A Note on Spanning Eulerian Graphs

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Abstract

A graph is called biclaw-free if it has no biclaw as an introduced subgraph. Lai and Yao [Discrete Math., 307 (2007) 1217] conjectured that every 2-connected biclaw-free graph G with $\delta(G) \geq 4$ has a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$. In this note, the conjecture is answered to the negative.

1 Introduction

We use [1] for terminology and notation not defined here and consider only finite, undirected graphs and we allow graphs to have multiple edges but no loops.

Let G be a graph and H be a connected subgraph of G. Denote by $d_H(v)$ the degree of v in subgraph H. We use $d(v)$ instead of $d_G(v)$ for a vertex v of G. The graph G/H is the multigraph obtained from G by replacing H by a vertex v_H such that the number of edges in G/H joining any $v \in V(G-H)$ to v_H in G/H equals the numbers of edges joining v in G to $V(H)$. Note that in contraction there may be multiple edges. A graph G is *eulerian* if G is connected and $d(v)$ is even for each vertex $v \in V(G)$. A graph is called *supereulerian* if it has a spanning eulerian subgraph.

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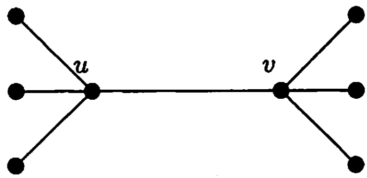

Fig 1. The biclaw and its center edge uv

A *claw* is a graph isomorphic to $K_{1,3}$. A *biclaw* is defined as the graph obtained from two vertex disjoint claws by adding an edge to the two vertices of degree 3 in each of the claws (see Fig 1). The *center edge* of a biclaw H is an edge uv such that $d_H(u) = d_H(v) = 4$. For each biclaw, there exists a unique center edge. A graph is called *biclaw-free* if it does not have a biclaw as an induced subgraph. A bipartite graph $G = (V_1, V_2)$ is called *balanced* if $|V_1| = |V_2|$.

Li [5] conjectured that there exists a constant c such that every connected bipartite biclaw-free graph G with $\delta(G) \geq c$ is hamiltonian. Lai and Yao [3] observed that if a bipartite graph G is hamiltonian, then G must be balanced. Naturally, Lai and Yao revised the conjecture as follows: There exists a constant c such that every connected balanced biclaw-free bipartite graph G with $\delta(G) \geq c$ is hamiltonian. Meanwhile, Lai and Yao [4] relaxed this conjecture and posed the following conjecture.

Conjecture 1.1 (Lai and Yao, [4]) *Every 2-connected biclaw-free graph G with $\delta(G) \geq 4$ has a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$.*

In the next section, we will show that Conjecture 1.1 fails in the sense that there exist infinite 2-connected biclaw-free graphs, both non-bipartite and balanced bipartite, with the minimum degree at least 4 which has no spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$.

2 Counterexamples

This section is devoted to showing a class of counterexamples to Conjecture 1.1. For this purpose, at first let G_m be the graph obtained by replacing each vertex v of the Petersen graph by a complete graph K_m, $m \geq 3$, such that three distinct edges incident with v are incident with three distinct vertices of K_m which replaces v in G_m, respectively (see Fig 2.).

Proposition 2.1 *For $m \geq 5$, G_m is a counterexample to Conjecture 1.1.*

Proof. For simplicity, let K_m^i be a copy of K_m for $i = 1, 2, \cdots, 10$. For $m \geq 5$, it is easy to see that G_m is 3-connected with $\delta(G_m) \geq 4$. Let $E_1 = \{e = uv : \text{both } u \text{ and } v \text{ are in } K_m^i \text{ for some } i\}$ and $E_2 = \{e = uv : u \text{ is in } K_m^i \text{ and } v \text{ is in } K_m^j,\ i \neq j\}$. Then $E(G_m) = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$.

Now we prove that G_m contains no biclaw. Suppose otherwise that G_m has a biclaw H with the center edge uv. If $uv \in E_1$, then $m \geq 6$. Let $N(u) \cap V(H) = \{u_1, u_2, u_3, v\}$ and $N(v) \cap V(H) = \{v_1, v_2, v_3, u\}$. Since K_m^i is a complete graph, there exist at least one vertex in $N(u) \cap V(H)$, say u_1 and at least one vertex v_1 in $N(v) \cap V(H)$ such that $u_1v_1 \in E(H)$, which contradicts to that uv is the center edge of H. If $uv \in E_2$, then all neighbors of u except v are in K_m^i and hence they are adjacent, which contradicts to that uv is the center edge of H. Thus, H is not a biclaw in G, a contradiction.

