Three types of edge-switchable kite systems *

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Abstract

Informally, a ϵ-switchable G-deisgn is a decomposition of the complete graph into subgraphs of isomorphic copies of G which have the property that they remain a G-decomposition when ϵ-edge switches are made to the subgraphs. This paper determines the spectrum of ϵ-switchable G-design where G is a kite (a triangle with an edge attached) and ϵ takes t-edge, h-edge and l-edge.

Key words: complete multipartite graph, G-design, switchable kite system, group divisible design

1 Introduction

Let G and H be simple finite graphs, and let λH denotes the graph H with each of its edges replicated λ times. The graph K_n denotes the complete graph with n vertices. The graph $K_{n_1,n_2,\ldots,n_t}$ denotes the complete multipartite graph with t partite sets of size $n_1, n_2, \ldots, n_t$ respectively. For convenience, we use $K_n \setminus K_m$ to denote the graph $K_{1,\ldots,1,m}$ with $n-m$ 1s.

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Note that $K_n \setminus K_m$ is sometimes referred to as a complete graph of order n with a hole of size m. A *λ-fold G-design* of λH, $(\lambda H, G)$-design, is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G whose edges partition the edges of λH. If H is a complete graph K_n, we refer to such a λ-fold G-design as one of order n. If $\lambda = 1$, we drop the term "1-fold". Let $\mathcal{H}$ be a partition of X into subsets called *holes*. Let $\mathcal{H} = \{X_1, X_2, \ldots, X_t\}$ with $|X_i| = n_i$ for $1 \leq i \leq t$. Let $K_{n_1, n_2, \ldots, n_t}$ on X with the i-part (or *i-hole*) on X_i. A *λ-fold holely G-design*, (G, λ)-HD, is a triple $(X, \mathcal{H}, \mathcal{B})$ where $(X, \mathcal{B})$ is a $(\lambda K_{n_1, n_2, \ldots, n_t}, G)$-design. The *hole-type* of the HD is $\{n_1, n_2, \ldots, n_t\}$. We usually use an "exponential" notation to describe hole-types: the hole-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, etc.

Let G be a simple finite graph. $V(G)$ and $E(G)$ denote the vertex-set and edge-set of G, respectively. For an edge $\epsilon \in E(G)$, $G - \epsilon$ denotes the graph obtained from G by deleting the edge ϵ. If $\epsilon' \notin E(G)$ and $V(\epsilon') \subseteq V(G)$, $G + \epsilon'$ denotes the graph obtained from G by adding the edge ϵ'. An edge ϵ of graph G is said to be *admissible* if the graph $(G - \epsilon) + \epsilon'$ is isomorphic to G for some edge $\epsilon' \notin E(G)$ and $V(\epsilon') \subseteq V(G)$.

Let $(X, \mathcal{H}, \mathcal{B})$ be a (G, λ)-HD and ϵ a fixed admissible edge of graph G. For each block $B \in \mathcal{B}$, delete an edge $\epsilon_B \in E(B)$ (called *ϵ-edge of B*) such that the graph $B - \epsilon_B$ is isomorphic to $G - \epsilon$. Define $D_\epsilon = \{\epsilon_B : B \in \mathcal{B}\}$ (called a *ϵ-set of* $(X, \mathcal{B})$). If there exists a bijection σ between $\mathcal{B}$ and D_ϵ such that $V(\sigma(B)) \subseteq V(B)$, $\sigma(B) \neq \epsilon_B$ and the graph $[(B - \epsilon_B) + \sigma(B)] \cong G$ for each $B \in \mathcal{B}$, then we call $(X, \mathcal{H}, \mathcal{B})$ *ϵ-switchable*. Note that if $\lambda > 1$, $\mathcal{B}$ and D_ϵ are multiset. The repeated elements in them are regarded as different.

For each $B \in \mathcal{B}$, $(B - \epsilon_B) + \sigma(B)$ is called an *ϵ-transformation* of B. Let $\mathcal{B}' = \{(B - \epsilon_B) + \sigma(B) : B \in \mathcal{B}\}$. It is easy to see that $(X, \mathcal{H}, \mathcal{B}')$ is also a (G, λ)-HD of the same type as $(X, \mathcal{H}, \mathcal{B})$. $(X, \mathcal{H}, \mathcal{B}')$ is said to be an *ϵ-transformed design* of $(X, \mathcal{H}, \mathcal{B})$.

For simple finite graphs G and H, we can similarly define ϵ-switchable $(\lambda H, G)$-design. The motivation for the concept of ϵ-switchable G-design can be found in [1].

In what follows we will denote the copy of $K_3 + e$ with vertices a, b, c, d and the dangling edge cd by $\{a, b, c - d\}$ and it is called a *kite*. In a kite $\{a, b, c - d\}$, edge $\{a, b\}$ is called *head-edge* (or h-edge); edges $\{a, c\}$ and $\{b, c\}$ are called *lateral edge* (or l-edge); edge $\{c, d\}$ is called *tail-edge* (or t-edge). A λ-fold $(K_3 + e)$-design of order v is called a *λ-fold kite system of order v*, and denoted by $KS(v, \lambda)$.

In this paper we will investigate the existence of the three types of ϵ-switchable kite systems when ϵ takes t-edge, h-edge and l-edge.

2 Working lemmas

A *group-divisible design* (or GDD) with index λ is a triple $(X, \mathcal{H}, \mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{H}$ is a partition of X into subsets called *groups*.

(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point.

(3) Every pair of points from distinct groups occurs in exactly λ blocks.

The group-type of a GDD $(X, \mathcal{H}, \mathcal{A})$ is the multiset $\{|H| : H \in \mathcal{H}\}$. We also usually use an "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k ...$ denotes i occurrences of 1, j occurrences of 2, etc. We call the GDD $(X, \mathcal{H}, \mathcal{A})$ a (K, λ)-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. A $(\{k\}, \lambda)$-GDD is briefly written as (k, λ)-GDD. A (v, K, λ)-PBD is a (K, λ)-GDD of type 1^v, and a $(v, \{k\}, \lambda)$-PBD is called a (v, k, λ)-BIBD.

The following construction is a variation of Wilson's Fundamental Construction in [5].

Construction 2.1 (Weighting Construction) *Let ϵ be a fixed admissible edge of graph G. Suppose $(X, \mathcal{H}, \mathcal{A})$ is a (K, λ)-GDD, and let $\omega : X \longmapsto Z^+ \cup \{0\}$ be any function (we refer to ω as a weighting). For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ "copies" of x. For every $A \in \mathcal{A}$, suppose that*

$$(\cup_{x\in A} S(x),\ \{S(x) : x \in A\},\ \mathcal{B}_A)$$

is an ϵ-switchable (G, μ)-HD with hole-type $\{\omega(x) : x \in A\}$ and bijection $\sigma_A : \mathcal{B}_A \mapsto D_\epsilon(A)$ where $D_\epsilon(A) = \{e_B : B \in \mathcal{B}_A\}$. Then

$$(\cup_{x\in X} S(x),\ \{\cup_{x\in H} S(x) :\ H \in \mathcal{H}\},\ \cup_{A\in\mathcal{A}} \mathcal{B}_A)$$

is an ϵ-switchable $(G, \lambda\mu)$-HD with hole-type $\{\sum_{x\in H} \omega(x) :\ H \in \mathcal{H}\}$ and bijection

$$\sigma : \cup_{A\in\mathcal{A}} \mathcal{B}_A \longmapsto \cup_{A\in\mathcal{A}} D_\epsilon(A)$$

where $\sigma(B) = \sigma_A(B)$ if there exists a block $A \in \mathcal{A}$ such that $B \in \mathcal{B}_A$.

