List-colouring the square of an outerplanar graph

Timothy J. Hetherington, Douglas R. Woodall*

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

Abstract

It is proved that if G is a $K_{2,3}$-minor-free graph with maximum degree Δ, then $\Delta + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2) \leqslant \Delta + 2$ if $\Delta \geqslant 3$, and $\mathrm{ch}(G^2) = \chi(G^2) = \Delta + 1$ if $\Delta \geqslant 6$. All inequalities here are sharp, even for outerplanar graphs.

Keywords: Choosability; Outerplanar graph; Minor-free graph; List square colouring

1 Introduction

We use standard terminology, as defined in the references: for example [5] or [9]. The *square* G^2 of a graph G has the same vertex-set as G, and two vertices are adjacent in G^2 if they are within distance two of each other in G.

There is great interest in discovering classes of graphs G for which the choosability or list chromatic number $\mathrm{ch}(G)$ is equal to the chromatic number $\chi(G)$. The *list-square-colouring conjecture* (*LSCC*) [5] is that, for every graph G, $\mathrm{ch}(G^2) = \chi(G^2)$. It is clear that this conjecture holds when the maximum degree $\Delta(G)$ of G is 0 or 1, and it can be deduced from the results of [7] when $\Delta(G) = 2$: see [4]. In general, it is easy to see that $\Delta(G) + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2)$.

It is well known that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. Squares of K_4-minor-free graphs were considered in [4]. For $K_{2,3}$-minor-free graphs we have the following result, which is the same as for the slightly smaller class of outerplanar graphs.

*Email: pmxtjh@nottingham.ac.uk, douglas.woodall@nottingham.ac.uk

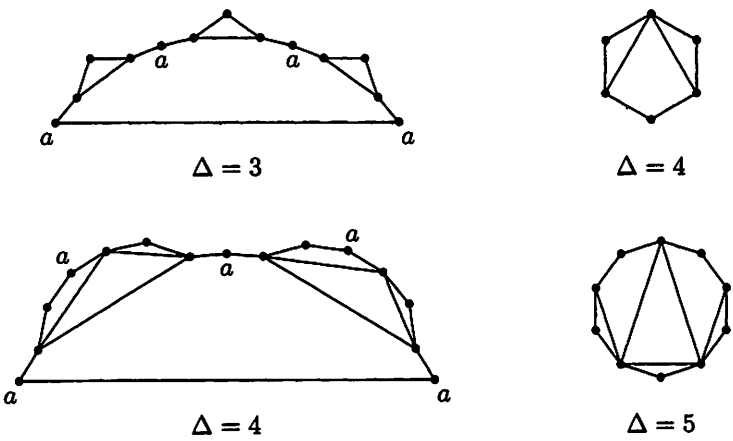

Fig. 1

Theorem 1. *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then $\Delta + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2) \leqslant \Delta + 2$ if $\Delta \geqslant 3$, and $\mathrm{ch}(G^2) = \chi(G^2) = \Delta + 1$ if $\Delta \geqslant 6$.*

We are indebted to the referee for telling us about reference [6], which led us indirectly to [1]. These papers contain alternative proofs of parts of Theorem 1 when G is outerplanar: [6] proves most of the results for $\chi(G^2)$, and [1] proves all of the results for $\chi(G^2)$ and also ('as a bonus') that $\mathrm{ch}(G^2) = \Delta + 1$ if $\Delta \geqslant 7$. Both of these papers were motivated by the conjecture of Wegner [8] that if G is a planar graph with maximum degree Δ then $\chi(G^2) \leqslant \Delta + 5$ if $4 \leqslant \Delta \leqslant 7$ and $\chi(G^2) \leqslant 3\Delta/2 + 1$ if $\Delta \geqslant 8$. Our motivation, the LSCC, is somewhat different.

