On the Vertex Geodomination Number of a Graph

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Abstract

For a connected graph G of order $p \geq 2$, a set $S \subseteq V(G)$ is an x-geodominating set of G if each vertex $v \in V(G)$ lies on an x-y geodesic for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-geodomination number of G, denoted by $g_x(G)$ or simply g_x. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set. A connected graph of order p with vertex geodomination numbers either $p-1$ or $p-2$ for every vertex is characterized. It is shown that there is no graph of order p with vertex geodomination number $p-2$ for every vertex. Also, for an even number p and an odd number n with $1 \leq n \leq p-1$, there exists a connected graph G of order p and $g_x(G) = n$ for every vertex x in G and for an odd number p and an even number n with $1 \leq n \leq p-1$, there exists a connected graph G of order p and $g_x(G) = n$ for every vertex x in G. It is shown that for any integer $n \geq 2$, there exists a connected regular as well as a non-regular graph G with $g_x(G) = n$ for every vertex x in G. For positive integers r, d and $n \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G of radius r, diameter d and $g_x(G) = n$ for every vertex x in G. Also, for integers p, d and n with $3 \leq d \leq p-1, 1 \leq n \leq p-1$ and $p-d-n+1 \geq 0$, there exists a graph G of order p, diameter d and $g_x(G) = n$ for some vertex x in G.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an *x-y geodesic.* A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. The *diameter diam* G of a connected graph G is the length of any longest geodesic. For any vertex u of G, the *eccentricity* of u is $e(u) = max\{d(u, v) : v \in V\}$. A vertex v of G such that $d(u, v) = e(u)$ is called an *eccentric vertex* of u. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete. A *nonseparable* graph is connected, nontrivial, and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. A *connected block graph* is a connected graph in which each of its blocks is complete. A *caterpillar* is a tree for which the removal of all the end vertices gives a path.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V, I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set.* The geodetic number of a graph was introduced in [1, 6] and further studied in [3]. It was shown in [6] that determining the geodetic number of a graph is an NP-hard problem. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart, and Zhang in [4], where a pair x, y of vertices in a nontrivial connected graph G is said to *geodominate a vertex* v of G if $v \in I[x, y]$, that is, v lies on an x-y geodesic of G. In [4], geodetic sets and the geodetic number were referred to as *geodominating sets* and *geodomination number* and it is this terminology that we adopt in this paper.

The concept of vertex geodomination number was introduced by Santhakumaran and Titus [8]. A vertex y in a connected graph G is said to *x-geodominate* a vertex u if u lies on an x-y geodesic. A set S of vertices of G is an *x-geodominating set* if each vertex $v \in V(G)$ is x-geodominated by some element of S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G, denoted by $g_x(G)$ or simply g_x. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set.

Every vertex of an x-y geodesic is x-geodominated by the vertex y. Since, by definition, a g_x-set is minimum, the vertex x and also the internal vertices of an x-y geodesic do not belong to a g_x-set. For the graph G given in Figure 1.1, $g_u(G) = 1, g_v(G) = 2, g_w(G) = 2, g_x(G) = 2$ and $g_y(G) = 1$ with minimum vertex geodominating sets $\{y\}, \{u, y\}, \{u, x\}, \{u, w\}$, and $\{u\}$ respectively.

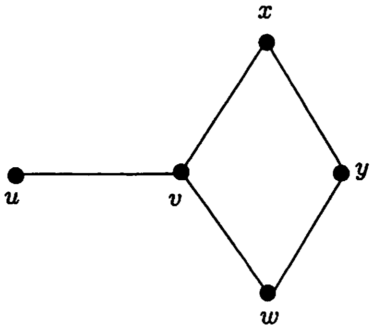

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Figure 1.1

It is proved in [8] that for any vertex x in G, the g_x-set is unique and $1 \leq g_x(G) \leq p - 1$ for every vertex x in G. We characterized graphs which realize the bounds. It is also proved that $g(G) \leq g_x(G) + 1$ for every vertex x in G and graphs with geodomination numbers 2 and p are characterized in terms of the vertex geodomination number. An elaborate study of results in vertex geodomination with several interesting applications is given in [8]. The following theorems will be used in the sequel.