Construction 2.2 (PBD-construction) *Let ϵ be a fixed admissible edge of graph G. Suppose that there exists a $(v, L, 1)$-PBD. For each $l \in L$, if there is an ϵ-switchable (G, λ)-HD of hole-type g^l, then so does an ϵ-switchable (G, λ)-HD of hole-type g^v.*

Proof A $(v, L, 1)$-PBD can be regarded as a $(L, 1)$-GDD with group type 1^v. Give each point of this GDD a weight of g, and apply the Weighting

Construction to get an ϵ-switchable (G, λ)-HD of hole-type g^v. The input designs are given by the hypothesis. $\diamond$

Construction 2.3 (Filling subdesigns) *Let a be a nonnegative integer and ϵ be a fixed admissible edge of graph G. Suppose that there exists an ϵ-switchable (G, λ)-HD of hole-type $\{n_1, n_2, \ldots, n_t\}$. If there is an ϵ-switchable $(\lambda(K_{n_i+a} \setminus K_a), G)$-design for each $1 \leq i \leq t-1$, then so does an ϵ-switchable $(\lambda(K_{v+a} \setminus K_{n_t+a}), G)$-design where $v = \sum_{i=1}^{t} n_i$. If further there exists an ϵ-switchable $(\lambda K_{n_t+a}, G)$-design, then an ϵ-switchable $(\lambda K_{v+a}, G)$-design exists.*

We quote the following known results for later use.

Lemma 2.4 [3] *Let g, t and u be nonnegative integers. There exists a $(3,1)$-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(1) *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*

(2) $u \leq g(t-1)$ *or* $gt = 0$;

(3) $g(t-1) + u \equiv 0 \pmod 2$ *or* $gt = 0$;

(4) $gt \equiv 0 \pmod 2$ *or* $u = 0$;

(5) $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 3$.

Lemma 2.5 ([2]) *There exists a $(v, \{3,4,5\}, 1)$-PBD for any integer $v \geq 3$ and $v \neq 6, 8$.*

3 t-switchable kite systems

Let $B = \{a, b, c - d\}$ where $\{c, d\}$ is the unique t-edge of the kite. The corresponding t-transformation of B is $\{a, c, b - d\}$, or $\{b, c, a - d\}$. In this section we always write down the corresponding t-transformation of B as the first one $\{a, c, b - d\}$ for convenience. A *t-switchable kite system of order v* is briefly denoted by $tSKS(v)$.

Lemma 3.1 *There exists a $tSKS(n)$ for $n =$ 8, 9, 16, 17.*

Proof $n = 8$: Let $X = Z_7 \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 8 with the base kite $\{0, 3, 1 - \infty\}$. The corresponding t-transformation of the base kite is $\{0, 1, 3 - \infty\}$. It is readily checked that $(X, \mathcal{B})$ is a $tSKS(8)$.

$n = 9$: Let $X = Z_9$ and $(X, \mathcal{B})$ be a kite system of order 9 with the base kite $\{0, 5, 3 - 4\}$. The corresponding t-transformation of the base kite is $\{0, 3, 5 - 4\}$. It is readily checked that $(X, \mathcal{B})$ is a $tSKS(9)$.

$n = 16$: Let $X = Z_{15} \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 16 with the base kites: $\{0, 6, 1 - \infty\}$, $\{0, 7, 3 - 5\}$. The corresponding t-transformations of the base kites are $\{0, 1, 6 - \infty\}$, $\{0, 3, 7 - 5\}$. It is readily checked that $(X, \mathcal{B})$ is a $tSKS(16)$.

$n = 17$: Let $X = Z_{17}$ and $(X, \mathcal{B})$ be a kite systems of order 17 with the base kites: $\{0, 8, 2 - 3\}$, $\{0, 7, 3 - 8\}$. The corresponding t-transformations of the base kites are $\{0, 2, 8 - 3\}$, $\{0, 3, 7 - 8\}$. It is readily checked that $(X, \mathcal{B})$ is a $tSKS(17)$. ◇

Lemma 3.2 *There exists a t-switchable $(K_3 + e, 1)$-HD of hole-types 4^3 and 2^5.*

Proof Hole-type 4^3: Let $\mathcal{H} = \{3Z_{12} + i : i = 0, 1, 2\}$. We construct a $(K_{4,4,4}, K_3 + e)$-design $(Z_{12}, \mathcal{H}, \mathcal{B})$ by the base kite $\{0, 5, 1 - 3\}$. The corresponding t-transformation of the base kite is $\{0, 1, 5 - 3\}$.

Hole-type 2^5: Let $X = Z_{10}$ and $\mathcal{H} = \{\{i, i + 5\} : 0 \leq i \leq 4\}$. We construct a $(K_{2,\ldots,2}, K_3 + e)$-design $(X, \mathcal{H}, \mathcal{B})$ with the base kite $\{3, 0, 2 - 6\}$. The corresponding t-transformation of the base kite is $\{2, 3, 0 - 6\}$. ◇

Lemma 3.3 *There exists a t-switchable $(K_3 + e, 1)$-HD of hole-types 8^3, 8^4, 8^5.*

Proof By Lemma 2.4 there exists a $(3, 1)$-GDD with group types 2^3 and 2^4. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get a t-switchable $(K_3 + e, 1)$-HD of hole-types 8^3 and 8^4. The input designs are from Lemma 3.2.

It is well known that there is a $(5, 1)$-GDD of group type 4^5. Give each point of the GDD a weight of 2, and similarly apply the Weighting Construction to get a t-switchable $(K_3 + e, 1)$-HD of hole-type 8^5. ◇

Theorem 3.4 *A $tSKS(v)$ exists if and only if $v \equiv 0, 1 \pmod 8$ and $v \geq 8$.*

Proof The necessity is obvious. The sufficiency follows as below. Let $v = 8t + a$ where $a = 0, 1$. For $t = 1, 2$, the conclusion follows from Lemma 3.1.

For $t \geq 3$ and $t \neq 6, 8$, by Lemma 2.5 there exists a $(t, \{3, 4, 5\}, 1)$-PBD. There exists a t-switchable $(K_3 + e, 1)$-HD of hole-types 8^3, 8^4 and 8^5 by Lemma 3.3. Apply Construction 2.2 with $g = 8$ to get a t-switchable $(K_3 + e, 1)$-HD of hole-type 8^t. By Construction 2.3 with $a = 0, 1$, there exists a $tSKS(8t + a)$. The needed $tSKS(8)$ and $tSKS(9)$ come from Lemma 3.1.

For $t = 6, 8$, by Lemma 2.4 there exists a $(3, 1)$-GDD with group types 4^3 and 4^4. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get a t-switchable $(K_3 + e, 1)$-HD of hole-types 16^3 and 16^4. The input designs are from Lemma 3.2. Applying again Construction 2.3 with $a = 0, 1$, there exists a $tSKS(8t + a)$. The needed $tSKS(16)$ and $tSKS(17)$ come from Lemma 3.1. ◇

4 h-switchable kite systems

Let $B = \{a, b, c - d\}$ where $\{a, b\}$ is the unique h-edge of the kite. The corresponding h-transformation of B is $\{b, d, c - a\}$, or $\{a, d, c - b\}$. In this section we always write down the corresponding h-transformation of B as the first one $\{b, d, c - a\}$ for convenience. An *h-switchable kite system of order v* is briefly denoted by $hSKS(v)$.

