MORE IDENTITIES ON THE TRIBONACCI NUMBERS

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ABSTRACT. In this paper, we use a simple method to derive different recurrence relations on the Tribonacci numbers and their sums. By using the companion matrices and generating matrices, we get more identities on the Tribonacci numbers and their sums, which are more general than that given in literature [E. Kilic, Tribonacci Sequences with Certain Indices and Their Sum, Ars Combinatoria 86 (2008), 13-22.].

1. INTRODUCTION

The Tribonacci sequence is like the Fibonacci sequence, but instead of starting with two predetermined terms, the sequence starts with three predetermined terms and each term afterwards is the sum of the preceding three terms, that is

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad n \geq 3 \tag{1.1}$$

where $T_0 = T_1 = 0$, $T_2 = 1$. The first few tribonacci numbers are:

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, \cdots$$

The tribonacci constant $\frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}$ is the ratio toward which adjacent tribonacci numbers tend. It is a root of the polynomial x^3-x^2-x-1, approximately 1.83929 , and also satisfies the equation $x^4 - 2x^3 + 1 = 0$.

In [1], the author derives new recurrence relations and generating matrices for the sums of usual Tribonacci numbers $\{S_n\}$ and $4n$ subscripted Tribonacci numbers $\{T_{4n}\}$, and their sums $\{S_{4n}\}$, where $S_n = \sum_{k=0}^{n} T_k$. In this paper, we intend to give the more identities on the Tribonacci numbers $\{T_{n+w}\}$, arbitrary subscripted Tribonacci numbers $\{T_{w(n+h)}\}$, and their sums $\{S_{n+w}\}, \{S_{w(n+h)}\}$, where w and h are arbitrary positive integers.

2. ANOTHER RECURRENCE RELATION

By the recurrence (1.1), we have two expressions: $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, and $T_{n-1} = T_{n-2} + T_{n-3} + T_{n-4}$, subtract the second expression from

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the first expression, we have another recurrence relation

$$T_n = 2T_{n-1} - T_{n-4}, \; n \geq 4 \tag{2.1}$$

with the initial conditions: $T_0 = T_1 = 0, T_2 = T_3 = 1$. This follows the result on paper [2].

Comparing (1.1) with (2.1), we find that there are three differences: the first is the initial conditions number, (1.1) is 3, but (2.1) is 4; the second is the order, (1.1) is 3, but (2.1) is 4; the third is characteristic equation, (1.1) is $x^3 - x^2 - x - 1 = 0$, but (2.1) is $x^4 - 2x^3 + 1 = 0$. Thus their companion matrices [3] are also different, denote them by A, B. respectively,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{2.2}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}. \tag{2.3}$$

From the recurrence (1.1) and the initial conditions, summing up the begin n recurrence relations, we can derive:

$$S_n = 1 + S_{n-1} + S_{n-2} + S_{n-3}, \; n \geq 3, \tag{2.4}$$

where $S_0 = S_1 = 0$, $S_2 = 1$. Subtract the relation $S_{n-1} = 1 + S_{n-2} + S_{n-3} + S_{n-4}, (n \geq 4)$ from (2.4), we can obtain

$$S_n = 2S_{n-1} - S_{n-4}, \; n \geq 4, \tag{2.5}$$

where $S_0 = S_1 = 0$, $S_2 = 1$, $S_3 = 2$. Its companion matrices also is B. The first few tribonacci numbers' sums are:

$$0, 0, 1, 2, 4, 8, 15, 28, 52, 96, 177, 326, 600, 1104, 2031, 3736, 6872, \cdots$$

Define the 3×3 matrix A_n and the 4×4 matrix B_n as shown:

$$A_n = \begin{bmatrix} T_n & T_{n+1} & T_{n+2} \\ T_{n+1} & T_{n+2} & T_{n+3} \\ T_{n+2} & T_{n+3} & T_{n+4} \end{bmatrix}, \tag{2.6}$$

$$B_n = \begin{bmatrix} S_n & S_{n+1} & S_{n+2} & S_{n+3} \\ S_{n+1} & S_{n+2} & S_{n+3} & S_{n+4} \\ S_{n+2} & S_{n+3} & S_{n+4} & S_{n+5} \\ S_{n+3} & S_{n+4} & S_{n+5} & S_{n+6} \end{bmatrix}. \tag{2.7}$$