Next, we prove that G_m has no spanning eulerian subgraph with maximum degree less than 4. To the contrary, suppose that H is such a subgraph. We contract each K_m^i to one vertex and the resulting graph is the Petersen graph. It is clear that G_m has a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$ if and only if the Petersen graph contains a spanning eulerian subgraph H_0 with maximum degree $\Delta(H_0) \leq 4$ but the Petersen graph has no such subgraph. This contradiction proves our proposition. ■

If $n = 10m$, then G_m is 3-connected and $\delta(G_m) = m - 1 = \frac{n}{10} - 1$. Proposition 2.1 tells us that when G is 3-connected and $\delta(G) \geq c$ for any constant c, Conjecture 1.1 still fails. One naturally ask whether

Conjecture 1.1 may hold for bipartite graphs. In fact, when $n \geq 3m$ and $m \geq 4$, $K_{m,n}$ is a counterexample to Conjecture 1.1(the proof is similar to that of Proposition 2.2). Proposition 2.2 shows a stronger result that Conjecture 1.1 does not hold for 2-connected biclaw-free balanced bipartite graphs.

Denote by $G_i = (V_{i1}, V_{i2})$ two copies of $K_{m,n}$ where $i = 1, 2$. Let $|V_{i1}| = m$ and $|V_{i2}| = n$ for $i = 1, 2$. We define a balanced bipartite graph $G_{m,n} = (V_1, V_2)$ as follows. Let $V_1 = V_{11} \cup V_{22} \cup \{u_1, u_2\}$ and $V_2 = V_{12} \cup V_{21} \cup \{v_1, v_2\}$. Pick $x_1, x_2 \in V_{12}$, $y_1, y_2 \in V_{11}, a_1, a_2 \in V_{21}, b_1, b_2 \in V_{22}$. Let $E(G_{m,n}) = E(G_1) \cup E(G_2) \cup \{u_1v_1, u_1v_2, u_2v_1, u_2v_2, v_1y_1, v_1y_2, u_1x_1, u_1x_2, v_2b_1, v_2b_2, u_2a_1, u_2a_2\}$. (see Fig 3.).

Proposition 2.2 *If $n \geq 3m$ and $m \geq 4$, then $G_{m,n}$ is also a counterexample to Conjecture 1.1.*

Proof. Clearly, $G_{m,n}$ is a 2-connected balanced bipartite graph with $\delta(G_{m,n}) \geq 4$. Now we show that $G_{m,n}$ is biclaw-free.

Suppose otherwise that $G_{m,n}$ has a biclaw H with the center edge uv. If $uv \in E(G_1)$, then let $N(u) = \{v, u_1, u_2, u_3\}$ and $N(v) = \{u, v_1, v_2, v_3\}$. Since $G_1 \cong K_{m,n}$, $u_1v_1 \in E(G_{m,n})$, which contradicts to that H is a biclaw. Similarly, $uv \notin E(G_2)$. Thus, $uv \notin E(G_1) \cup E(G_2)$. It follows that $uv \in \{u_1v_1, u_1v_2, u_2v_1, u_2v_2, u_1x_1, u_1x_2, v_1y_1, v_1y_2, u_2a_1, u_2a_2, v_2b_1, v_2b_2\}$. By symmetry, $uv \in \{u_1v_1, u_1v_2, u_1x_1\}$.

Suppose that $uv = u_1v_1$. We assume, without loss of generality, that $u = u_1$ and $v = v_1$. Then $N(u) = \{v_1, x_1, x_2, v_2\}$ and $N(v) = \{y_1, y_2, u_1, u_2\}$. Since $x_1y_1, x_2y_1 \in E(G_{m,n})$, uv is not the center edge of H, a contradiction. Since $v_1u_2 \in E(G_{m,n})$, u_1v_2 is not the center edge of H. Thus, $uv = u_1x_1$. We assume that $u = u_1$ and $v = x_1$. Then $N(u_1) = \{x_1, x_2, v_1, v_2\}$ and $N(x_1) = V_{11} \cup \{u_1\}$. Since $y_1x_2 \in E(G_{m,n})$, u_1x_1 is not the center edge of H, a contradiction.

We then show that $G_{m,n}$ has no spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$. Suppose otherwise that $G_{m,n}$ has such a subgraph H. Since $\{u_1v_2, u_2v_1\}$ is a 2-edge-cut of $G_{m,n}$, $u_1v_2, u_2v_1 \in E(H)$. Clearly, both $d_H(u_1)$ and $d_H(v_1)$ are even. $u_1v_1 \notin E(H)$ if $d_H(u_1) = d_H(v_1) = 2$ and $u_1v_1 \in E(H)$ otherwise.