Lemma 4.1 *There does not exist an $hSKS(8)$.*

Proof Let $(X, \mathcal{B})$ be an h-switchable kite system of order 8. For each kite $B = \{a, b, c - d\}$ of $\mathcal{B}$, the corresponding h-transformation of B is $B_h = \{b, d, c - a\}$. Let $\mathcal{B}' = \{B_h : B \in \mathcal{B}\}$. Then $(X, \mathcal{B}')$ is also a kite system of order 8.

For any $x \in X$, denote by $d_i(x)$, $i = 1, 3$, the number of kites in $\mathcal{B}$ in which the degree of x is i. Similarly, denote by $d_2(x)$ the number of kites $B \in \mathcal{B}$ such that the degree of x is 2 in both B and its corresponding transformation B_h; denote by $d_2'(x)$ the number of kites $B \in \mathcal{B}$ satisfying that the degree of x is 2 in B, but 1 in its corresponding transformation B_h.

Since both $(X, \mathcal{B})$ and $(X, \mathcal{B}')$ are kite systems of order 8, we have

$$d_1(x) + 3d_3(x) + 2[d_2(x) + d_2'(x)] = 7,$$
$$d_2'(x) + 3d_3(x) + 2[d_2(x) + d_1(x)] = 7.$$

Solving the equations gives $d_2(x) = 2$ for any $x \in X$. This implies that the number of kites in $(X, \mathcal{B})$ would be at least 16. It is impossible. ◊

Lemma 4.2 *There exists an $hSKS(n)$ for $n = 9$, 16, 17.*

Proof $n = 9$: Let $X = Z_9$ and $(X, \mathcal{B})$ be a kite system of order 9 with the base kite $\{0, 3, 2 - 6\}$. The corresponding h-transformation of the base kite is $\{3, 6, 2 - 0\}$. It is readily checked that $(X, \mathcal{B})$ is an $hSKS(9)$.

$n = 16$: Let $X = (Z_5 \times \{0, 1, 2\}) \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 16 with the base kites in mod $(5, -)$: $\{2_0, 3_1, \infty - 3_2\}$, $\{3_1, 3_2, 4_2 - 4_0\}$, $\{0_0, 4_2, 1_0 - 2_2\}$, $\{2_2, 4_2, 0_1 - 1_1\}$, $\{4_2, 1_1, 2_0 - 4_1\}$, $\{1_1, 4_1, 1_0 - 3_0\}$. For each base kite $B = \{a, b, c - d\}$, take the corresponding h-transformation of B as $\{b, d, c - a\}$. It is readily checked that $(X, \mathcal{B})$ is an $hSKS(16)$.

$n = 17$: Let $X = Z_{17}$ and $(X, \mathcal{B})$ be a kite system of order 17 with the base kites: $\{0, 10, 4 - 5\}$, $\{0, 12, 3 - 5\}$. The corresponding h-transformations of the base kites are $\{5, 10, 4 - 0\}$, $\{5, 12, 3 - 0\}$. It is readily checked that $(X, \mathcal{B})$ is an $hSKS(17)$. ◊

Lemma 4.3 *There exists an h-switchable $(K_{24} \setminus K_9, K_3 + e)$-design.*

Proof Let $Y = \{\infty_1, \infty_2, \ldots, \infty_9\}$. We construct a $(K_{24} \setminus K_9, K_3 + e)$-design $(Z_{15} \cup Y, \mathcal{B})$ with hole Y and 60 kites. The first 15 kites are obtained by developing the base kite $\{3, 7, 0 - 9\}$ in Z_{15}. The other 45 kites are listed as below:

$\{2, 0, \infty_1 - 4\}$,	$\{11, 9, \infty_1 - 13\}$,	$\{5, 3, \infty_1 - 7\}$,	$\{8, 6, \infty_1 - 10\}$,
$\{14, 12, \infty_1 - 1\}$,	$\{13, 11, \infty_2 - 0\}$,	$\{4, 2, \infty_2 - 6\}$,	$\{7, 5, \infty_2 - 9\}$,
$\{10, 8, \infty_2 - 12\}$,	$\{1, 14, \infty_2 - 3\}$,	$\{0, 13, \infty_3 - 2\}$,	$\{9, 7, \infty_3 - 11\}$,
$\{3, 1, \infty_3 - 5\}$,	$\{6, 4, \infty_3 - 8\}$,	$\{12, 10, \infty_3 - 14\}$,	$\{0, 1, \infty_4 - 2\}$,
$\{3, 4, \infty_4 - 5\}$,	$\{6, 7, \infty_4 - 8\}$,	$\{9, 10, \infty_4 - 11\}$,	$\{12, 13, \infty_4 - 14\}$,
$\{13, 14, \infty_5 - 0\}$,	$\{1, 2, \infty_5 - 3\}$,	$\{4, 5, \infty_5 - 6\}$,	$\{7, 8, \infty_5 - 9\}$,
$\{10, 11, \infty_5 - 12\}$,	$\{14, 0, \infty_6 - 1\}$,	$\{2, 3, \infty_6 - 4\}$,	$\{5, 6, \infty_6 - 7\}$,
$\{8, 9, \infty_6 - 10\}$,	$\{11, 12, \infty_6 - 13\}$,	$\{0, 5, \infty_7 - 10\}$,	$\{1, 6, \infty_7 - 11\}$,
$\{2, 7, \infty_7 - 12\}$,	$\{3, 8, \infty_7 - 13\}$,	$\{4, 9, \infty_7 - 14\}$,	$\{5, 10, \infty_8 - 0\}$,
$\{6, 11, \infty_8 - 1\}$,	$\{7, 12, \infty_8 - 2\}$,	$\{8, 13, \infty_8 - 3\}$,	$\{9, 14, \infty_8 - 4\}$,
$\{10, 0, \infty_9 - 5\}$,	$\{11, 1, \infty_9 - 6\}$,	$\{12, 2, \infty_9 - 7\}$,	$\{13, 3, \infty_9 - 8\}$,
$\{14, 4, \infty_9 - 9\}$.			

For each kite $B = \{a, b, c - d\}$, take the corresponding h-transformation of B as $\{b, d, c - a\}$. It is readily checked that $(Z_{15} \cup Y, \mathcal{B})$ is h-switchable. ◇

A *Kirkman triple system of order* v (briefly $KTS(v)$) is a $(v, 3, 1)$-BIBD $(X, \mathcal{B})$ together with a partition R of the set of triples $\mathcal{B}$ into subsets R_1, R_2, ..., R_n called *parallel classes* such that each R_i ($i = 1, 2, \ldots, n$) is a partition of X. It is well known that a $KTS(v)$ exists if and only if $v \equiv 3 \pmod 6$ (see [4]).