When $3 \leqslant \Delta \leqslant 5$, the upper bound on $\mathrm{ch}(G^2)$ in Theorem 1 is sharp even for $\chi(G^2)$, and even for the smaller class of outerplanar graphs, as shown by the graphs in Fig. 1. For each of the cases $\Delta = 3$ and $\Delta = 4$ there is an infinite family of minimal (under subgraph-inclusion) extremal examples. One member of each family is shown in Fig. 1; in each case, if only $\Delta + 1$ colours are available, then all the vertices labelled a have to have the same colour, which gives a contradiction on the bottom edge. Fig. 1 also shows the smallest extremal example with $\Delta = 4$ and a smallest known extremal example with $\Delta = 5$; in fact, for $\Delta = 5$ we know of only two minimal extremal examples, both of order 10.

For the case $\Delta = 6$, the proof that $\mathrm{ch}(G^2) = \Delta + 1$ in Theorem 1 is exceptionally long and involved, and so we omit it from this paper, instead proving only that $\mathrm{ch}(G^2) \leqslant \Delta + 2 = 8$; the proof that $\mathrm{ch}(G^2) = \Delta + 1 = 7$ is included in the first author's doctoral thesis [3]. Since $\Delta(G)+1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2)$ for every graph G, in order to prove this weaker version of Theorem 1 it suffices to prove the following.

Theorem 2. *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then $\mathrm{ch}(G^2) \leqslant \Delta+2$ if $\Delta \geqslant 3$, and $\mathrm{ch}(G^2) \leqslant \Delta+1$ if $\Delta \geqslant 7$.*

The rest of this paper is devoted to a proof of Theorem 2. We will need the following simple lemma.

Lemma 1. *Let G be a $K_{2,3}$-minor-free graph. Then each block of G is either K_4-minor-free (and hence outerplanar) or else isomorphic to K_4.*

Proof. Suppose B is a block of G that has a K_4 minor. Since $\Delta(K_4)=3$, it follows that B has a subgraph H homeomorphic to K_4. Since any graph obtained by subdividing an edge of K_4, or by adding a path joining two vertices of K_4, has a $K_{2,3}$ minor, it follows that $H \cong K_4$ and $B=H$. □

As usual, $d(v)=d_G(v)$ will denote the degree of vertex v in graph G.

2 The start of the proof

Fix the value of $\Delta \geqslant 3$, and suppose if possible that G is a $K_{2,3}$-minor-free graph with maximum degree *at most* Δ and with as few vertices as possible such that $\mathrm{ch}(G^2) > \Delta+2$ or $\Delta+1$ as appropriate. By Lemma 1, every block of G is outerplanar or isomorphic to K_4. Clearly G is connected and is not K_2. If G is 2-connected, let $B := G$ and let z_0 be an arbitrary vertex of G; otherwise, let B be an endblock of G with cutvertex z_0. Assume that every vertex v of G is given a list $L(v)$ of $\Delta+2$ or $\Delta+1$ colours, as appropriate, such that G^2 has no proper colouring from these lists.

Claim 2.1. *Not every vertex of $B - z_0$ is adjacent to z_0.*

Proof. Suppose it is. Then every vertex of $B - z_0$ has degree at most Δ in G^2, since all its neighbours in G^2 are in the closed neighbourhood of z_0 in G. Thus we can colour $(G-(B-z_0))^2$ from its lists by the minimality of G, and then colour all the remaining vertices. This contradiction proves Claim 2.1. □

Claim 2.2. *G does not contain three vertices u, v, w of degree 2 such that $uv, vw \in E(G)$.*

Proof. Suppose it does. Then $d_{G^2}(v) \leqslant 4$. Let $H := G - v + uw$, so that H is $K_{2,3}$-minor-free and $G^2 - v \subseteq H^2$. Then we can colour G^2 from its lists by first colouring $G^2 - v$ and then colouring v. This is the required contradiction. □

It follows from Claim 2.1 that $B \not\cong K_2$, K_3 or K_4; thus B is an outerplanar graph that is 2-connected but not complete, and consists of a cycle C with chords. (A *chord* is an edge that joins two nonconsecutive vertices of the cycle.) Claim 2.2 shows that C has at least one chord.

