(2,1)-TOTAL LABELING OF PLANAR GRAPHS WITH LARGE GIRTH AND LOW MAXIMUM DEGREE *

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abstract A $(2,1)$-total labeling of a graph G is a labeling of vertices and edges,such that (1) any two adjacent vertices of G receive distinct integers, (2) any two adjacent edges receive distinct integers,and (3) a vertex and its incident edges receive integers that differ by at least 2 in absolute value .The span of a $(2,1)$-total labeling is the difference between the maximum label and the minimum label.We note the minimum span λ_2^T. In this paper,we prove that if G is a planar graph with $\Delta \leq 3$ and girth $g \geq 18$,then $\lambda_2^T(G) \leq 5$.If G is a planar graph with $\Delta \leq 4$ and girth $g \geq 12$,then $\lambda_2^T(G) \leq 7$.

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1 Introduction

The channel assignment problem involves assigning radio channels to transmitters with the constraint that if transmitters are close geographically, interferences will occur if they attempt to transmit on frequencies which are close in the radio spectrum.The problem of channel assignment has been modelled by a particular coloring,the $L(2,1)-$labeling introduced by Griggs and Yeh[1],which is a vertex-coloring such that two neighbors must have

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colors at distance 2 and two vertices at distance 2 must have colors at distance 1.The incidence graph of a graph G is the graph obtained by replacing each edge of G by a path of length 2.An $L(p,1)-$labeling of the incidence graph of G corresponds to a particular total coloring of the graph G introduced by Havet and Yu[2, 3]: the $L(p,1)-$total labeling.When $p=2$,that is $L(2,1)-$total labeling.A $L(2,1)-$total labeling of a graph $G=(V,E)$ is a function $c{:}V\bigcup E\rightarrow\{0,1,\cdots\}$ verifying:

(a) $\forall(u,v)\in V^2{:}uv\in E\Rightarrow c(u)\neq c(v)$;

(b) $\forall(u,v,w)\in V^3{:}uv\in E,uw\in E\Rightarrow c(uv)\neq c(uw)$;

(c) $\forall(u,v)\in V^2{:}uv\in E\Rightarrow|c(u)-c(uv)|\geq 2$.

The width of the minimum range of labels required for such a labeling of G is called the $(2,1)-$total number and is denoted by $\lambda_2^T(G)$.

In what follows,we will consider only simple graphs.A $k-vertex$ will be a vertex of degree k.The girth $g=g(G)$ of a graph G is the length of its shortest cycle(if any).Acyclic graphs are considered to have infinite girth.From now on,we will consider the set of labels as an interval of integers beginning by zero.

2 Main results

Conjecture (*Havat* and *Yu*[4]). *Let G be a graph with $\Delta(G)\leq 3,G\neq K_4$,then $\lambda_2^T(G)\leq 5$.*

Lemma 1 (*Havat* and *Yu*[4]). *If $\Delta(G)\geq 2$,then $\lambda_2^T(G)\leq 2\Delta$.*

Lemma 2 (*Bazzaro*[5]). *Let G be a planar graph with girth g and having m edges,then $m\leq\frac{g(n-2)}{g-2}$.*

Theorem 1. *Let G be a planar graph with $\Delta\leq 3$ and girth $g\geq 18$,then $\lambda_2^T(G)\leq 5$.*

Proof. We will prove Theorem 1 by contradiction.Suppose $H=(V,E)$ is a minimal counterexample to Theorem 1 with $\Delta(G)\leq 3,g\geq 18$ and $\lambda_2^T=6$ having a minimum number of vertices and edges,with n vertices,m edges.

It is clear that the counterexample H is connected.Otherwise,there is a component of H,which is a smaller counterexample.

Claim 1. *H has no 1-vertex.*

Proof. Suppose that H contains a 1-vertex v linked to a vertex w.Let $H'=H\backslash\{v\}$.Observe that $\Delta(H')\leq 3,g\geq 18$ and the number of vertices of H' is less than the number of vertices of H.So,by minimality of H,H' has a $(2,1)-$total labeling in $[0,5]$.