Define Γ to be the graph from $G_{m,n}$ by deleting u_2, v_2 and all vertices of G_2 and by contracting an edge u_1v_1. We also define H_1 to be the subgraph of Γ induced by the edge set $E(H) \cap E(\Gamma)$. Then H_1 is a spanning eulerian subgraph of Γ with $\Delta(H_1) \leq 4$.

Let x be the average degree of the vertices in $V(H_1) \cap V_{11}$. When $d_H(u_1) = d_H(v_1) = 2$ or $d_H(u_1) = d_H(v_1) = 4$,

$$mx - \epsilon \geq 2n - \epsilon \Longrightarrow x \geq \frac{2n}{m} \geq 6,$$

where $\epsilon = 2$ if $d_H(u_1) = d_H(v_1) = 2$;$\epsilon = 4$ if $d_H(u_1) = d_H(v_1) = 4$.

When $d_H(u_1) = 4$ and $d_H(v_1) = 2$,

$$mx \geq 2n - 2 \Longrightarrow x \geq \frac{2n-2}{m} \geq \frac{6m-2}{m} > 5.$$

When $d_H(u_1) = 2$ and $d_H(v_1) = 4$,

$$mx - 2 \geq 2n \Longrightarrow x \geq \frac{2n+2}{m} \geq 6.$$

All cases contradict to $\Delta(H_1) \leq 4$. This contradiction proves our proposition. ■

Catlin [2] introduced the concept of collapsible graphs. Let G be a graph and let $O(G)$ to denote the set of odd degree vertices of G. A graph G is called *collapsible* if for every even subset R of $V(G)$ there is a spanning connected subgraph Γ_R of G such that $O(\Gamma_R) = R$. Thus, if G is collapsible, then G is supereulerian and hence G is 2-edge connected.

Lai and Yao [4] proved that every connected bipartite biclaw-free graph G with $\delta(G) \geq 5$ is collapsible. When G is supereulerian, G may not contain a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$. We notice that $G_{m,n}$ is a 2-edge connected balanced bipartite biclaw-free graph with $\delta(G_m) = 4$ in Proposition 2.2. We construct a graph $G^*_{m,n}$ from $G_{m,n}$ by deleting an edge u_1v_1 and adding u_1a_1 and v_1b_1. It is easy to check that $G^*_{m,n}$ is 4-edge connected balanced bipartite biclaw-free with $\delta(G) \geq 4$. By the technique in the proof of Proposition 2.2, we can prove that $G^*_{m,n}$ has no a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$. Motivated by G_m and $G^*_{m,n}$, we ask whether every 3-edge connected balanced bipartite biclaw-free graph G with $\delta(G) \geq 5$ has a spanning eulerian subgraph H with maximum degree $\Delta(H) \leq 4$.

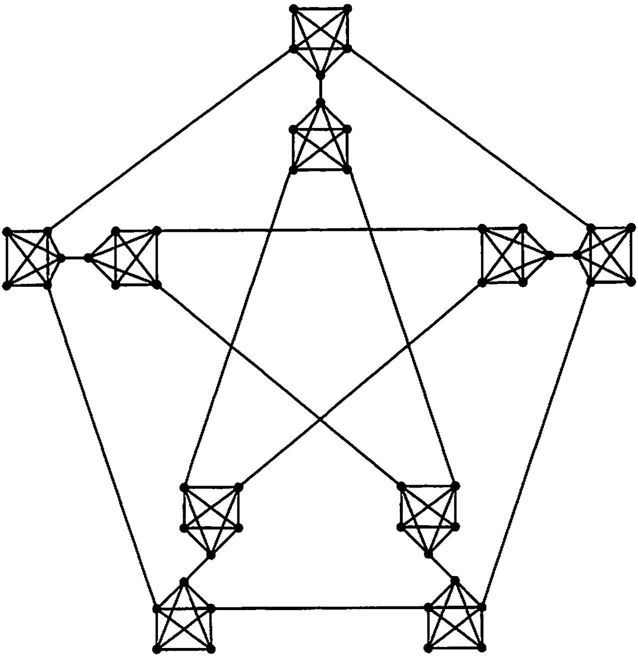

Fig 2. G_5

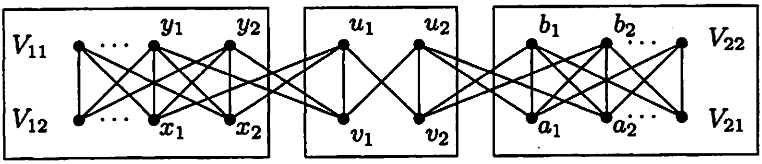

Fig 3. $G_{m,n}$

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