loops. □

4 Relations between the hyperbolicity constant, edges and order of a graph

Since Theorem 4 gives that K_n (for every $n \geq 4$) and $K_{m,n}$ (for every $m, n \geq 2$) are graphs with $\delta = k$, one can think that any graph obtained by adding edges to $K_{m,n}$ must satisfy $\delta = k$. However, this is false, as shows the following result.

Proposition 26. *Let G be a graph obtained from $K_{m,n}$ $(m, n \geq 2)$ by adding some edges. Assume that there exist vertices u, v, w, in the same part of $K_{m,n}$ with $[u, v] \in E(G)$ and $[u, w], [v, w] \notin E(G)$. Then $\delta(G) = 5k/4$ and* $\operatorname{diam} G = 5k/2$.

Proof. Let us consider vertices x, y, in the other part of $K_{m,n}$, and the cycle with length 5 given by $\{u, x, w, y, v, u\}$. Since $[w, x], [w, y] \in E(G)$ and $[u, w], [v, w] \notin E(G)$, Theorem 12 implies that $\delta(G) \geq 5k/4$.

Therefore, Lemma 6 gives $5k/4 \leq \delta(G) \leq \frac{1}{2} \operatorname{diam} G$ and $\operatorname{diam} G \geq 5k/2$. Let p be the midpoint of $[u, v]$; then $d(p, w) = d(p, \{u, v\}) + d(\{u, v\}, w) = k/2 + 2k = 5k/2$. Hence, $\operatorname{diam} G = 5k/2$, and applying Theorem 6 again, we obtain $\delta(G) \leq \frac{1}{2} \operatorname{diam} G = 5k/4$. Then we conclude $\delta(G) = 5k/4$. □

On the other side, if we have "many edges" in the graph, then we have $\delta = k$.

We denote by $\deg(v)$ the degree of a vertex in a graph.

Proposition 27. *Let G be a graph with edges of length k and with $n \geq 4$ vertices. If $\deg(v) \geq n - 2$ for every vertex $v \in V(G)$, then $\delta(G) = k$ and $\operatorname{diam} G = 2k$.*

Proof. First of all we are going to prove that $\operatorname{diam} G = 2$.

Let us consider points $x, y \in G$ and the edges $[x_1, x_2], [y_1, y_2] \in E(G)$ with $x \in [x_1, x_2], y \in [y_1, y_2]$. Without loss of generality we can assume that $t := d(x, x_1)$ and $s := d(y, y_1)$ verify $0 \leq t \leq s \leq k/2$. If $[x_1, y_1] \in E(G)$, then $d(x, y) \leq d(x, x_1) + d(x_1, y_1) + d(y_1, y) = t + k + s \leq 2k$. If $[x_1, y_1] \notin E(G)$, then $[x_1, y_2] \in E(G)$ since $\deg(x_1) \geq n - 2$, and consequently, $d(x, y) \leq d(x, x_1) + d(x_1, y_2) + d(y_2, y) = t + 2k - s \leq 2k$ since $t \leq s$. Therefore, $\operatorname{diam} G \leq 2k$.

Consider now two edges $[a_1, a_2], [b_1, b_2] \in E(G)$, with $a_i \neq b_j$, $1 \leq i, j \leq 2$ (we can do it since $n \geq 4$), and their midpoints a of $[a_1, a_2]$ and b of $[b_1, b_2]$. Then $d(a, b) = k/2 + d(\{a_1, a_2\}, \{b_1, b_2\}) + k/2 \geq k/2 + k + k/2 = 2k$. Hence, $\operatorname{diam} G = 2k$.