Assume that B is embedded in the plane with C bounding the outside face. In [2], a *cap* is defined to be a region R of the plane that is bounded by a segment of C and one chord u_1u_2. We modify the definition slightly here by insisting also that z_0 is not in the interior of this segment; so z_0 is either u_1 or u_2 or is not in R. We call u_1 and u_2 the *endvertices* of R. By an abuse of terminology, the subgraph of B induced by all vertices in R will also be referred to as a *cap*. We will refer to an edge of C as a *trivial cap* or a *0-cap*. For $i \geqslant 1$, an *i-cap* is a cap that properly contains an $(i-1)$-cap and is minimal with this property.

The proof now divides into two cases.

3 Proof that $\mathrm{ch}(G^2) \leqslant \Delta + 2$

In this section we assume that every vertex v of G has a list $L(v)$ of $\Delta + 2$ colours, and G^2 is not colourable from these lists, but if H is any $K_{2,3}$-minor-free graph with maximum degree at most Δ and fewer vertices than G then $\mathrm{ch}(H^2) \leqslant \Delta + 2$.

Claim 3.1. *Every 1-cap in B is a triangle xu_1u_2 where $d_G(x) = 2$ and $d_G(u_i) \geqslant 4$ $(i = 1, 2)$.*

Proof. By definition, a 1-cap is a region bounded by a chord u_1u_2 and a segment $u_1x_1 \dots x_ru_2$ of C, where $d_G(x_i) = 2$ for each i. By Claim 2.2, $r \leqslant 2$. So if Claim 3.1 is false then either $r = 2$, or $r = 1$ and $d_G(u_j) \leqslant 3$ for some $j \in \{1, 2\}$. But in either case $G^2 - x_1 = (G - x_1)^2$ and $d_{G^2}(x_1) \leqslant \Delta + 1$, and so we can colour G^2 from its lists by first colouring $(G - x_1)^2$ (by the minimality of G) and then colouring x_1. This contradiction proves Claim 3.1. □

Claim 3.2. *B has a cap that is not a 1-cap.*

Proof. Suppose that every cap in B is a 1-cap. Then z_0 is not the endvertex of a chord, since a chord z_0y bounds two caps, and if both of these caps are 1-caps then $d_G(y) = 3$, contrary to Claim 3.1. Also, at most two chords of C are incident with any one vertex, since if there were three (or more) chords incident with the same vertex then the middle one (or more) of these chords would bound a cap that is not a 1-cap. It follows from this and Claim 3.1 that the endvertices of every chord have degree exactly 4 in G. The chords therefore form a cycle inside C, every edge of

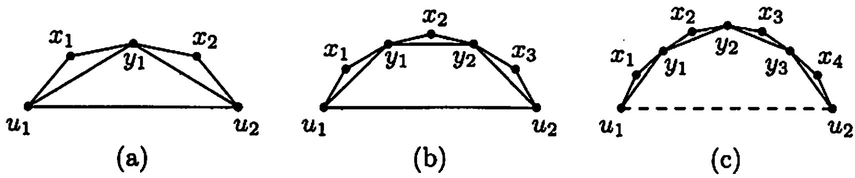

Fig. 2

which joins vertices that are distance 2 apart around C, except possibly for the edge e of the cycle that bounds a face of B with z_0 in its boundary. Now, a cap cannot contain z_0 by definition, except as an endvertex of its chord, which we have already shown to be impossible. Thus there is a unique cap bounded by e, and this cap contains all the 1-caps in B and so is not a 1-cap itself. This contradiction proves Claim 3.2. □

Claim 3.3. *Every 2-cap in B looks like one of the caps in the sequence of which Figs* 2(a) *and* 2(b) *are the first two members.*

Proof. Let R be a 2-cap that is bounded by a chord u_1u_2 and a segment of C. Since R properly contains a 1-cap and is minimal with this property, there is at least one chord inside R, and *every* such chord cuts off a 1-cap. So the chords inside R can be enumerated as $l_1r_1, \ldots, l_kr_k$, where the vertices

$$u_1, l_1, r_1, \ldots, l_k, r_k, u_2$$

occur in that order round C, but possibly $u_1 = l_1$, or $r_i = l_{i+1}$ for some i, or $r_k = u_2$. In fact, since $d(l_i) \geqslant 4$ and $d(r_i) \geqslant 4$ for each i by Claim 3.1, necessarily $u_1 = l_1$, and $r_i = l_{i+1}$ for *every* i, and $r_k = u_2$, since otherwise $d(l_i) = 3$ or $d(r_i) = 3$ for some i. Since, by Claim 3.1 again, every chord l_ir_i cuts off a triangle from R, the proof of Claim 3.3 is complete. □

It follows from Claims 3.2 and 3.3 that $\Delta \geqslant 4$ and B contains one of the configurations H shown in Fig. 2, where the dashed edge may or may not be present in Fig. 2(c), and z_0 is either u_1 or u_2 or is not in H.