Lemma 4.4 *There exists an h-switchable $(K_{32} \setminus K_{17}, K_3 + e)$-design.*

Proof Let $I_{15} = \{1, 2, \ldots, 15\}$. A $KTS(15)$ $(I_{15}, \mathcal{A})$ with parallel classes $\{\mathcal{P}_i : 1 \le i \le 7\}$ is listed as follows:

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$
1 14 15	1 6 7	1 8 9	1 10 11	1 12 13	1 2 3	1 4 5
2 5 7	2 8 10	2 13 15	2 12 14	2 4 6	4 10 14	2 9 11
3 10 13	3 9 14	3 4 7	3 5 6	3 11 15	5 8 13	3 8 12
4 9 12	4 11 13	5 11 14	4 8 15	5 9 10	6 9 15	6 13 14
6 8 11	5 12 15	6 10 12	7 9 13	7 8 14	7 11 12	7 10 15

Let $Y = \{\infty_1, \infty_2, \ldots, \infty_{17}\}$. We construct a $(K_{32} \setminus K_{17}, K_3 + e)$-design $(I_{15} \cup Y, \mathcal{B})$ with hole Y and 90 kites. The three parallel classes $\mathcal{P}_i$, $5 \le i \le 7$ serve to give the following 30 kites:

$\{2, 3, 1 - 12\}$, $\{14, 4, 10 - 5\}$, $\{5, 13, 8 - 14\}$, $\{15, 9, 6 - 2\}$,
$\{12, 7, 11 - 15\}$, $\{12, 13, \infty_1 - 6\}$, $\{5, 9, \infty_1 - 11\}$, $\{14, 7, \infty_1 - 10\}$,
$\{2, 4, \infty_1 - 1\}$, $\{15, 3, \infty_1 - 8\}$, $\{1, 13, \infty_2 - 5\}$, $\{10, 9, \infty_2 - 15\}$,
$\{8, 7, \infty_2 - 12\}$, $\{6, 4, \infty_2 - 14\}$, $\{11, 3, \infty_2 - 2\}$, $\{1, 5, \infty_3 - 9\}$,
$\{10, 15, \infty_3 - 3\}$, $\{8, 12, \infty_3 - 13\}$, $\{6, 14, \infty_3 - 7\}$, $\{11, 2, \infty_3 - 4\}$,
$\{4, 1, \infty_4 - 5\}$, $\{7, 10, \infty_4 - 15\}$, $\{3, 8, \infty_4 - 12\}$, $\{13, 6, \infty_4 - 14\}$,
$\{9, 11, \infty_4 - 2\}$, $\{5, 4, \infty_5 - 6\}$, $\{15, 7, \infty_5 - 8\}$, $\{12, 3, \infty_5 - 11\}$,
$\{14, 13, \infty_5 - 1\}$, $\{2, 9, \infty_5 - 10\}$.

The other 60 kites are constructed as follows:

$$\{a, b, \infty_{3i+3} - c\}, \quad \{b, c, \infty_{3i+4} - a\}, \quad \{c, a, \infty_{3i+5} - b\},$$

for each $1 \le i \le 4$ and each $\{a, b, c\} \in \mathcal{P}_i$.

For each kite $B = \{a, b, c - d\}$, take the corresponding h-transformation of B as $\{b, d, c - a\}$. It is readily checked that $(Z_{15} \cup Y, \mathcal{B})$ is h-switchable. ◇

Lemma 4.5 *There exists an* $hSKS(40)$.

Proof Let $X = (Z_{13} \times \{0,1,2\}) \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 40 with the base kites in mod $(13, -)$:

$$\begin{array}{ll}
\{(1,0),(3,1),\infty-(10,2)\}, & \{(0,0),(1,0),(3,0)-(3,1)\}, \\
\{(0,0),(4,0),(8,1)-(5,0)\}, & \{(0,0),(5,0),(6,1)-(9,0)\}, \\
\{(0,0),(6,0),(5,1)-(11,0)\}, & \{(0,1),(2,0),(2,2)-(8,0)\}, \\
\{(0,0),(9,1),(9,2)-(11,0)\}, & \{(0,1),(1,1),(4,2)-(5,0)\}, \\
\{(0,1),(2,1),(6,1)-(3,1)\}, & \{(0,1),(5,1),(6,2)-(7,1)\}, \\
\{(0,2),(2,1),(12,2)-(7,1)\}, & \{(0,0),(4,2),(1,2)-(6,1)\}, \\
\{(0,0),(2,2),(6,2)-(11,0)\}, & \{(0,2),(10,0),(7,2)-(12,2)\}, \\
\{(0,1),(7,2),(9,2)-(4,0)\}. &
\end{array}$$

For each base kite $B = \{a, b, c - d\}$, take the corresponding h-transformation of B as $\{b, d, c - a\}$. It is readily checked that $(X, \mathcal{B})$ is an $hSKS(40)$. ◇

Lemma 4.6 *There exists an* $hSKS(56)$.

Proof Let $X = (Z_{11} \times \{0,1,2,3,4\}) \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 56 with the base kites in mod $(11, -)$:

$$\begin{array}{ll}
\{(9,2),(10,4),\infty-(0,0)\}, & \{(0,3),\infty,(8,1)-(2,3)\}, \\
\{(0,0),(0,2),(1,0)-(1,4)\}, & \{(0,1),(0,0),(2,0)-(0,2)\}, \\
\{(0,1),(1,0),(4,0)-(1,1)\}, & \{(0,0),(4,0),(5,1)-(3,1)\}, \\
\{(0,0),(5,0),(6,2)-(9,0)\}, & \{(0,1),(5,0),(1,2)-(10,0)\}, \\
\{(0,0),(2,1),(2,3)-(7,0)\}, & \{(0,0),(3,1),(5,2)-(1,0)\}, \\
\{(0,0),(4,1),(5,3)-(1,0)\}, & \{(0,1),(1,1),(0,2)-(8,0)\}, \\
\{(0,1),(3,1),(7,2)-(4,1)\}, & \{(0,1),(4,1),(6,3)-(7,1)\}, \\
\{(0,1),(5,1),(5,4)-(9,1)\}, & \{(0,2),(2,1),(0,3)-(7,1)\}, \\
\{(0,1),(5,2),(8,4)-(7,1)\}, & \{(0,1),(6,2),(8,3)-(1,1)\}, \\
\{(0,1),(8,2),(4,4)-(2,1)\}, & \{(0,2),(1,2),(10,4)-(4,1)\}, \\
\{(0,2),(2,2),(6,2)-(3,2)\}, & \{(0,3),(1,2),(4,3)-(3,2)\}, \\
\{(0,0),(0,3),(8,3)-(1,2)\}, & \{(0,0),(1,3),(10,3)-(1,0)\}, \\
\{(0,0),(3,3),(3,4)-(2,0)\}, & \{(0,0),(7,3),(8,4)-(4,0)\}, \\
\{(0,2),(4,3),(6,4)-(8,0)\}, & \{(0,2),(8,3),(9,3)-(4,2)\}, \\
\{(0,3),(5,2),(7,4)-(2,3)\}, & \{(0,4),(3,3),(2,4)-(8,2)\}, \\
\{(0,4),(2,3),(7,3)-(10,4)\}, & \{(0,2),(0,4),(8,4)-(2,3)\}, \\
\{(0,1),(10,4),(3,4)-(10,2)\}, & \{(0,0),(7,4),(6,4)-(8,1)\}, \\
\{(0,0),(10,4),(5,4)-(3,0)\}. &
\end{array}$$

For each base kite $B = \{a, b, c - d\}$, take the corresponding h-transformation of B as $\{b, d, c - a\}$. It is readily checked that $(X, \mathcal{B})$ is an $hSKS(56)$. ◇

Lemma 4.7 *There exists an h-switchable $(K_3+e,1)$-HD of hole-types 4^3 and 2^5.*