Lemma 6 gives now that $\delta(G) \leq \frac{1}{2} \operatorname{diam} G = k$. We have $\deg(v) \geq n - 2 \geq n/2$ for every vertex $v \in V(G)$, since $n \geq 4$; hence, by Dirac's Theorem (see [9]) there exists a Hamiltonian cycle g in G. Since g is a cycle in G with length $L(g) = nk \geq 4k$, Theorem 7 implies that $\delta(G) \geq k$. Therefore, $\delta(G) = k$. □

Theorem 28. *For each $n \geq 5$ and $1 \leq m \leq n - 2$, let $G_{n,m}$ be the graph with edges of length k obtained by removing m edges starting in the same vertex from the complete graph K_n. Then $\delta(G_{n,m}) = k$ if $m = 1$ or $m = n - 2$, and $\delta(G_{n,m}) = 5k/4$ if $1 < m < n - 2$.*

Proof. If $m = 1$, then Proposition 27 gives that $\delta(G_{n,1}) = k$. If $m = n - 2$, then the canonical tree-decomposition of $G_{n,m}$ has two components: K_{n-1} and an edge; therefore, Lemma 8 implies $\delta(G_{n,n-2}) = \delta(K_{n-1})$; finally, Theorem 4 gives that $\delta(G_{n,n-2}) = k$.

Assume now that $1 < m < n - 2$. Let $u \in V(G_{n,m})$ be the vertex with less degree. Let us consider $v, w \in V(G_{n,m})$ two vertices with $d(u, v) > 1$

and $d(u,w) > 1$, and $v', w' \in V(G_{n,m})$ two vertices with $d(u,v') = 1$ and $d(u,w') = 1$. If g is the cycle $g := \{u, v', v, w, w'\}$, then Theorem 12 gives $\delta(G_{n,m}) \geq 5k/4$.

We prove now that $\operatorname{diam} G_{n,m} = 5k/2$. It easy to check that $\operatorname{diam} G_{n,m} = \max\{d(u,p) : p \in G_{n,m}\}$ and that $d(u,q) = 5k/2$ if q is the midpoint of v and w. If p belongs to some edge starting in u, then $d(u,p) \leq k$; If p does not belong to the edges starting in u, then $d(v',p) \leq 3k/2$ and $d(u,p) \leq 5k/2$. Then $\operatorname{diam} G_{n,m} = 5k/2$ and Lemma 6 gives $\delta(G_{n,m}) \leq 5k/4$. Hence, $\delta(G_{n,m}) = 5k/4$. □

The following family of graphs allows to characterize the extremal graphs in Theorem 30. Let $\mathcal{F}_n$ be the set of Hamiltonian graphs G of order n with every edge of length k and such that there exists a Hamiltonian cycle G_0 which is the union of two geodesics Γ_1, Γ_2 in G with length $nk/2$ such that the midpoint x_0 of Γ_1 satisfies $d_G(x_0, \Gamma_2) = nk/4$.

Note that the endpoints of Γ_1, Γ_2 do not have to belong to $V(G)$.

We have a precise description of $\mathcal{F}_n$.

Proposition 29. *For $n \geq 3$, let us consider the cycle graph C_n with edges of length k. Fix a vertex $z \in V(C_n)$ and the geodesics (in C_n) Γ_1^*, Γ_2^* with lengths $nk/2$ joining the vertex z and the point w and $C_n = \Gamma_1^* \cup \Gamma_2^*$. Denote by w_i^j the vertex in Γ_i^* with $d(w_i^j, z) = jk$, for $i = 1, 2$ and $j \geq 1$ (with $w_i^j \neq w$).*

- *If n is even, we have $1 \leq j \leq n/2 - 1$. Then a graph belongs to $\mathcal{F}_n$ if and only if it is isomorphic (and hence, isometric) to a graph obtained by adding to C_n any amount of multiple edges and/or loops and a subset (proper or not) of either*

$$\{\, [w_1^1, w_2^1], [w_1^1, w_2^2], [w_1^{n/2-1}, w_2^{n/2-1}], [w_1^{n/2-1}, w_2^{n/2-2}] \,\}$$

or

$$\{\, [z, w_2^2], [w_1^{n/2-1}, w_2^{n/2-1}] \,\}.$$

- *If n is odd, we have $1 \leq j \leq (n-1)/2$. Then a graph belongs to $\mathcal{F}_n$ if and only if it is isomorphic (and hence, isometric) to a graph obtained by adding to C_n any amount of multiple edges and/or loops and a subset (proper or not) of*