Suppose first that B contains H as in Fig. 2(a). Then we can colour $(G - x_1)^2$ from its lists by the minimality of G, and then colour x_1, since $d_{G^2}(x_1) = d_G(u_1) + 1 \leqslant \Delta + 1$. This is the required contradiction.

Suppose next that B contains H as in Fig. 2(b). Colour the graph $(G - \{x_1, x_2, x_3, y_1, y_2\})^2$ from its lists, and for each uncoloured vertex w let $L'(w)$ denote the 'residual list' of colours in $L(w)$ that are not used on any G^2-neighbour of w and so are still available for use on w. Then $|L'(w)| \geqslant (\Delta + 2) - (\Delta - 1) = 3$ if $w \in \{x_1, y_1, y_2, x_3\}$, and $|L'(x_2)| \geqslant (\Delta + 2) - 2 \geqslant 4$. So if we try to colour the vertices in the order

$$x_1, y_1, y_2, x_3, x_2, \tag{1}$$

Fig. 3

it is only at x_2 that we may fail. If $L'(x_1) \cap L'(x_3) \neq \emptyset$, give x_1 and x_3 the same colour; then y_1, y_2 and x_2 can be coloured in the same order as in (1). If however $L'(x_1) \cap L'(x_3) = \emptyset$, then either $|L'(x_2)| \geqslant 6$, or else x_1, say, has a usable colour c_1 not in $L'(x_2)$; in either case, the vertices can be coloured in the order (1), with x_1 receiving colour c_1 if it exists.

Suppose finally that B contains H as in Fig. 2(c). Colour the graph $(G-(V(H)\backslash\{u_1,u_2\}))^2$ from its lists, and let each uncoloured vertex w have residual list $L'(w)$. Then $|L'(w)| \geqslant 3$ if $w \in \{x_1, y_1, y_3, x_4\}$, $|L'(y_2)| \geqslant 4$, and $|L'(w)| \geqslant 5$ if $w \in \{x_2, x_3\}$. So if we try to colour the vertices in the order

$$y_1, x_4, y_3, y_2, x_1, x_2, x_3, \qquad (2)$$

it is only at x_3 that we may fail. If $L'(y_1) \cap L'(x_4) \neq \emptyset$, give y_1 and x_4 the same colour, then colour the remaining vertices in the order (2). If however $L'(y_1) \cap L'(x_4) = \emptyset$, then either $|L'(x_3)| \geqslant 6$, or else y_1 or x_4 has a usable colour c_1 not in $L'(x_3)$; in either case, the vertices can be coloured in the order (2), with y_1 or x_4 receiving colour c_1 if it exists.

In every case we have obtained a contradiction, and so we have proved that $\mathrm{ch}(G^2) \leqslant \Delta + 2$ for all $\Delta \geqslant 3$.

4 Proof that $\mathrm{ch}(G^2) \leqslant \Delta + 1$ when $\Delta \geqslant 7$

In this section we assume that every vertex v of G has a list $L(v)$ of $\Delta + 1$ colours, and G^2 is not colourable from these lists, but if H is any $K_{2,3}$-minor-free graph with maximum degree at most Δ and fewer vertices than G then $\mathrm{ch}(H^2) \leqslant \Delta + 1$. To begin with we assume only that $\Delta \geqslant 6$; we will not use the fact that $\Delta \geqslant 7$ until Claim 4.4.