Now,we label the vertices and edges of H with the corresponding labels of the vertices and edges of H'.Then we extend this labeling:to label vw,we have six labels and at most five labels may be forbidden(three labels may be forbidden by the label of w and two labels may be forbidden by the incident edges of w).So,there remains at least one label for vw.Finally,to label the vertex v ,we have at least two choices(That is 6-3-1=2.).This implies that we have a $(2,1)-$total labeling of H in $[0,5]$ which is a contradiction. ∎

Claim 2. *Each 2-vertex of H is adjacent to at least a 3-vertex.*
Proof. Suppose that H contains a 2-vertex v adjacent to no 3-vertex,by Claim 1,v is adjacent to two 2-vertex u and w.

The graph $H' = H\backslash\{v\}$ has a maximum degree $\Delta(H') \leq 3$ and $g \geq$ 18,by minimality of H,H' has a $(2,1)-$total labeling in $[0,5]$,i.e.There is a function $f : V(H')\bigcup E(H') \to \{0,1,2,3,4,5\}$.

We label the vertices and edges of H with the corresponding labels of the vertices and edges of H'.Then we extend this labeling in order to $(2,1)-$total label H with six labels.We erase the labels of u and w,then label u, v, w, uv and vw.

Let $uv = e_1, vw = e_2$,and the other edge incident to $u(w)$ in H is $e_3(e_4)$.We consider two cases.

(R_1) $f(e_3) = f(e_4) = 5$.For the vertex u,the forbidden labels are 4,5 and one label belong to $\{0,1,2,3\}$,then we have at least three available labels in $\{0,1,2,3\}$ for u.Similarly,we have at least three available labels in $\{0,1,2,3\}$ for w.So,we have at least one available label in $\{0,1\}(\{2,3\})$ for $u(w)$.We consider the following four subcases:

Case 1:$f(u) = 0, f(w) = 2$.Let $f(v) = 1, f(e_1) = 3, f(e_2) = 4$.
Case 2:$f(u) = 0, f(w) = 3$.Let $f(v) = 4, f(e_1) = 2, f(e_2) = 0$.
Case 3:$f(u) = 1, f(w) = 2$.Let $f(v) = 0, f(e_1) = 3, f(e_2) = 4$. Case 4:$f(u) = 1, f(w) = 3$.Let $f(v) = 2, f(e_1) = 4, f(e_2) = 0$.

So,if $f(e_3) = f(e_4) = 5$,we have a $(2,1)-$total labeling of H in $[0,5]$.

(R_2) At least one of $f(e_3)$ and $f(e_4)$ don't equal to 5.We have at least two available labels in $[0,5]$ for $u(w)$,and $(f(u), f(w))$ may have the following different values:

(0,0),(0,1),(0,2),(0,3),(0,4),(0,5), (1,0),(1,1),(1,2),(1,3),(1,4),(1,5),
(2,0),(2,1),(2,2),(2,3),(2,4),(2,5), (3,0),(3,1),(3,2),(3,3),(3,4),(3,5),
(4,0),(4,1),(4,2),(4,3),(4,4),(4,5), (5,0),(5,1),(5,2),(5,3),(5,4),(5,5).

If $(f(u), f(w)) = (q,p)$,then when label v, e_1 and e_2 in $[0,5]$,we only need to meet:$|f(e_1) - q| \geq 2, f(e_1) \neq f(e_3); f(e_2) \neq f(e_1), |f(e_2) - p| \geq 2, f(e_2) \neq f(e_4); f(v) \neq p, q, |f(v) - f(e_i)| \geq 2, i = 1,2$.

If $(f(u), f(w)) = (p,q)$,then when label v, e_1 and e_2 in $[0,5]$,we only need to meet:$|f(e_2) - q| \geq 2, f(e_2) \neq f(e_4); f(e_1) \neq f(e_2), |f(e_1) - p| \geq 2, f(e_1) \neq f(e_3); f(v) \neq p, q, |f(v) - f(e_i)| \geq 2, i = 1,2$.

So,it is clear that the relabeling case $(f(u), f(w)) = (q,p)$ is similar to the the relabeling case $(f(u), f(w)) = (p,q)$.Thus,we just need to discuss the case $(f(u), f(w)) = (p,q)$.