$$\{\, [w_1^1, w_2^1], [w_1^1, w_2^2], [w_1^{(n-1)/2}, w_2^{(n-1)/2-1}] \,\}.$$

Proof. Assume that a graph G belongs to $\mathcal{F}_n$. We deal first with the case n even. Let x and y be the points at distance $nk/2$ in G_0 joined by the geodesics (in G) Γ_1, Γ_2 with $G_0 = \Gamma_1 \cup \Gamma_2$. Denote by v_i^1 the nearest vertex

in Γ_i to x (different from x if $x \in V(G)$) and $t_i := d(v_i^1, x) \in (0, k]$, for $i = 1, 2$ (notice that $t_1 = k$ if and only if $t_2 = k$; otherwise, $t_1 + t_2 = k$). Denote also by v_i^j the vertex in Γ_i with $d(v_i^j, v_i^1) = (j-1)k$ and $d(v_i^j, x) > (j-1)k$, for $i = 1, 2$ and $2 \leq j \leq j_x$, where $v_i^{j_x}$ is the nearest vertex in Γ_i to y (different from y if $y \in V(G)$) (notice that $d(v_1^{j_x}, y) = t_2$). If $t_1 = k$, then $j_x = n/2 - 1$; if $t_1 \in (0, k)$, then $j_x = n/2$.

First of all, note that if we add to a graph in $\mathcal{F}_n$ any amount of multiple edges and/or loops, then we also obtain a graph in $\mathcal{F}_n$. Therefore, without loss of generality we can assume that G has not multiple edges or loops.

Note also that in G there is not an edge joining two non-adjacent vertices in the same geodesic Γ_i, for $i = 1, 2$, since if there is such edge then Γ_i would not be a geodesic.

If $t_1 \in (0, k/2)$, then in $E(G) \setminus E(G_0)$ there is not an edge starting in v_1^1 (note that $[v_1^1, v_2^1] \in E(G_0) \subseteq E(G)$): if $[v_1^1, v_2^j] \in E(G)$ with $j \geq 2$, then $2k - t_1 \leq jk - t_1 = d(v_2^j, x) \leq d(v_2^j, v_1^1) + d(v_1^1, x) = k + t_1$ and we deduce $t_1 \geq k/2$, which is a contradiction. Hence, $\deg(v_1^1) = 2$.

If $t_1 \in [k/2, k)$, then it is possible to have $[v_1^1, v_2^2] \in E(G)$, but there is not any edge starting in v_1^1 and finishing in v_2^j with $j > 2$: if $[v_1^1, v_2^j] \in E(G)$ with $j > 2$, then $3k - t_1 \leq jk - t_1 = d(v_2^j, x) \leq d(v_2^j, v_1^1) + d(v_1^1, x) = k + t_1$ and we deduce $t_1 \geq k$, which is a contradiction. Hence, $\deg(v_1^1) \leq 3$.

A similar argument gives that if $t_1 = k$, then it is possible to have $[v_1^1, v_2^1], [v_1^1, v_2^2] \in E(G)$, but there is not any edge starting in v_1^1 and finishing in v_2^j with $j > 2k$.

We have similar results for the vertex $v_1^{j_x}$.

If a vertex v_1^j with $2 \leq j \leq j_x - 1$ has $\deg(v_1^j) \geq 3$, then it belongs to an edge finishing in some vertex of Γ_2. But this is not possible, since then $d_G(x_0, \Gamma_2) < nk/4$. Hence, every vertex v_1^j with $2 \leq j \leq j_x - 1$ has degree 2, and consequently, every vertex v_2^j with $2 < j < j_x - 1$ has degree 2 also.

Therefore, we have proved the result if $t_1 = k$ by identifying x and z, and considering the first set of edges in the statement of the Proposition. It suffices to deal with the case $t_1 = t_2 = k/2$, since this case allows more edges: if $t_1 \in (0, k/2)$, then $\deg(v_1^1) = 2$ and if $t_2 \in (0, k/2)$, then $\deg(v_1^{j_x}) = 2$. In this case, it is possible to have $[v_1^1, v_2^2], [v_1^{n/2}, v_2^{n/2-1}] \in E(G)$; hence, we have proved the result by identifying v_1^1 and z, and considering the second set of edges in the statement of the Proposition.