Using direct computation, we have $A_n = AA_{n-1}$, $B_n = BB_{n-1}$. by the induction on n, one obtains that:

$$A_n = AA_{n-1} = A^2 A_{n-2} = \cdots = A^n A_0, \tag{2.8}$$

$$B_n = BB_{n-1} = B^2 B_{n-2} = \cdots = B^n B_0. \tag{2.9}$$

By the definition of the sums of the Tribonacci numbers, $S_n = T_0 + T_1 + \cdots + T_{k-4} + T_{k-3} + T_{k-2} + T_{n-1} + T_n = S_{n-4} + (T_{k-3} + T_{k-2} + T_{n-1}) + T_n = S_{n-4} + 2T_n$, then we get

$$T_n = \frac{1}{2}(S_n - S_{n-4}) \tag{2.10}$$

3. Identities on the Tribonacci numbers and their sums

Suppose we want the identity of the form ($w > 2$ is an arbitrary given positive integer)

$$T_{n+w} = x_1 T_n + x_2 T_{n+1} + x_3 T_{n+2}, \tag{3.1}$$

By the recurrence theory, there are: $T_{n+w+1} = x_1 T_{n+1} + x_2 T_{n+2} + x_3 T_{n+3}$, $T_{n+w+2} = x_1 T_{n+2} + x_2 T_{n+3} + x_3 T_{n+4}$, thus the constants x_1, x_2, x_3 follow the system:

$$\begin{bmatrix} T_n & T_{n+1} & T_{n+2} \\ T_{n+1} & T_{n+2} & T_{n+3} \\ T_{n+2} & T_{n+3} & T_{n+4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{n+w} \\ T_{n+w+1} \\ T_{n+w+2} \end{bmatrix}.$$

From (2.8) we use elementary row operations on the augmented matrix [4] A_0^*:

$$A_0^* = \begin{bmatrix} T_0 & T_1 & T_2 & T_w \\ T_1 & T_2 & T_3 & T_{w+1} \\ T_2 & T_3 & T_4 & T_{w+2} \end{bmatrix}, \tag{3.2}$$

to obtain a classes of identities (for w various values):

$T_{n+5} = 2T_n + 3T_{n+1} + 4T_{n+2}$;
$T_{n+6} = 4T_n + 6T_{n+1} + 7T_{n+2}$;
$T_{n+7} = 7T_n + 11T_{n+1} + 13T_{n+2}$;
$T_{n+8} = 13T_n + 20T_{n+1} + 24T_{n+2}$;
$T_{n+9} = 24T_n + 37T_{n+1} + 44T_{n+2}$;
...

We use the principle of induction on w to get

$$T_{n+w} = T_{w-1}T_n + (T_{w-2} + T_{w-1})T_{n+1} + T_w T_{n+2},\ n \geq 0,\ w \geq 2. \tag{3.3}$$

For the sums of the Tribonacci numbers $\{S_{n+w}\}$, we use the same method, use elementary row operations on the augmented matrix B_0^*

$$B_0^* = \begin{bmatrix} S_0 & S_1 & S_2 & S_3 & S_w \\ S_1 & S_2 & S_3 & S_4 & S_{w+1} \\ S_2 & S_3 & S_4 & S_5 & S_{w+2} \\ S_3 & S_4 & S_5 & S_6 & S_{w+3} \end{bmatrix}, \tag{3.4}$$

for various positive integer w, we get identities:

$$S_{n+6} = -4S_n - 2S_{n+1} - S_{n+2} + 8S_{n+3}$$
$$S_{n+7} = -8S_n - 4S_{n+1} - 2S_{n+2} + 15S_{n+3};$$
$$S_{n+8} = -15S_n - 8S_{n+1} - 4S_{n+2} + 28S_{n+3};$$
$$S_{n+9} = -28S_n - 15S_{n+1} - 8S_{n+2} + 52S_{n+3};$$
$$S_{n+10} = -52S_n - 28S_{n+1} - 15S_{n+2} + 96S_{n+3};$$

...

We use the principle of induction on w to get

$$S_{n+w} = -S_{w-2}S_n - S_{w-3}S_{n+1} - S_{w-4}S_{n+2} + S_{w-1}S_{n+3},\ n \geq 0,\ w \geq 4. \tag{3.5}$$

4. Identities on the arbitrary subscripted Tribonacci numbers and their sums

Suppose we want the identity of the form (w and $h > 2$ are arbitrary given positive integers)

$$T_{w(n+h)} = x_1 T_{wn} + x_2 T_{w(n+1)} + x_3 T_{w(n+2)}, \tag{4.1}$$

we construct the 3×3 generating matrix C and $A1_n$:

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_1 & x_2 & x_3 \end{bmatrix} \tag{4.2}$$

$$A1_n = \begin{bmatrix} T_{wn} & T_{w(n+1)} & T_{w(n+2)} \\ T_{w(n+1)} & T_{w(n+2)} & T_{w(n+3)} \\ T_{w(n+2)} & T_{w(n+3)} & T_{w(n+4)} \end{bmatrix} \tag{4.3}$$