So,we just need to discuss the cases when the possible values of $(f(u), f(w))$ are as follows:

(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),
(1,1),(1,2),(1,3),(1,4),(1,5),
(2,2),(2,3),(2,4),(2,5),
(3,3),(3,4),(3,5),
(4,4),(4,5),
(5,5).

If $0 \leftrightarrow 5, 1 \leftrightarrow 4, 2 \leftrightarrow 3$, then for any two labels x and y, if $d(x,y) \geq 2(1)$,then after replacement,$d(x,y) \geq 2(1)$.So after replacement,the $(2,1)-$total labeling of H in $[0,5]$ is also a $(2,1)-$total labeling of H in $[0,5]$.

Hence,among the possible values of $(f(u),f(w))$,the similar relabeling cases are as follows:

(1,5) and (4,0);(2,4) and (3,1);(2,5) and (3,0);(3,3) and (2,2);(3,4) and (2,1);(3,5) and (2,0);(4,4) and (1,1);(4,5) and (1,0);(5,5) and (0,0).

Because the relabeling case $(f(u),f(w))=(3,1)$ is similar to the relabeling case $(f(u),f(w))=(1,3)$,the relabeling case $(f(u),f(w))=(2,4)$ is similar to the relabeling case $(f(u),f(w))=(1,3)$.In the same way,we know the relabeling case $(f(u),f(w))=(2,5)$ is similar to the relabeling case $(f(u),f(w))=(0,3)$.The relabeling case $(f(u),f(w))=(1,5)$ is similar to the relabeling case $(f(u),f(w))=(0,4)$.

So we just need to discuss the cases when the possible values of $(f(u),f(w))$ are as follows:

1.(0,0); 2.(0,1); 3.(0,2); 4.(0,3); 5.(0,4); 6.(0,5); 7.(1,1); 8.(1,2); 9.(1,3); 10.(1,4); 11.(2,2); 12.(2,3).

When $f(u)$=0,as we have at least two available labels in $[0,5]$ for w,so we can let $f(w)\neq 5$.Then there is a case of the possible value of $(f(u),f(w))$, which is one of the former five cases.So we don't need to discuss the 6th case.

When $f(u)$=2,as we have at least two available labels in $[0,5]$ for w,so we can let $f(w)\neq 3$.Then the possible values of $f(w)$ are 0,1,2,4 and 5.Next,we consider the following:

(i)If $f(w)=0$. Then $(f(u),f(w))=(2,0)$ and the relabeling case $(f(u),f(w))=(0,2)$ is similar to the 3rd case. (ii)If $f(w)=1$. Then $(f(u),f(w))=(2,1)$ and the relabeling case $(f(u),f(w))=(2,1)$ is similar to the 8th case. (iii)If $f(w)=2$. Then $(f(u),f(w))=(2,2)$ and the relabeling case $(f(u),f(w))=(2,2)$ is similar to the 11th case. (iv) If $f(w)=4$.Then $(f(u),f(w))=(2,4)$ and the relabeling case $(f(u),f(w))=(2,4)$ is similar to the 9th case. (v) If $f(w)=5$.Then $(f(u),f(w))=(2,5)$ and the relabeling case $(f(u),f(w))=(2,5)$ is similar to the 4th case.

So,we don't need to discuss the 12th case.Thus,we only need to discuss the following ten cases:

Case 1:$(f(u),f(w))=(0,0)$.Let $f(v)=1,f(e_1)\in\{3,4\}-\{f(e_3)\},f(e_2)\in\{3,4,5\}-\{f(e_1),f(e_4)\}$.

Case 2:$(f(u),f(w))=(0,1)$.If $f(e_4)=3$,let $f(v)=2,f(e_1)\in\{4,5\}-\{f(e_3)\},f(e_2)\in\{4,5\}-\{f(e_1)\}$.Otherwise,we divide into two subcases.

Case 2.1:$f(e_3)=3$.Let $f(v)=2,f(e_1)\in\{4,5\}-\{f(e_2)\},f(e_2)\in\{4,5\}-\{f(e_4)\}$.

Case 2.2:$f(e_3)\neq 3$.we divide into two subcases again.

Case 2.2.1:$f(e_3)\neq 2$.Let $f(v)=5,f(e_1)=2,f(e_2)=3$.