Claim 4.1. *Every vertex of degree 2 in G has degree at least $\Delta + 1$ in G^2.*

Proof. Let v be a vertex of degree 2 in G with neighbours u, w, and suppose that $d_{G^2}(v) \leqslant \Delta$. Let $H := G - v$ if $uw \in E(G)$ and let $H := G - v + uw$ otherwise. Then H is $K_{2,3}$-minor-free and $G^2 - v \subseteq H^2$ and $\mathrm{ch}(G^2 - v) \leqslant \mathrm{ch}(H^2) \leqslant \Delta + 1$ by the minimality of G. So we can colour G^2 from its lists by first colouring $G^2 - v$ and then colouring v. This is the required contradiction. □

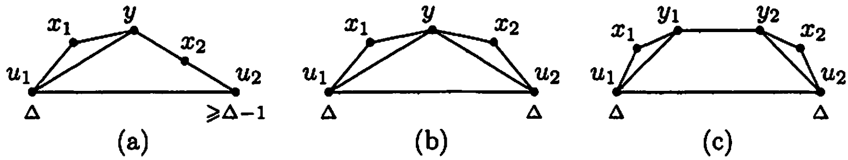

Fig. 4

Claim 4.2. *Every* 1*-cap in* B *has the form shown in Fig.* 3(a) *or* 3(b), *where* $d_G(u_1) + d_G(u_2) \geqslant \Delta + 3$ *in Fig.* 3(a), *and* $d_G(u_1) = d_G(u_2) = \Delta$ *in Fig.* 3(b).

Proof. The first part of the statement follows immediately from Claim 2.2 and the definition of a 1-cap. To prove the second part, note that, by Claim 4.1,

$$\Delta + 1 \leqslant d_{G^2}(x) \leqslant d_G(u_1) + d_G(u_2) - 2$$

in Fig. 3(a), and

$$\Delta + 1 \leqslant d_{G^2}(x_i) = d_G(u_i) + 1 \qquad (i = 1, 2)$$

in Fig. 3(b). □

Claim 4.3. *Every* 2*-cap in* B *has one of the forms shown in Fig.* 4, *where the degrees of* u_1 *and* u_2 *are restricted as specified.*

Proof. Let R be a 2-cap that is bounded by a chord u_1u_2 and a segment of C. As in the proof of Claim 3.3, the chords inside R can be enumerated as $l_1r_1, \ldots, l_kr_k$, where the vertices

$$u_1, l_1, r_1, \ldots, l_k, r_k, u_2$$

occur in that order round C, but possibly $u_1 = l_1$, or $r_i = l_{i+1}$ for some i, or $r_k = u_2$. Thus every vertex of R other than u_1 and u_2 has degree at most 4 in G. It follows from the degree conditions in Claim 4.2 that R contains no 1-cap of the type in Fig. 3(b), and also, since $\Delta + 3 \geqslant 9 > 2 \cdot 4$, any 1-cap in R of the type in Fig. 3(a) must share an endvertex with R. Thus $k = 1$ or 2.

If $k = 1$ then R is as in Fig. 4(a) (or its reflection). Note that if there were no vertex x_2, just a single edge yu_2, then we would have $d_{G^2}(x_1) = d_G(u_1) \leqslant \Delta$, and if there were a further vertex x_3 subdividing the edge x_2u_2 then we would have $d_{G^2}(x_2) = 5 < \Delta$, contradicting Claim 4.1 in each case. The degree conditions in Fig. 4(a) also follow from Claim 4.1, because $d_{G^2}(x_1) = d_G(u_1) + 1$ and $d_{G^2}(x_2) = d_G(u_2) + 2$.

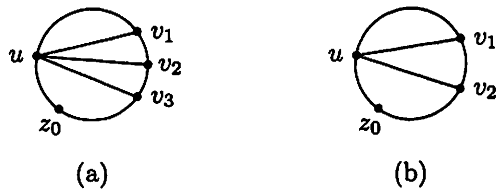

Fig. 5

So suppose $k = 2$. Then R is as in Fig. 4(b) or 4(c). Note that if there were a further vertex w subdividing the edge y_1y_2 in Fig. 4(c) then we would have $d_{G^2}(w) = 6 \leqslant \Delta$, contrary to Claim 4.1. The degree conditions in the figures again follow from Claim 4.1, because $d_{G^2}(x_i) = d_G(u_i) + 1$ $(i = 1, 2)$ in each case. □

From now on, we will assume that $\Delta \geqslant 7$.