Using direct computation, we have $A1_n = CA1_{n-1}$. by the induction on n, one obtains that:

$$A1_n = CA1_{n-1} = C^2 A1_{n-2} = \cdots = C^n A1_0. \tag{4.4}$$

Thus, in order to solve the constants x_1, x_2, x_3, we use elementary row operations on the augmented matrix $A1_0^*$,

$$A1_0^* = \begin{bmatrix} T_0 & T_w & T_{2w} & T_{wh} \\ T_w & T_{2w} & T_{3w} & T_{w(h+1)} \\ T_{2w} & T_{3w} & T_{4w} & T_{w(h+2)} \end{bmatrix}$$

to obtain a classes of identities (for w and h various value):

$T_{2(n+9)} = 1316T_{2n} + 1705T_{2(n+1)} + 4452T_{2(n+2)}$;

$T_{3(n+5)} = 44T_{3n} - 213T_{3(n+1)} + 274T_{3(n+2)}$;

$T_{4(n+4)} = 11T_{4n} + 56T_{4(n+1)} + 126T_{4(n+2)}$;

$T_{5(n+5)} = 442T_{5n} + 463T_{5(n+1)} + 9304T_{5(n+2)}$;

$T_{7(n+8)} = 1825767089T_{7n} + 27412145228T_{7(n+1)} + 130014406756T_{7(n+2)}$;

...

Especially, there are:

$T_{2n} = 3T_{2(n-1)} + T_{2(n-2)} + T_{2(n-3)}$;

$T_{3n} = 7T_{3(n-1)} - 5T_{3(n-2)} + T_{3(n-3)}$;

$T_{4n} = 11T_{4(n-1)} + 5T_{4(n-2)} + T_{4(n-3)}$;

$T_{5n} = 21T_{5(n-1)} + T_{5(n-2)} + T_{5(n-3)}$;

$T_{6n} = 39T_{6(n-1)} - 11T_{6(n-2)} + T_{6(n-3)}$;

...

Suppose we want the identity of the form (w and $h > 3$ are arbitrary given positive integers)

$$S_{w(n+h)} = x_1 S_{wn} + x_2 S_{w(n+1)} + x_3 S_{w(n+2)} + x_4 S_{w(n+3)}, \qquad (4.5)$$

we can construct the 4×4 generating matrix D and $B1_n$:

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix},$$

$$B1_n = \begin{bmatrix} S_{wn} & S_{w(n+1)} & S_{w(n+2)} & S_{w(n+3)} \\ S_{w(n+1)} & S_{w(n+2)} & S_{w(n+3)} & S_{w(n+4)} \\ S_{w(n+2)} & S_{w(n+3)} & S_{w(n+4)} & S_{w(n+5)} \\ S_{w(n+3)} & S_{w(n+4)} & S_{w(n+5)} & S_{w(n+6)} \end{bmatrix} \qquad (4.6)$$

Using direct computation, we have $B1_n = DB1_{n-1}$. by the induction on n, one obtains that:

$$B1_n = DB1_{n-1} = D^2 B1_{n-2} = \cdots = D^n B1_0. \qquad (4.7)$$

Hence, in order to solve the constants $x_1, x_2, x_{3,}, x_4$, we use elementary row operations on the augmented matrix $B1_0^*$,

$$B1_0^* = \begin{bmatrix} S_0 & S_w & S_{2w} & S_{3w} & S_{wh} \\ S_w & S_{2w} & S_{3w} & S_{4w} & S_{w(h+1)} \\ S_{2w} & S_{3w} & S_{4w} & S_{5w} & S_{w(h+2)} \\ S_{3w} & S_{4w} & S_{5w} & S_{6w} & S_{w(h+3)} \end{bmatrix}, \qquad (4.8)$$

for w and h various values, we obtain a classes of identities:

$S_{2(n+9)} = -552S_{2n} - 163S_{2(n+1)} - 1152S_{2(n+2)} + 1868S_{2(n+3)}$;

$S_{3(n+5)} = -8S_{3n} + 47S_{3(n+1)} - 90S_{3(n+2)} + 52S_{3(n+3)}$;

$S_{4(n+6)} = -138S_{4n} - 564S_{4(n+1)} - 877S_{4(n+2)} + 1580S_{4(n+3)}$;

$S_{5(n+7)} = -9768S_{5n} - 464S_{5(n+1)} - 195382S_{5(n+2)} + 205615S_{5(n+3)}$;

...

Especially, there are

$S_{2n} = 4S_{2(n-1)} - 2S_{2(n-2)} - S_{2(n-4)}, n \geq 4$;

$S_{3n} = 8S_{3(n-1)} - 12S_{3(n-2)} + 6S_{3(n-3)} - S_{3(n-4)}, n \geq 4$;

$S_{4n} = 12S_{4(n-1)} - 6S_{4(n-2)} - 4S_{4(n-3)} - S_{4(n-4)}, n \geq 4$;

...

This shows that our identities are more general than that given in paper [1].

5. Conclusion

In this paper, we use a simple method to give different recurrence relations on the Tribonacci numbers and their sums, we use the companion matrices and generating matrices to derive more identities on the Tribonacci numbers and their sums, which are more general than that given in paper [1].

References

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