Case 2.2.2:$f(e_3)=2$.Let $f(v)=2,f(e_1)\in\{4,5\}-\{f(e_2)\},f(e_2)\in\{4,5\}-\{f(e_4)\}$.

Case 3:$(f(u),f(w))=(0,2)$.Let $f(v)=1,f(e_1)\in\{3,4,5\}-\{f(e_2),f(e_3)\}$, $f(e_2)\in\{4,5\}-\{f(e_4)\}$.

Case 4:$(f(u),f(w))=(0,3)$.Let $f(v)=5,f(e_1)\in\{2,3\}-\{f(e_3)\},f(e_2)\in\{0,1\}-\{f(e_4)\}$.

Case 5:$(f(u),f(w))=(0,4)$.Let $f(v)=5,f(e_1)\in\{2,3\}-\{f(e_3)\},f(e_2)\in\{0,1\}-\{f(e_4)\}$.

Case 6:$(f(u),f(w))=(1,1)$.Let $f(v)=0,f(e_1)\in\{4,5\}-\{f(e_3)\},f(e_2)\in\{3,4,5\}-\{f(e_1),f(e_4)\}$.

Case 7:$(f(u),f(w))=(1,2)$.Let $f(v)=0, f(e_1)\in\{3,4,5\}-\{f(e_2),f(e_3)\}$, $f(e_2)\in\{4,5\}-\{f(e_4)\}$.

Case 8:$(f(u),f(w))=(1,3)$.If $f(e_4)\neq 0$,let $f(v)=2, f(e_1)\in\{4,5\}-\{f(e_3)\}, f(e_2)=0$.Otherwise,let $f(v)=0, f(e_1)\in\{3,4\}-\{f(e_3)\}, f(e_2)=5$.

Case 9:$(f(u),f(w))=(1,4)$.If $f(e_4)\neq 2$,let $f(v)=0, f(e_1)\in\{4,5\}-\{f(e_3)\}, f(e_2)=2$.Otherwise,let $f(v)=2, f(e_1)\in\{4,5\}-\{f(e_3)\}, f(e_2)=0$.

Case 10:$(f(u),f(w))=(2,2)$.If $f(e_3)=4$,let $f(v)=3, f(e_1)\in\{0,5\}-\{f(e_2)\}, f(e_2)\in\{0,5\}-\{f(e_4)\}$.Otherwise,we divide into two subcases.

Case 10.1:$f(e_4)\neq 5$.Let $f(v)=0, f(e_1)=4, f(e_2)=5$.

Case 10.2:$f(e_4)=5$.As $f(e_3)$ and $f(e_4)$ don't simultaneously equal to 5,so $f(e_3)\neq 5$.Let $f(v)=0, f(e_1)=5, f(e_2)=4$.

So,if $f(e_3)$ and $f(e_4)$ don't simultaneously equal to 5,then we have a $(2,1)-$total labeling of H in $[0,5]$.

With these cases,we have a $(2,1)-$total labeling of H in $[0,5]$ which is a contradiction to the hypothesis. ∎

We complete the proof of Theorem 1 using a discharging procedure.We denote by $w(v)$ the initial charge of the vertex v and $w^*(v)$ the new charge after the procedure.

We assign to each vertex v the charge $w(v)=d(v)-\frac{9}{4}$.Hence,the total charge of the vertices of H is

$$\sum_{v\in V(H)} w(v)=\sum_{v\in V(H)}(d(v)-\frac{9}{4})=\sum_{v\in V(H)} d(v)-\frac{9}{4}n=2m-\frac{9}{4}n.$$

Now,if $g\geq 18$,by Lemma 2,$m\leq\frac{9(n-2)}{8}$;that implies $\sum_{v\in V(H)} w(v)\leq-\frac{9}{2}<0$.

We shall now redistribute the charges without changing the total charge. The rule is the following:

Rule: Each 3-vertex gives $\frac{1}{4}$ to each adjacent 2-vertex.