We deal now with the case n odd. We use the same notation that in the case n even. The previous arguments give that if $t_1 = k$ and we identify x and z, then G is isomorphic to a graph obtained by adding to C_n any amount of multiple edges and/or loops and a subset (proper or not) of

$$\{ [w_1^1, w_2^1], [w_1^1, w_2^2], [w_1^{(n-1)/2}, w_2^{(n-1)/2-1}] \}$$

(note that in this case w is the midpoint of $[w_1^{(n-1)/2}, w_2^{(n-1)/2}]$). Now, by symmetry, it suffices to deal with the case $t_1 \in (k/2, k)$, and then $d(v_1^{(n-1)/2}, y) \in (k/2, k)$. In this case, the previous arguments give that it is possible to have $[v_1^1, v_2^2], [v_1^{(n-1)/2}, v_2^{(n-1)/2}] \in E(G)$; hence, we have proved the result by identifying v_2^1 and z.

It is not difficult to check that if a graph can be obtained by adding to C_n some of those edges, then it belongs to $\mathcal{F}_n$. □

The following result gives an optimal inequality between the order of a graph and its hyperbolicity constant. Furthermore, it is useful in the study of the hyperbolicity of complement of graphs (see [4]).

Theorem 30. *Let G be any graph with n vertices. If every edge has length k, then $\delta(G) \leq nk/4$. Moreover, if $n \geq 3$ we have $\delta(G) = nk/4$ if and only if $G \in \mathcal{F}_n$; if $n = 2$, $\delta(G) = k/2$ if and only if G has a multiple edge; if $n = 1$, $\delta(G) = k/4$ if and only if G has a loop.*

Proof. It is not difficult to check the result for $n = 1$ and $n = 2$. Assume now that $n \geq 3$.

Let $T = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be any geodesic triangle in G. By Lemma 18, we can assume that T is a cycle. Since G has n vertices and $L(e) = k$ for every $e \in E(G)$, we have $L(T) \leq n$ and $L(\gamma_j) \leq nk/2$ for $j = 1, 2, 3$. If $\gamma_1 = [xy]$ and $p \in \gamma_1$, then $d(p, \gamma_2 \cup \gamma_3) \leq d(p, \{x, y\}) \leq nk/4$. Hence $\delta(T) \leq nk/4$, and consequently $\delta(G) \leq nk/4$.

If $\delta(G) = nk/4$, then every inequality in the previous argument must be equality. Hence, there exists a geodesic triangle T in G with $L(T) = nk$, and there exists a geodesic γ_1 in T such that $L(\gamma_1) = nk/2$. Without loss of generality we can assume that the midpoint x_0 of γ_1 verifies $d_G(x_0, \gamma_2 \cup \gamma_3) = nk/4$; since T is a cycle, then $L(\gamma_2 \cup \gamma_3) = nk/2 = L(\gamma_1)$; consequently, $\Gamma_2 := \gamma_2 \cup \gamma_3$ is a geodesic; we define $\Gamma_1 := \gamma_1$, and $G_0 := \Gamma_1 \cup \Gamma_2 = T$. Then $d_G(x_0, \Gamma_2) = nk/4$. Furthermore, $L(G_0) = L(T) = nk$, and this implies $V(G) = V(G_0)$ and G_0 isomorphic to C_n. Consequently, $G \in \mathcal{F}_n$.

If $G \in \mathcal{F}_n$, then there exists a bigon $B = \{\Gamma_1, \Gamma_2\}$ such that $L(\Gamma_1) = L(\Gamma_2) = nk/2$ and the midpoint x_0 of Γ_1 satisfies $d_G(x_0, \Gamma_2) = nk/4$. Then we deduce that $\delta(T) \geq nk/4$ and $\delta(G) \geq nk/4$. Since we have proved $\delta(G) \leq nk/4$, we conclude $\delta(G) = nk/4$. □

Note that it is not possible to obtain the reverse inequality $\delta(G) \geq cnk$ for some positive constant c, since any tree T has $\delta(T) = 0$.

