AN EXTENSION OF F_1, F_2, F_3 APPELL'S HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The main aim of this paper is to construct an extension of Appell's hypergeometric functions by means of modified Beta functions $B(x,y;p)$. We give integral representations for these functions and obtain some relations for these functions and extended Gauss hypergeometric function via decomposition operators defined by Burchnall and Chaundy. Furthermore we present some transformation formulas for the first and second kind of extended Appell's hypergeometric functions. Also, we give some relations between first kind of extended Appell's hypergeometric functions,Whittaker and Modified Bessel functions.

1. INTRODUCTION

Classical Gauss hypergeometric function defined by [1, p. 1]

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \left((\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \right) \qquad (1.1)$$

which is usually denoted by ${}_2F_1(a,b;c;z)$. This hypergeometric function satisfies following hypergeometric equation[1, p. 2];

$$z(1-z)\frac{d^2y}{dz^2} + \{c-(a+b+1)z\}\frac{dy}{dz} - aby = 0. \qquad (1.2)$$

The Gauss hypergeometric function can be given by integral representation as follows[1, p. 4]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \qquad (1.3)$$

$$(\Re(c) > \Re(b) > 0 \quad ; \quad |\arg(1-z) < \pi|).$$

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Since Euler's Beta function $B(x,y)$ has the integral representation

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=B(y,x) \quad ; \quad (\Re(x)>0,\Re(y)>0) \tag{1.4}$$

using a series expansion of $(1-zt)^{-a}$ in (1.3), we can also write the Gauss hypergeometric function in terms of the beta function as follows:

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n B(b+n,c-b)\frac{z^n}{n!} \qquad (1.5)\\ & (|z|<1 \ , \ \Re(c)>\Re(b)>0). \end{aligned}$$

Over six decades ago, Burchnall and Chaundy [10, 11] presented a number of expansion and decomposition formulas for double hypergeometric functions. Their method is based on the following inverse pairs of symbolic operators:

$$\begin{aligned} \nabla(h) &= \frac{\Gamma(h)\Gamma(h+\delta_1+\delta_2)}{\Gamma(h+\delta_1)\Gamma(h+\delta_2)}=\sum_{k=0}^{\infty}\frac{(-\delta_1)_k(-\delta_2)_k}{(h)_k\,k!}\\ \Delta(h) &= \frac{\Gamma(h+\delta_1)\Gamma(h+\delta_2)}{\Gamma(h)\Gamma(h+\delta_1+\delta_2)}=\sum_{k=0}^{\infty}\frac{(-\delta_1)_k(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\,k!}\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{(h)_{2k}(-\delta_1)_k(-\delta_2)_k}{(h+k-1)_k(\delta_1+h)_k(\delta_2+h)_k\,k!}\\ & \left(\delta_1=x\frac{d}{dx},\delta_2=y\frac{d}{dy}\right). \end{aligned}$$

In this paper, we consider a different extension which is based on the extension of the beta function by introducing an extra parameter p as follows

$$\begin{aligned} B(x,y;p) &= \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)dt=B(y,x;p),\\ & (\Re(p)>0) \qquad (1.6) \end{aligned}$$

which has already been found to be useful in various applications. For the properties of the generalized beta function defined by (1.6), we refer the reader to [3].

Chaudhry [2] has introduced the following extended Gauss hypergeometric function

$$\begin{aligned} F_p(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n B(b+n,c-b;p)\frac{z^n}{n!} \qquad (1.7)\\ & (|z|<1 \ , \ \Re(c)>\Re(b)>0) \end{aligned}$$

and he has obtained integral representation, differential formulas, Mellin transforms for the extended Gauss hypergeometric functions $F_p(a,b;c;z)$. In this paper we define extended F_1 and F_3 hypergeometric functions using the relations (1.5),(1.6) and Appell's hypergeometric functions. We further consider the extension of F_2 function defined in [9]. We recall the F_1, F_2 and F_3 Appell's hypergeometric functions as defined by

$$F_1(a,b,b';c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}\,m!n!}x^m y^n \qquad (|x|,|y|<1), \tag{1.8}$$

$$F_2(a,b,b';c,c';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n\,m!n!}x^m y^n \qquad (|x|+|y|<1), \tag{1.9}$$

$$F_3(a,a',b,b';c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}\,m!n!}x^m y^n \qquad (|x|,|y|<1). \tag{1.10}$$