Proof Hole-type 4^3: Let $X=\{x,y,z\}\times Z_4$ and $\mathcal{H}=\{\{i\}\times Z_4 : i\in\{x,y,z\}\}$. We construct a $(K_{4,4,4},K_3+e)$-design $(X,\mathcal{H},\mathcal{B})$ by listing its kites as follows:

$$\begin{array}{llll}\{y_0,z_2,x_0-y_3\}, & \{z_2,y_3,x_2-z_1\}, & \{y_3,z_1,x_3-y_0\}, & \{z_1,y_0,x_1-z_2\},\\ \{z_0,x_2,y_0-z_3\}, & \{x_2,z_3,y_2-x_1\}, & \{z_3,x_1,y_3-z_0\}, & \{x_1,z_0,y_1-x_2\},\\ \{x_0,y_2,z_0-x_3\}, & \{y_2,x_3,z_2-y_1\}, & \{x_3,y_1,z_3-x_0\}, & \{y_1,x_0,z_1-y_2\}.\end{array}$$

For each kite $B=\{a,b,c-d\}$, take the corresponding h-transformation of B as $\{b,d,c-a\}$. It is readily checked that $(X,\mathcal{H},\mathcal{B})$ is h-switchable.

Hole-type 2^5: Let $X=Z_{10}$ and $\mathcal{H}=\{\{i,i+5\} : 0\le i\le 4\}$. We construct a $(K_{2,\dots,2},K_3+e)$-design $(X,\mathcal{H},\mathcal{B})$ with the base kite $\{0,7,8-4\}$. The corresponding h-transformation of the base kite is $\{7,4,8-0\}$. It is readily checked that $(X,\mathcal{H},\mathcal{B})$ is h-switchable. ◇

Lemma 4.8 *There exists an h-switchable $(K_3+e,1)$-HD of hole-types 8^3, 8^4 and 8^5; There exists an h-switchable $(K_3+e,1)$-HD of hole-types 16^3, 16^4 and 16^5.*

Proof The proof of the first assertion is similar as that of Lemma 3.3.

The second assertion is proved as follows: By Lemma 2.4 there exists a $(3,1)$-GDD with group types 4^3 and 4^4. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get an h-switchable $(K_3+e,1)$-HD of hole-types 16^3 and 16^4. The needed h-switchable $(K_3+e,1)$-HD of hole-type 4^3 is from Lemma 4.7.

It is well known that there is a $(5,1)$-GDD of group type 8^5. Give each point of the GDD a weight of 2, and similarly apply the Weighting Construction to get an h-switchable $(K_3+e,1)$-HD of hole-type 16^5. The needed h-switchable $(K_3+e,1)$-HD of hole-type 2^5 is from Lemma 4.7. ◇

Lemma 4.9 *There exists an $hSKS(v)$ for any integer $v\equiv 1 \pmod 8$ and $v\ge 9$.*

Proof Let $v=8t+1$ where $t\ge 1$. The conclusion follows by Lemma 4.2 for $t=1,2$. For $t\ge 3$ and $t\ne 6,8$, by Lemma 2.5 there exists a $(t,\{3,4,5\},1)$-PBD. There exists an h-switchable $(K_3+e,1)$-HD of hole-types 8^3, 8^4 and 8^5 by Lemma 4.8. Apply Construction 2.2 with $g=8$ to get an h-switchable $(K_3+e,1)$-HD of hole-type 8^t. By Construction 2.3 with $a=1$, there exists an $hSKS(8t+1)$. The needed $hSKS(9)$ comes from Lemma 4.2.

We remain to deal with $t=6,8$. By Lemma 2.4 there exists a $(3,1)$-GDD with group type $4^{t/2}$. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get an h-switchable $(K_3 + e, 1)$-HD of hole-type $16^{t/2}$. The input design is from Lemma 4.7. Applying again Construction 2.3 with $a = 1$, there exists an $hSKS(8t + 1)$. The needed $hSKS(17)$ comes from Lemma 4.2. ◇

Lemma 4.10 *There exists an $hSKS(v)$ for any integer $v \equiv 0 \pmod 8$ and $v \geq 16$.*

Proof For $v = 16$, the conclusion follows from Lemma 4.2. For $v = 24$, there exists an h-switchable $(K_{24} \setminus K_9, K_3 + e)$-design by Lemma 4.3. By Construction 2.3 with an $hSKS(9)$ from Lemma 4.2, there exists an $hSKS(24)$.

For $v = 32$, there exists an h-switchable $(K_{32} \setminus K_{17}, K_3 + e)$-design by Lemma 4.4. By Construction 2.3 with an $hSKS(17)$ from Lemma 4.2, there exists an $hSKS(32)$.

For $v = 40$, the conclusion follows by Lemma 4.5.

Next we deal with the case $v \geq 48$. Let $v = 16t + \delta$ where $t \geq 3$ and $\delta = 0, 8$. We divide the problem into three cases.

Case 1: $v = 16t$ where $t \geq 3$. For $t \neq 6, 8$, by Lemma 2.5 there exists a $(t, \{3, 4, 5\}, 1)$-PBD. There exists an h-switchable $(K_3 + e, 1)$-HD of hole-types 16^3, 16^4 and 16^5 by Lemma 4.8. Apply Construction 2.2 with $g = 16$ to get an h-switchable $(K_3 + e, 1)$-HD of hole-type 16^t. By Construction 2.3 with $a = 0$, there exists an $hSKS(16t)$. The needed $hSKS(16)$ comes from Lemma 4.2.

We remain to deal with $t = 6, 8$. By Lemma 2.4 there exists a $(3, 1)$-GDD with group type $4^{t/2}$. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an h-switchable $(K_3 + e, 1)$-HD of hole-type $32^{t/2}$. The needed h-switchable $(K_3 + e, 1)$-HD of hole-type 8^3 is from Lemma 4.8. Applying again Construction 2.3 with $a = 0$, there exists an $hSKS(16t)$. The needed $hSKS(32)$ exists as pointed before.

Case 2: $v = 16t + 8$ where $t \equiv 1, 2 \pmod 3$ and $t \geq 3$. By Lemma 2.4 there exists a $(3, 1)$-GDD with group type $4^{t-1}6^1$. By Lemma 4.7 there exists an h-switchable $(K_3 + e, 1)$-HD of type 4^3. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get an h-switchable $(K_3 + e, 1)$-HD of hole-type $16^{t-1}24^1$. Applying Construction 2.3, there exists an $hSKS(16t + 8)$. The needed $hSKS(16)$ and $hSKS(24)$ exist as above.

Case 3: $v = 16t + 8$ where $t \equiv 0 \pmod 3$ and $t \geq 3$.

For $t = 3$, the conclusion follows from Lemma 4.6.

For $t = 6$ (i.e., $v = 13 \times 8$), there exists a $(4, 1)$-GDD of type 4^4. Take a block B of this GDD, and give three point of B a weight of zero and each other point of the GDD a weight of 8. By Lemma 4.8 there exists an h-switchable $(K_3 + e, 1)$-HD of types 8^3 and 8^4. We apply the Weighting Construction to get an h-switchable $(K_3 + e, 1)$-HD of hole-type $24^3 32^1$. Applying Construction 2.3, there exists an $hSKS(13 \times 8)$. The needed $hSKS(24)$ and $hSKS(32)$ come as above.