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References

[1] Alvarez, V., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Denjoy domains, *Geom. Dedicata* **121** (2006), 221-245.

[2] Balogh, Z. M. and Buckley, S. M., Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003), 261-301.

[3] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic Graphs. Preprint.

[4] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E., Hyperbolicity and complement of graphs. Preprint.

[5] Bonk, M., Heinonen, J. and Koskela, P., *Uniformizing Gromov hyperbolic spaces.* Astérisque **270** (2001).

[6] Bowditch, B. H., Notes on Gromov's hyperobolicity criterion for pathmetric spaces. Group theory from a geometrical viewpoint, Trieste, 1990 (ed. E. Ghys, A. Haefliger and A. Verjovsky; World Scientific, River Edge, NJ, 1991) 64-167.

[7] Brinkmann, G., Koolen J. and Moulton, V., On the hyperbolicity of chordal graphs, *Ann. Comb.* **5** (2001), 61-69.

[8] Chepoi, V., Dragan, F. F., Estellon, B., Habib, M. and Vaxes Y., Notes on diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs, *Electr. Notes Discrete Math.* **31** (2008), 231-234.

[9] Dirac, G .A., Généralisation du théorème de Menger, *C .R. Acad. Sci. Paris* **250** (26) (1960), 4252-4253.

[10] Frigerio, R. and Sisto, A., Characterizing hyperbolic spaces and real trees, *Geom Dedicata* **142** (2009), 139-149.

[11] Ghys, E. and de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[12] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[13] Gromov, M. (with appendices by M. Katz, P. Pansu and S. Semmes), Metric Structures for Riemannian and Non-Riemannnian Spaces. Progress in Mathematics, vol. 152. Birkhäuser, 1999.

[14] Hästö, P. A., Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics, *Proc. Amer. Math. Soc.* **134** (2006), 1137–1142.

[15] Hästö, P. A., Lindén, H., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics. To appear in *J. Math. Soc. Japan.*

[16] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, *Bull. London Math. Soc.* **42** (2010), 282-294.

[17] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Comparative Gromov hyperbolicity results for the hyperbolic and quasihyperbolic metrics, *Complex Var. Elliptic Eq.* **55** (2010), 127-135.

[18] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Uniformly separated sets and Gromov hyperbolicity of domains with the quasihyperbolic metric, *Medit. J. Math.* **8** (2011), 47-65.

[19] Jonckheere, E. A., Controle du trafic sur les reseaux a geometrie hyperbolique–Une approche mathematique a la securite de l'acheminement de l'information, *Journal Europeen de Systemes Automatises* **37(2)** (2003), 145-159.

[20] Jonckheere, E. A. and Lohsoonthorn, P., Geometry of network security, *American Control Conference* **ACC** (2004), 111-151.

[21] Jonckheere, E. A., Lohsoonthorn, P. and Bonahon, F., Scaled Gromov hyperbolic graphs, *Journal of Graph Theory* **2** (2007), 157-180.

[22] Kanai, M., Rough isometries and combinatorial approximations of geometries of noncompact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.

[23] Koolen, J. H. and Moulton, V., Hyperbolic Bridged Graphs, *Europ. J. Combinatorics* **23** (2002), 683-699.

[24] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity in cartesian product graphs, *Proc. Indian Acad. Sci. (Mathematical Sciences)* **120** (2010), 1-17.

[25] Oshika, K., Discrete groups, AMS Bookstore, 2002.

[26] Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.

[27] Portilla, A., Rodríguez, J. M. and Tourís, E., Stability of Gromov hyperbolicity, *J. Advanced Math. Studies* **2** (2009), 77-96.

[28] Portilla, A. and Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83-110.

[29] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, *Discrete Math.* **311** (2011), 211-219.

[30] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.

[31] Rodríguez, J. M. and Tourís, E., A new characterization of Gromov hyperbolicity for Riemann surfaces, *Publ. Mat.* **50** (2006), 249-278.

[32] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica* **23** (2007), 209-228.

[33] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces. To appear in *J. Math. Anal. Appl.*