Now,we computer the new charge $w^*(v)$ of each vertex v.Let v be a vertex with degree k:

- If $k=2$,then v started with a charge $w(v)=2-\frac{9}{4}=-\frac{1}{4}$ has gained at least $\frac{1}{4}$,so its new charge $w^*(v)\geq-\frac{1}{4}+\frac{1}{4}\geq 0$.
- If $k=3$,then v started with a charge $w(v)=3-\frac{9}{4}=\frac{3}{4}$ has given at most $\frac{3}{4}$,so its new charge $w^*(v)\geq\frac{3}{4}-\frac{3}{4}\geq 0$.

After the redistribution,the total sum of the charges is positive or null.The contradiction with $\sum_{v\in V(H)} w(v)=\sum_{v\in V(H)} w^*(v)<0$ completes the proof. ∎

Theorem 2. *Let G be a planar graph with $\Delta\leq 4$ and girth $g\geq 12$,then $\lambda_2^T(G)\leq 7$.*

Proof. We prove it by contradiction. Suppose $H=(V,E)$ is a minimal counterexample with $\Delta(G)\le 4, g\ge 12$ and $\lambda_2^T=8$ having a minimum number of vertices and edges, with n vertices, m edges.

By the method of the proof of Theorem 1, we can prove that the counterexample H is connected and has no 1-vertex.

Claim . *Each 2-vertex of H is adjacent to at least a 4-vertex.*

Proof. Suppose that H contains a 2-vertex v adjacent to u and w, where $d(u)<4$ and $d(w)<4$.

The graph $H'=H\backslash\{v\}$ has a maximum degree $\Delta(H')\le 4$ and $g\ge 12$, by minimality of H, H' has a $(2,1)-$total labeling in $[0,7]$, i.e. there is a function $f: V(H')\bigcup E(H')\to\{0,1,2,\dots,7\}$.

We label the vertices and edges of H with the corresponding labels of the vertices and edges of H'. Then we extend this labeling in order to $(2,1)-$total label H with eight labels. We label v, uv and vw.

Let $uv=e_1, vw=e_2$. To label e_1, we have eight labels and at most five labels may be forbidden(three labels may be forbidden by the label of u and two labels may be forbidden by the incident edges of u). So, there remains at least three labels for e_1. Similarly, we have at least two(That is 8-3-2-1=2) choices for e_2. We denote by $A(e_1)(A(e_2))$ the available label set of $e_1(e_2)$. It is easy to see $|A(e_1)|\ge 3and|A(e_2)|\ge 2$. We shall now prove that there are $f(e_1)$ and $f(e_2)$, which are meeting $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}|\le 5$. Then at most seven(That is 2+5=7.) labels may be forbidden for v. So we have at least one available label for v. Thus, we have a $(2,1)-$total labeling of H in $[0,7]$ which is a contradiction to the hypothesis. This implies that the Claim is correct. We consider three cases.

Case 1: At least one of e_1 and e_2 can be labeled by 0 or 7. Then it is clear that $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}|\le 5$.

Case 2: Both e_1 and e_2 can't be labeled by 0 or 7, but at least one of them can be labeled by 3 or 4. We consider the following three subcases.

Case 2.1: $3\in A(e_1)$. Let $f(e_1)=3$, and as $|A(e_2)|\ge 2$, so $I=A(e_2)\bigcap\{1,2,4,5\}\ne\emptyset$. Let $f(e_2)\in I$, then $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}|\le 5$.

Case 2.2: $4\in A(e_1)$. Let $f(e_1)=4$, and as $|A(e_2)|\ge 2$, so $I=A(e_2)\bigcap\{2,3,5,6\}\ne\emptyset$. Let $f(e_2)\in I$, then $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}|\le 5$.

Case 2.3: $k\in A(e_2), k=3$ or $k=4$. In the same way, we prove that there are $f(e_1)$ and $f(e_2)$, which are meeting $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}|\le 5$.

Case 3: Both e_1 and e_2 can't be labeled by anyone of 0,3,4,7. Then $A(e_i)\subseteq\{1,2,5,6\}, i=1,2$. At first, we don't require $f(e_1)\ne f(e_2)$, then at most five(That is 3+2=5.) labels may be forbidden for e_2. So we have at least three available labels for e_2, i.e. $|A(e_2)|\ge 3$. Finally, we look for suitable $f(e_1)$ and $f(e_2)$, which are meeting $f(e_1)\ne f(e_2)$.