For $t \geq 9$, by Lemma 2.4 there exists a (3, 1)-GDD with group type $4^{t-3}14^1$. By Lemma 4.7 there exists an h-switchable $(K_3 + e, 1)$-HD of type 4^3. Give each point of the GDD a weight of 4, and apply the Weighting Construction to get an h-switchable $(K_3 + e, 1)$-HD of hole-type $16^{t-3}56^1$. Applying Construction 2.3, there exists an $hSKS(16t+8)$. The existence of an $hSKS(16)$ and an $hSKS(56)$ are pointed as before. ◇

Theorem 4.11 *An $hSKS(v)$ exists if and only if $v \equiv 0, 1 \pmod 8$ and $v > 8$.*

Proof The necessity follows from Lemma 4.1 and the necessary condition of the existence of a kite system of order v. The sufficiency follows by Lemmas 4.9 and 4.10. ◇

5 l-switchable kite systems

Let $B = \{a, b, c - d\}$ where $\{a, c\}$ and $\{b, c\}$ are the only two l-edge of the kite. The corresponding l-transformation of B is $\{c, d, b - a\}$, or $\{c, d, a - b\}$. In this section we always write down the corresponding l-transformation of B as the first one $\{c, d, b - a\}$ for convenience. An l-switchable kite system of order v is briefly denoted by $lSKS(v)$.

Lemma 5.1 *There does not exist an $lSKS(8)$.*

Proof Let $(X, \mathcal{B})$ be an l-switchable kite system of order 8. For each kite $B = \{a, b, c-d\}$ of $\mathcal{B}$, the corresponding l-transformation of B is $B_l = \{c, d, b-a\}$. Let $\mathcal{B}' = \{B_l : B \in \mathcal{B}\}$. Then $(X, \mathcal{B}')$ is also a kite system of order 8.

For any $x \in X$, denote by $d_i(x)$, $i = 1, 3$, the number of kites in $\mathcal{B}$ in which the degree of x is i. Similarly, denote by $d_2(x)$ the number of kites $B \in \mathcal{B}$ satisfying that the degree of x is 2 in B, but 1 in its corresponding transformation B_l; denote by $d_2'(x)$ the number of kites $B \in \mathcal{B}$ satisfying that the degree of x is 2 in B, but 3 in its corresponding transformation B_l.

Since both $(X, \mathcal{B})$ and $(X, \mathcal{B}')$ are kite systems of order 8, by counting the pairs containing x in both $\mathcal{B}$ and $\mathcal{B}'$ we have

$$d_1(x) + 2[d_2(x) + d_2'(x)] + 3d_3(x) = 7,$$
$$d_2(x) + 2[d_3(x) + d_1(x)] + 3d_2'(x) = 7.$$

Solving the equations gives $d_1(x) \geq 1$ for any $x \in X$. This implies that the number of kites in $(X, \mathcal{B})$ would be at least 8. It is impossible. ◇

Lemma 5.2 *There exists an $lSKS(n)$ for $n = 9$, 16, 17.*

Proof $n = 9$: Let $X = Z_9$ and $(X, \mathcal{B})$ be a kite system of order 9 with the base kite $\{0, 5, 2 - 3\}$. The corresponding l-transformation of the base kite is $\{2, 3, 5 - 0\}$. It is readily checked that $(X, \mathcal{B})$ is an $lSKS(9)$.

$n = 16$: Let $X = (Z_3 \times \{0, 1, 2, 3, 4\}) \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 16 with the base kites in mod $(3, -)$:

$$\begin{array}{ll}
\{(0,2),(1,3),\infty-(1,1)\}, & \{(0,1),(0,0),(0,3)-(2,0)\},\\
\{(0,0),(1,1),(2,0)-(1,4)\}, & \{(0,4),(1,1),(0,1)-(1,2)\},\\
\{(0,1),(2,2),(0,2)-(2,0)\}, & \{(0,2),(2,3),(0,0)-(0,4)\},\\
\{(0,4),(1,3),(2,3)-(0,1)\}, & \{(0,1),(1,4),(1,3)-(1,2)\},\\
\{(0,4),(2,4),(0,2)-(1,0)\}, & \{(0,0),\infty,(1,4)-(0,2)\}.
\end{array}$$

For each base kite $B = \{a, b, c - d\}$, take the corresponding l-transformation of B as $\{c, d, b - a\}$. It is readily checked that $(X, \mathcal{B})$ is an $lSKS(16)$.

$n = 17$: Let $X = Z_{17}$ and $(X, \mathcal{B})$ be a kite system of order 17 with the base kites: $\{0, 6, 2 - 3\}$, $\{0, 8, 3 - 10\}$. The corresponding l-transformations of the base kites are $\{2, 3, 6 - 0\}$, $\{3, 10, 8 - 0\}$. It is readily checked that $(X, \mathcal{B})$ is an $lSKS(17)$. ◇

Lemma 5.3 *There exists an l-switchable $(K_3 + e, 1)$-HD of hole-types 8^3, 8^4 and 2^5.*

Proof Hole-type 8^3: Let $\mathcal{H} = \{3Z_{24} + i : i = 0, 1, 2\}$. We construct a $(K_{8,8,8}, K_3 + e)$-design $(Z_{24}, \mathcal{H}, \mathcal{B})$ by listing its base kites: $\{1, 5, 0 - 7\}$, $\{2, 10, 0 - 11\}$. The corresponding l-transformations of the base kites are: $\{0, 7, 5 - 1\}$, $\{0, 11, 10 - 2\}$. It is readily checked that $(Z_{24}, \mathcal{H}, \mathcal{B})$ is l-switchable.

Hole-type 8^4: Let $\mathcal{H} = \{4Z_{32} + i : i = 0, 1, 2, 3\}$. We construct a $(K_{8,8,8,8}, K_3 + e)$-design $(Z_{32}, \mathcal{H}, \mathcal{B})$ by listing its base kites: $\{0, 1, 3 - 10\}$, $\{0, 15, 9 - 20\}$, $\{0, 18, 5 - 15\}$. The corresponding l-transformations of the base kites are: $\{3, 10, 1 - 0\}$, $\{9, 20, 15 - 0\}$, $\{5, 15, 18 - 0\}$. It is readily checked that $(Z_{32}, \mathcal{H}, \mathcal{B})$ is l-switchable.

Hole-type 2^5: Let $X = Z_{10}$ and $\mathcal{H} = \{\{i, i + 5\} : 0 \le i \le 4\}$. We construct a $(K_{2,\ldots,2}, K_3 + e)$-design $(X, \mathcal{H}, \mathcal{B})$ with the base kite: $\{0, 4, 1 - 3\}$. The corresponding l-transformation of the base kite is $\{1, 3, 4 - 0\}$. It is readily checked that $(Z_{10}, \mathcal{H}, \mathcal{B})$ is l-switchable. ◇

Lemma 5.4 *There exists an l-switchable $(K_{24} \setminus K_9, K_3 + e)$-design.*

Proof Let $Y = \{\infty_1, \infty_2, \ldots, \infty_9\}$. We construct a $(K_{24} \setminus K_9, K_3 + e)$-design $(Z_{15} \cup Y, \mathcal{B})$ with hole Y and 60 kites. The first 15 kites are obtained by developing the base kite $\{7, 6, 0 - 13\}$ in Z_{15}. Let $G = (V, E)$ be the graph where $V = Z_{15}$

and $E = \{\{x, x+d\} : x \in Z_{15}, d = 3,4,5\}$. In order to obtain the other 45 kites, partition the graph G into the following 3 2-factors (note that the size of cycles in each 2-factor is $3k$ where $k = 1, 3$, or 4):

F_1: $(0,11,14,10,6,9,5,1,4)$, $(12,7,2)$, $(3,13,8)$;

F_2: $(12,8,4,7,3,14,2,13,9)$, $(0,10,5)$, $(6,1,11)$;

F_3: $(0,12,1,13,10,7,11,8,5,2,6,3)$, $(9,4,14)$;

and for each 2-factor arrange three infinity points as follows: for every $3k$-cycle $(x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{k-1}, y_{k-1}, z_{k-1})$ of F_1 form the kites

$$\{\infty_1, y_i, x_i - \infty_2\},\quad \{\infty_2, z_i, y_i - \infty_3\},\quad \{\infty_3, x_{i+1}, z_i - \infty_1\},$$

where $i = 0, 1, \ldots, k-1$ and the subscript $i+1$ is reduced modulo k. Similarly, arrange $\infty_4, \infty_5, \infty_6$ with F_2 and $\infty_7, \infty_8, \infty_9$ with F_3.