Claim 4.4. *Every nontrivial cap in B is a 1-cap or a 2-cap.*

Proof. Suppose this is not true. Then B contains a 3-cap. Let R be a 3-cap in B, with endvertices u_1, u_2. The chords inside R divide R into faces. Let f be the face with u_1u_2 in its boundary. There are three possible types for every other edge of f: it may be an edge of C, or a chord cutting off a 1-cap, or a chord cutting off a 2-cap. There must be at least one edge of f that is a chord cutting off a 2-cap, since otherwise R would itself be a 1-cap or a 2-cap. So let u, v, w be three consecutive vertices in the boundary of f, where uv is a chord cutting off a 2-cap. Then $d_G(v) \leqslant 6$, since the cap cut off by uv, and the cap (possibly a 0-cap) cut off by vw, each contribute at most 3 to the degree of v. Since $\Delta \geqslant 7$, and in view of the degrees indicated in Fig. 4, the only possibility is that uv cuts off a 2-cap of the type in Fig. 4(a), and $d_G(v) = \Delta - 1$. But then this cap contributes only 2 to the degree of v, so that $d_G(v) \leqslant 5 < \Delta - 1$. This contradiction completes the proof of Claim 4.4. □

Claim 4.5. *$\Delta(B) \leqslant 6$, and if u is a vertex of B that is adjacent to z_0 then $d_G(u) \leqslant 5$.*

Proof. Suppose that $u \in V(B)$ and $d_B(u) \geqslant 7$, or $uz_0 \in E(B)$ and $d_B(u) = 6$. Then there are chords uv_1, uv_2 and uv_3 as shown in Fig. 5(a), where z_0 lies in the closed segment of C between u and v_3 that does not contain v_1 and v_2 ('closed' meaning that possibly $z_0 = u$ or $z_0 = v_3$). The chord uv_1 cuts off a cap R_1 which, by Claim 4.4, is a 1-cap or a 2-cap. The chord uv_2 cuts off a cap R_2 that properly contains R_1 and so must be a 2-cap. The chord uv_3 cuts off a cap that properly contains R_2 and so is neither a 1-cap nor a 2-cap. This contradicts Claim 4.4. □

Claim 4.6. *Every nontrivial cap in B is a* 1*-cap.*

Proof. Suppose this is not true. Then, by Claim 4.4, B contains a 2-cap. Suppose there is a 2-cap in B with endvertices u_1, u_2, where w.l.o.g. $u_2 \neq z_0$. Then $d_G(u_2) = d_B(u_2) \leqslant 6$ by Claim 4.5, while $d_G(u_2) \geqslant \Delta - 1 \geqslant 6$ by the degree constraints in Fig. 4. The only possibility is that $d_G(u_2) = \Delta - 1 = 6$. This is impossible if $u_1 = z_0$, since then Claim 4.5 implies that $d_G(u_2) \leqslant 5$. So $u_1 \neq z_0$. But then the same argument as for u_2 shows that $d_G(u_1) = \Delta - 1$, which is impossible since every 2-cap in Fig. 4 has at least one endvertex with degree Δ. □

Claim 4.7. $\Delta(B) \leqslant 4$.

Proof. Suppose that $u \in V(B)$ and $d_B(u) \geqslant 5$. Then there are chords uv_1 and uv_2 as shown in Fig. 5(b), where z_0 lies in the closed segment of C between u and v_2 that does not contain v_1. The chord uv_1 cuts off a cap R_1 which, by Claim 4.6, is a 1-cap. The chord uv_2 cuts off a cap that properly contains R_1 and so is not a 1-cap. This contradicts Claim 4.6. □

It is now easy to finish the proof. It follows from Claims 4.6 and 4.7 and the degree conditions in Claim 4.2 that every nontrivial cap in B is a 1-cap of the type in Fig. 3(a) with z_0 as one endvertex. But then B consists of a quadrilateral z_0xyzz_0 with one chord z_0y, and this contradicts Claim 2.1. This finally completes the proof of Theorem 2.

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