Now, for $1\le i\le 2$, all the possible cases of $A(e_i)$ are as follows: $\{1,2,5\}\subseteq A(e_i)$; $\{1,2,6\}\subseteq A(e_i)$; $\{1,5,6\}\subseteq A(e_i)$; $\{2,5,6\}\subseteq A(e_i)$. We consider the following two cases.

Case 3.1: $\{1,2,k\}\subseteq A(e_1), k=5$ or $k=6$. If $\{2,5,6\}\subseteq A(e_2)$ or $\{1,2,k\}\subseteq A(e_2), k=5$ or $k=6$, let $f(e_1)=1, f(e_2)=2$. If $\{1,5,6\}\subseteq$

$A(e_2)$,let $f(e_1)=2, f(e_2)=1$.

Case 3.2:$\{k,5,6\} \subseteq A(e_1), k=1$ or $k=2$.If $\{1,2,6\} \subseteq A(e_2)$ or $\{k,5,6\} \subseteq A(e_2), k=1$ or $k=2$,let $f(e_1)=5, f(e_2)=6$.If $\{1,2,5\} \subseteq A(e_2)$,let $f(e_1)=6, f(e_2)=5$.

With these cases,we prove that there are $f(e_1)$ and $f(e_2)$,which are meeting $|\bigcup_{1\le i\le 2}\bigcup_{-2+1\le j\le 2-1}\{f(e_i)+j\}| \le 5$.By above discussion,we have a $(2,1)-$total labeling of H in $[0,7]$ which is a contradiction to the hypothesis. ∎

We complete the proof using a discharging procedure.We denote by $w(v)$ the initial charge of the vertex v and $w^*(v)$ the new charge after the procedure.

We assign to each vertex v the charge $w(v)=d(v)-\frac{12}{5}$.Hence,the total charge of the vertices of H is

$$\sum_{v\in V(H)} w(v) = \sum_{v\in V(H)} (d(v)-\frac{12}{5}) = \sum_{v\in V(H)} d(v) - \frac{12}{5}n = 2m - \frac{12}{5}n.$$

Now,if $g \ge 12$,by Lemma 2,$m \le \frac{6(n-2)}{5}$;that implies $\sum_{v\in V(H)} w(v) \le -\frac{24}{5} < 0$.

We shall now redistribute the charges without changing the total charge. The rule is the following:

Rule: Each 4-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.

Now,we computer the new charge $w^*(v)$ of each vertex v.Let v be a vertex with degree k:

- If $k=2$,then v started with a charge $w(v)=2-\frac{12}{5}=-\frac{2}{5}$ has gained at least $\frac{2}{5}$,so its new charge $w^*(v)\ge -\frac{2}{5}+\frac{2}{5}\ge 0$.
- If $k=3$,then for v,the initial charge is equal to the new charge,i.e.$w^*(v)=w(v)=3-\frac{12}{5}\ge 0$.
- If $k=4$,then v started with a charge $w(v)=4-\frac{12}{5}=\frac{8}{5}$ has given at most $\frac{8}{5}$,so its new charge $w^*(v)\ge \frac{8}{5}-\frac{8}{5}\ge 0$.

After the redistribution,the total sum of the charges is positive or null.The contradiction with $\sum_{v\in V(H)} w(v) = \sum_{v\in V(H)} w^*(v) < 0$ completes the proof. ∎

References

[1] Griggs J R, Yeh R K, Labeling graphs with a condition at distance two,SIAM J. Discrete Mathematics.1992,5:586-595.

[2] F.Havet,M.-L.Yu,(d,1)-Total labeling of graphs, Technical Report 4650,INRIA,2002.

[3] F.Havet (d,1)-Total labeling of graphs.Workshop on Graphs and Algorithems,Dijon,France.2003.

[4] F.Havet,M.-L.Yu,(p,1)-Total labeling of graphs,Discrete Mathematics.2007.

[5] F.Bazzaro,M.Montassier,A.Raspaud.(d,1)-Total labeling of planar graph with large girth and high maximum degree[J],Discrete Math,2007,307:2141-2151.

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