For each kite $B = \{a, b, c - d\}$, the corresponding l-transformation of B is $\{c, d, b - a\}$. It is readily checked that $(Z_{15} \cup Y, \mathcal{B})$ is l-switchable. ◊

Lemma 5.5 *There exists an l-switchable $(K_{32} \setminus K_{17}, K_3 + e)$-design.*

Proof Let $Y = \{\infty_0, \infty_1, \ldots, \infty_{16}\}$. We construct a $(K_{32} \setminus K_{17}, K_3 + e)$-design $(Z_{15} \cup Y, \mathcal{B})$ with hole Y and 90 kites. Firstly, we arrange $\infty_0, \infty_1, \infty_2, \infty_3, \infty_4$ and the differences 3, 6, and 7 of Z_{15} into the following 30 kites: $\{i, i+3, \infty_i - (i+9)\}$ and $\{i+1, \infty_i, (i+7) - i\}$ where $i \in Z_{15}$ and the subscripts of the infinite points are taken modulo 5.

Note that each of the remaining 4 differences gives a 2-factor of Z_{15} which is the union of $3k$-cycles, for $k = 1$ or 5. Then the graph $G = (V, E)$, where $V = Z_{15}$ and $E = \{\{x, x+d\} : x \in Z_{15}, d = 1,2,4,5\}$, can be partitioned into 4 2-factors, say, F_1, F_2, F_3 and F_4. In order to obtain the other 60 kites, for each 2-factor we arrange three of the remaining 12 infinity points as follows. For every $3k$-cycle $(x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{k-1}, y_{k-1}, z_{k-1})$ of F_1 form the kites

$$\{\infty_5, y_j, x_j - \infty_6\},\quad \{\infty_6, z_j, y_j - \infty_7\},\quad \{\infty_7, x_{j+1}, z_j - \infty_5\},$$

where $j = 0, 1, \ldots, k-1$ and the subscript $j+1$ is reduced modulo k. Similarly, arrange $\infty_8, \infty_9, \infty_{10}$ with F_2, $\infty_{11}, \infty_{12}, \infty_{13}$ with F_3 and $\infty_{14}, \infty_{15}, \infty_{16}$ with F_4.

For each kite $B = \{a, b, c - d\}$, the corresponding l-transformation of B is $\{c, d, b - a\}$. It is readily checked that $(Z_{15} \cup Y, \mathcal{B})$ is l-switchable. ◊

Lemma 5.6 *There exists an $lSKS(40)$.*

Proof Let $X = (Z_{13} \times \{0,1,2\}) \cup \{\infty\}$ and $(X, \mathcal{B})$ be a kite system of order 40 with the base kites in mod $(13, -)$:

$$\begin{array}{ll}
\{(0,1),(2,2),(5,0)-\infty\}, & \{\infty,(5,1),(2,2)-(10,0)\}, \\
\{(0,0),(1,0),(3,0)-(3,1)\}, & \{(0,1),(2,0),(11,0)-(12,1)\}, \\
\{(0,1),(8,0),(3,0)-(7,1)\}, & \{(0,0),(6,0),(12,1)-(0,1)\}, \\
\{(0,0),(3,1),(7,1)-(9,1)\}, & \{(0,1),(0,2),(6,1)-(10,0)\}, \\
\{(0,0),(0,2),(3,2)-(4,0)\}, & \{(0,0),(1,2),(9,2)-(3,0)\}, \\
\{(0,0),(2,2),(11,2)-(12,1)\}, & \{(0,2),(2,1),(10,1)-(6,2)\}, \\
\{(0,1),(3,1),(4,2)-(11,2)\}, & \{(0,2),(11,2),(5,1)-(10,2)\}, \\
\{(0,2),(5,0),(12,2)-(8,0)\}. &
\end{array}$$

For each base kite $B = \{a, b, c-d\}$, take the corresponding l-transformation of B as $\{c, d, b-a\}$. It is readily checked that $(X, \mathcal{B})$ is an $lSKS(40)$. ◇

Lemma 5.7 *There exists an* $lSKS(56)$.

Proof Let $Y = \{\infty_1, \infty_2, \ldots, \infty_{17}\}$. We construct a $(K_{56} \setminus K_{17}, K_3 + e)$-design $((Z_{13} \times I_3) \cup Y, \mathcal{B})$ with hole Y and with the base kites in mod $(13, -)$:

$$\begin{array}{ll}
\{(5,1),(0,0),(11,2)-(2,1)\}, & \{(11,1),(6,2),(9,0)-(0,1)\}, \\
\{(1,0),(3,0),(0,0)-(4,0)\}, & \{(4,1),(1,1),(0,1)-(5,1)\}, \\
\{(4,2),(3,2),(0,2)-(7,2)\}, & \{(0,0),(7,0),(6,1)-(0,1)\}, \\
\{(0,1),(11,1),(2,2)-(0,2)\}, & \{(0,2),(8,2),(5,0)-(0,0)\}, \\
\{(7,1),(0,0),(7,2)-(0,1)\}, & \{\infty_1,(6,1),(5,0)-(6,2)\}, \\
\{\infty_2,(0,0),(4,2)-\infty_1\}, & \{\infty_3,(4,2),(1,1)-\infty_2\}, \\
\{\infty_4,(0,1),(4,0)-\infty_3\}, & \{\infty_5,(0,0),(0,2)-\infty_4\}, \\
\{\infty_6,(1,2),(0,1)-\infty_5\}, & \{\infty_7,(1,1),(4,0)-\infty_6\}, \\
\{\infty_8,(4,0),(0,2)-\infty_7\}, & \{\infty_9,(1,2),(4,1)-\infty_8\}, \\
\{\infty_{10},(0,1),(2,0)-\infty_9\}, & \{\infty_{11},(0,0),(2,2)-\infty_{10}\}, \\
\{\infty_{12},(7,2),(2,1)-\infty_{11}\}, & \{\infty_{13},(2,1),(7,0)-\infty_{12}\}, \\
\{\infty_{14},(2,0),(7,2)-\infty_{13}\}, & \{\infty_{15},(0,2),(2,1)-\infty_{14}\}, \\
\{\infty_{16},(4,1),(1,0)-\infty_{15}\}, & \{\infty_{17},(7,0),(0,2)-\infty_{16}\}, \\
\{(0,0),(12,2),(0,1)-\infty_{17}\}. &
\end{array}$$

For each base kite $B = \{a, b, c-d\}$, take the corresponding l-transformation of B as $\{c, d, b-a\}$. It is readily checked that $((Z_{13} \times I_3) \cup Y, \mathcal{B})$ is l-switchable. By Construction 2.3 with an $lSKS(17)$ from Lemma 5.2, there exists an $lSKS(56)$. ◇

Lemma 5.8 *There exists an l-switchable $(K_3+e,1)$-HD of hole-types* 8^3, 8^4 *and* 8^5; *There exists an l-switchable $(K_3+e,1)$-HD of hole-types* 16^3, 16^4 *and* 16^5.