2. AN EXTENSION FOR F_1, F_2 AND F_3 APPELL'S HYPERGEOMETRIC FUNCTIONS

The proposed extension (Extended Appell's hypergeometric functions) of the former Appell's hypergeometric functions can be written by using (1.6) in (1.8), (1.9) and (1.10) as follows:

$$\begin{aligned} F_{1,p}(a,b,b';c;x,y) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \\ &\times \sum_{m,n=0}^{\infty}(a)_{m+n}B(b+m,c-b+n;p) \\ &\times B(b'+n,c-b-b';p)\frac{x^m y^n}{m!n!} \end{aligned} \tag{2.1}$$

$$(p\geq 0, |x|,|y|<1, \Re(c)>\Re(b)>\Re(b')>0)$$

$$F_{2,p}(a;b,b';c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')} \times \sum_{m,n=0}^{\infty} (a)_{m+n} B(b+m, c-b; p) \times B(b'+n, c'-b'; p) \frac{x^m y^n}{m!n!} \tag{2.2}$$

$$\left(\begin{array}{c} p \geq 0, |x| + |y| < 1, \\ \Re(c) > \Re(b) > 0, \Re(c') > \Re(b') > 0 \end{array} \right)$$

$$F_{3,p}(a,a';b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \sum_{m,n=0}^{\infty} (a)_m (a')_n B(b+m, c-b+n; p) \times B(b'+n, c-b-b'; p) \frac{x^m y^n}{m!n!} \tag{2.3}$$

$$\left(\begin{array}{c} p \geq 0, |x|, |y| < 1, \\ \Re(c) > \Re(b) > \Re(b') > 0 \end{array} \right)$$

also for $p = 0$, these functions are reduced to the usual Appell's hypergeometric functions. Note that the function $F_{2,p}$ was defined in [9].

Since the coefficient of $x^m y^n$ for $F_4(a,b;c,c';x,y)$ can't be written as a product of two Beta functions, the fourth kind of Appell's hypergeometric function $F_4(a,b;c,c';x,y)$ can not be extended.

3. INTEGRAL REPRESENTATIONS FOR THE EXTENDED APPELL'S HYPERGEOMETRIC FUNCTIONS

The extended Appell's hypergeometric functions can be provided with integral representations by using (1.6) in (2.1),(2.2) and (2.3). For these functions, we have the following representations:

$$F_{1,p}(a,b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \iint u^{b-1} v^{b'-1} (1-u-v)^{c-b-b'-1} (1-ux-vy)^{-a} \times \exp\left[-p\left(\frac{1}{u(1-u)} + \frac{(1-u)^2}{v(1-u-v)}\right)\right] dudv \tag{3.1}$$

$$(p \geq 0, \Re(c) > \Re(b) > \Re(b') > 0)$$

$$F_{2,p}(a;b,b';c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')}$$
$$\times\int_0^1\int_0^1 u^{b-1}(1-u)^{c-b-1}$$
$$\times v^{b'-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}$$
$$\times\exp\left[-p\left(\frac{1}{u(1-u)}+\frac{1}{v(1-v)}\right)\right]dudv \quad (3.2)$$
$$(p\geq 0, \Re(c)>\Re(b)>0, \Re(c')>\Re(b')>0)$$

$$F_{3,p}(a,a';b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}$$
$$\times\iint u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}$$
$$\times(1-ux)^{-a}(1-vy)^{-a'}$$
$$\times\exp\left[-p\left(\frac{1}{u(1-u)}+\frac{(1-u)^2}{v(1-u-v)}\right)\right]dudv$$
$$(p\geq 0, \Re(c)>\Re(b)>\Re(b')>0). \quad (3.3)$$

$F_{1,p}$ and $F_{3,p}$ extended Appell's hypergeometric functions are taken over the triangle domain

$$u\geq 0, v\geq 0, u+v\leq 1.$$

The first kind of extended Appell's hypergeometric function $F_{1,p}$ can also be expressed by a simple integral, the formula being

$$F_{1,p}(a,b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}(1-ty)^{-b'}$$
$$\times\exp\left(-\frac{p}{t(1-t)}\right)dt \quad (3.4)$$
$$(p\geq 0, \Re(c)>\Re(b)>0).$$

Note that the integral representation of the function $F_{2,p}$ was obtained by Özarslan and Özergin [9].

4. DECOMPOSITIONS AND RECURRENCE FORMULAS

Decompositions for extended Appell's hypergeometric functions (2.1),(2.2) and (2.3) have been found by means of the operator $\nabla(a)$. We find the following decompositions applying Burchnall and Chaundy's method [10,11]

$$F_{1,p}(a,b,b';c;x,y) = \nabla(a)\, F_{3,p}(a,a;b,b';c;x,y) \quad (4.1)$$
$$F_{2,p}(a;b,b';c,c';x,y) = \nabla(a)\, F_p(a,b;c;x)\, F_p(a,b';c';y). \quad (4.2)$$

It's not possible to apply the operator $\Delta(h)$ to extended Appell's hypergeometric functions, since there is no relationship between extended beta and gamma function as known between usual beta and gamma function.