Proof By Lemma 5.3 there exists an l-switchable $(K_3+e,1)$-HD of hole-types 8^3 and 8^4. Note that a $(5,1)$-GDD of group type 4^5 exists. Give each point of the

GDD a weight of 2 and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type 8^5. The needed l-switchable $(K_3 + e, 1)$-HD of hole-type 2^5 is from Lemma 5.3.

The second assertion is proved as follows: By Lemma 2.4 there exists a $(3, 1)$-GDD with group types 2^3 and 2^4. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-types 16^3 and 16^4. The needed l-switchable $(K_3 + e, 1)$-HD of hole-type 8^3 is from Lemma 5.3.

It is well known that there is a $(5, 1)$-GDD of group type 8^5. Give each point of the GDD a weight of 2, and similarly apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type 16^5. The needed l-switchable $(K_3 + e, 1)$-HD of hole-type 2^5 is from Lemma 5.3. ◇

Lemma 5.9 *There exists an $lSKS(v)$ for any integer $v \equiv 1 \pmod 8$ and $v \geq 9$.*

Proof Let $v = 8t + 1$ where $t \geq 1$. The conclusion follows by Lemma 5.2 for $t = 1, 2$. For $t \geq 3$ and $t \neq 6, 8$, by Lemma 2.5 there exists a $(t, \{3, 4, 5\}, 1)$-PBD. There exists an l-switchable $(K_3 + e, 1)$-HD of hole-types 8^3, 8^4 and 8^5 by Lemma 5.8. Apply Construction 2.2 with $g = 8$ to get an l-switchable $(K_3 + e, 1)$-HD of hole-type 8^t. By Construction 2.3 with $a = 1$, there exists an $lSKS(8t + 1)$. The needed $lSKS(9)$ comes from Lemma 5.2.

We remain to deal with $t = 6, 8$. By Lemma 2.4 there exists a $(3, 1)$-GDD with group type $2^{t/2}$. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type $16^{t/2}$. The input design is from Lemma 5.3. Applying again Construction 2.3 with $a = 1$, there exists an $lSKS(8t + 1)$. The needed $lSKS(17)$ comes from Lemma 5.2. ◇

Lemma 5.10 *There exists an $lSKS(v)$ for any integer $v \equiv 0 \pmod 8$ and $v \geq$ 16.*

Proof For $v = 16$, the conclusion follows from Lemma 5.2. For $v = 24$, there exists an l-switchable $(K_{24} \setminus K_9, K_3 + e)$-design by Lemma 5.4. By Construction 2.3 with an $lSKS(9)$ from Lemma 5.2, there exists an $lSKS(24)$.

For $v = 32$, there exists an l-switchable $(K_{32} \setminus K_{17}, K_3 + e)$-design by Lemma 5.5. By Construction 2.3 with an $lSKS(17)$ from Lemma 5.2, there exists an $lSKS(32)$.

For $v = 40$, the conclusion follows by Lemma 5.6.

Next we deal with the case $v \geq 48$. Let $v = 16t + \delta$ where $t \geq 3$ and $\delta = 0, 8$. We divide the problem into two cases.

Case 1: $v = 16t$ where $t \geq 3$. For $t \neq 6, 8$, by Lemma 2.5 there exists a $(t, \{3, 4, 5\}, 1)$-PBD. There exists an l-switchable $(K_3 + e, 1)$-HD of hole-types 16^3, 16^4 and 16^5 by Lemma 5.8. Apply Construction 2.2 with $g = 16$ to get an l-switchable $(K_3 + e, 1)$-HD of hole-type 16^t. By Construction 2.3 with $a = 0$, there exists an $lSKS(16t)$. The needed $lSKS(16)$ comes from Lemma 5.2.

We remain to deal with $t = 6, 8$. By Lemma 2.4 there exists a $(3, 1)$-GDD with group type $4^{t/2}$. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type $32^{t/2}$. The needed l-switchable $(K_3 + e, 1)$-HD of hole-type 8^3 is from Lemma 5.8. Applying again Construction 2.3 with $a = 0$, there exists an $lSKS(16t)$. The needed $lSKS(32)$ exists as pointed before.

Case 2: $v = 16t + 8$ where $t \geq 3$. For $t = 3$, the conclusion follows from Lemma 5.7.

For $t = 4$ (i.e., $v = 9 \times 8$), by Lemma 2.4 there exists a $(3, 1)$-GDD with group type 3^3. By Lemma 5.3 there exists an l-switchable $(K_3 + e, 1)$-HD of type 8^3. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type 24^3. Applying Construction 2.3, there exists an $lSKS(9 \times 8)$. The needed $lSKS(24)$ exists as above.

For $t = 5$ (i.e., $v = 11 \times 8$), there exists a $(4, 1)$-GDD of type 3^4. Take a block B of this GDD, and give one point of B a weight of zero and each other point of the GDD a weight of 8. By Lemma 5.8 there exists an l-switchable $(K_3 + e, 1)$-HD of types 8^3 and 8^4. We apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type $24^3 16^1$. Applying Construction 2.3, there exists an $lSKS(11 \times 8)$. The needed $lSKS(16)$ and $lSKS(24)$ come as above.

For $t = 6$ (i.e., $v = 13 \times 8$), there exists a $(4, 1)$-GDD of type 4^4. Take a block B of this GDD, and give three point of B a weight of zero and each other point of the GDD a weight of 8. By Lemma 5.8 there exists an l-switchable $(K_3 + e, 1)$-HD of types 8^3 and 8^4. We apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type $24^3 32^1$. Applying Construction 2.3, there exists an $lSKS(13 \times 8)$. The needed $lSKS(24)$ and $lSKS(32)$ come as above.

Next we deal with the case of $t \geq 7$. Let $t = 3s + i$ where $s \geq 2$ and $i = 1, 2, 3$. By Lemma 2.4 there exists a $(3, 1)$-GDD with group type $3^{2s}(2i+1)^1$. By Lemma 5.3 there exists an l-switchable $(K_3 + e, 1)$-HD of type 8^3. Give each point of the GDD a weight of 8, and apply the Weighting Construction to get an l-switchable $(K_3 + e, 1)$-HD of hole-type $24^{2s}(8(2i + 1))^1$. Applying Construction 2.3, there exists an $lSKS(8(6s + 2i + 1))$, i.e., $lSKS(16t + 8)$. The needed $lSKS(24)$ and $lSKS(8(2i + 1))$ $(i = 1, 2, 3)$ exist as above. ◇

Theorem 5.11 *An* $lSKS(v)$ *exists if and only if* $v \equiv 0, 1 \pmod 8$ *and* $v > 8$.

Proof The necessity follows from Lemma 5.1 and the necessary condition of the existence of a kite system of order v. The sufficiency follows by Lemmas 5.9 and 5.10. ◇

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