Furthermore, by putting $x = y$ in (3.4), we find the following recurrence relation.

$$F_{1,p}(a,b,b';c;x,x) = F_p(b+b',a;c;x). \tag{4.3}$$

5. SOME TRANSFORMATIONS OF THE FUNCTIONS $F_{1,p}$ AND $F_{2,p}$

We consider the integral

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}(1-ty)^{-b'}\exp(-\frac{p}{t(1-t)})dt$$

which occurs in (3.4). Making the substitution

$$t = 1-u,$$

we obtain the formula

$$\begin{aligned} F_{1,p}(a,b,b';c;x,y) &= (1-x)^{-b}(1-y)^{-b'} \\ &\times F_{1,p}(c-a,b,b';c;-\frac{x}{1-x},-\frac{y}{1-y}). \end{aligned} \tag{5.1}$$

Similarly, by considering the double integral

$$\int_0^1\int_0^1 u^{b-1}(1-u)^{c-b-1}v^{b'-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}$$
$$\times\exp\left[-p(\frac{1}{u(1-u)}+\frac{1}{v(1-v)})\right]dudv$$

which occurs in (3.2), and making the following substitutions respectively

$$\begin{aligned} u &= 1-u' \ , \ v = v' \\ u &= u' \quad\ \ , \ v = 1-v' \\ u &= 1-u' \ , \ v = 1-v', \end{aligned}$$

we have

$$F_{2,p}(a;b,b';c,c';x,y) = (1-x)^{-a} \times F_{2,p}(a;c-b,b';c,c';-\frac{x}{1-x},-\frac{y}{1-x}) \quad (5.2)$$

$$F_{2,p}(a;b,b';c,c';x,y) = (1-y)^{-a} \times F_{2,p}(a;b,c-b';c,c';\frac{x}{1-y},-\frac{y}{1-y}) \quad (5.3)$$

$$F_{2,p}(a;b,b';c,c';x,y) = (1-x-y)^{-a} \times F_{2,p}(a;c-b,b';c,c';-\frac{x}{1-x},-\frac{y}{1-x}) \quad (5.4)$$

But, there are no similar transformations for the $F_{3,p}$ function.

6. SUMMATION FORMULA

In this section, we give some relations between the first kind of extended Appell's hypergeometric functions and Whittaker, Modified Bessel functions. We consider simple integral representation of $F_{1,p}$ given by (3.4). Setting $x=1, y=1$ in (3.4), we have

$$\begin{aligned} F_{1,p}(a,b,b';c;1,1) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-t)^{-b}(1-t)^{b'}\exp(-\frac{p}{t(1-t)})dt \\ &= \frac{1}{B(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-b-b'-1}\exp(-\frac{p}{t(1-t)})dt \\ &= \frac{B(a,c-a-b-b';p)}{B(a,c-a)} \qquad (6.1) \end{aligned}$$

$$(p>0 \ , \ p=0 \ \Re(c-a-b-b')>0).$$

Taking $c=b+b'+2a$ in (6.1), it is simplified to Whittaker function $W_{k,m}(z)$ [3]

$$\begin{aligned} F_{1,p}(a,b,b';b+b'+2a;1,1) &= \frac{B(a,a;p)}{B(a,b+b'+a)} \\ &= \frac{\sqrt{\pi}p^{\frac{(a-1)}{2}}e^{-2p}}{2^a B(a,b+b'+a)} W_{-a/2,a/2}(4p) \quad (6.2) \end{aligned}$$

$$(\mathrm{Re}(p)>0)$$

which gives the relation between extended Appell's hypergeometric function and Whittaker function. Similarly, getting $c=b+b'$ in (6.1), we can connect the first kind of extended Appell's hypergeometric function with the modified Bessel function $K_\nu(z)$ [3] as follows:

$$F_{1,p}(a,b,b';b+b';1,1) = \frac{B(a,-a;p)}{B(a,b+b'-a)} = \frac{2\exp(-2p)}{B(a,b+b'-a)}K_a(2p). \tag{6.3}$$

Setting $c = b+b'-n$ and $c = 2a+b+b'+n$ respectively in (6.1), we find

$$F_{1,p}(a,b,b';b+b'-n;1,1) = \frac{B(a,-a-n;p)}{B(a,b+b'-a-n)} = \frac{2\exp(-2p)\sum_{k=0}^{n}\binom{n}{k}K_{n+k}(2p)}{B(a,b+b'-a-n)} \tag{6.4}$$

and

$$F_{1,p}(a,b,b';2a+b+b'+n;1,1) = \frac{B(a,a+n;p)}{B(a,a+b+b'+n)} = \frac{(\sqrt{\pi}\exp(-2p))\frac{1}{2}n}{B(a,a+b+b'+n)}$$
$$\times\sum_{k=0}^{[\frac{n}{2}]}2^{-a-k}p^{\frac{(a+k-1)}{2}}\frac{(-1)^k}{n-k}\binom{n-k}{k}W_{-(a+k)/2,(a+k)/2}(4p)\ . \tag{6.5}$$

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