Theorem 1.1. [5] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every u-w path.*

(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every u-w path.*

Theorem 1.2. [5] *Every nontrivial connected graph has at least two vertices which are not cut vertices.*

Theorem 1.3. [5] *Let G be a connected graph with at least three vertices. The following statements are equivalent:*

(i) *G is a block.*

(ii) *Every two vertices of G lie on a common cycle.*

Theorem 1.4. [2] *Let G be a connected graph of order $p \geq 3$. Then $g(G) = p - 1$ if and only if $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$.*

Theorem 1.5. [8] *Let G be a connected graph.*

(i) *Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to the g_x-set for any vertex x in G.*

(ii) *For any vertex x, eccentric vertices of x belong to the g_x-set.*

(iii) *No cut vertex of G belongs to any g_x-set.*

Theorem 1.6. [8] *Let G be a connected graph. For a vertex x in G, $g_x(G) = 1$ if and only if there is a unique eccentric vertex y of x such that every vertex of G is on an x-y geodesic.*

Theorem 1.7. [8] *For any graph $G, g_x(G) = p - 1$ if and only if deg $x = p - 1$.*

Theorem 1.8. [8] *A graph G is complete if and only if $g_x(G) = p - 1$ for every vertex x in G.*

Throughout the following G denotes a connected graph with at least two vertices.

2. Some Results on the Vertex Geodomination Number

In this section we characterize connected graphs G of order p having vertex geodomination number $g_x(G)$ equaling either $p-1$ or $p-2$ for every vertex x in G. Further, we discuss the realization of a graph with vertex geodomonation number n such that $1 \leq n \leq p - 1$.

Theorem 2.1. *Let G be a connected graph with number of cut vertices k. Then every vertex of G is either a cut vertex or a simplicial vertex if and only if $g_x(G) = p - k$ or $p - k - 1$ for any vertex x in G.*

Proof. Let G be a connected graph with every vertex of G is either a cut vertex or a simplicial vertex. Since x does not belong to the g_x-set of G, it follows from Theorem 1.5 that $g_x(G) = p - k$ or $p - k - 1$ according as x is a cut vertex or a simplicial vertex.

Conversely, suppose $g_x(G) = p - k$ or $p - k - 1$ for any vertex x in G. Suppose there is a vertex x in G which is neither a cut vertex nor a simplicial vertex. Since x is not a simplicial vertex, $N(x)$ does not induce a complete subgraph and hence there exist u and v in $N(x)$ such that $d(u, v) = 2$. Also, since x is not a cut vertex of $G, G - \{x\}$ is connected and hence there exists a u-v geodesic say $P : u, u_1, \ldots, u_n, v$ in $G - \{x\}$. Then $P \cup \{v, x, u\}$ is a shortest cycle, say C, containing both the vertices u and v with length at least 4 in G.

Case 1. Suppose either u or v is not a cut vertex of G. Assume that u is not a cut vertex of G. Clearly, x lies on a u-v geodesic and hence u and x do not belong to the g_u-set. Thus by Theorem 1.5, $g_u(G) \leq p - k - 2$, which is a contradiction to the assumption.

Case 2. Suppose both u and v are cut vertices of G. By Theorem 1.1, there exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for vertices $u_1 \in U$ and $w_1 \in W$, the vertex v is on every u_1-w_1 path. Without loss of generality, assume that $x \in U$. Let y be a vertex in W with maximum distance from v in W. By choice of y, y is not a cut vertex of G. Since the order of the cycle C is at least 4, the vertices x and y do not belong to the g_y-set and hence by Theorem 1.5, $g_y(G) \leq p - k - 2$, which is a contradiction to the assumption. Hence every vertex of G is either a cut vertex or a simplicial vertex. □

Corollary 2.2. *Let G be a connected block graph with number of cut vertices k. Then $g_x(G) = p - k$ or $p - k - 1$ for any vertex x in G.*

Proof. Let G be a connected block graph. Then every vertex of G is either a cut vertex or a simplicial vertex and hence by Theorem 2.1, $g_x(G) = p - k$ or $p - k - 1$ for any vertex x in G. □

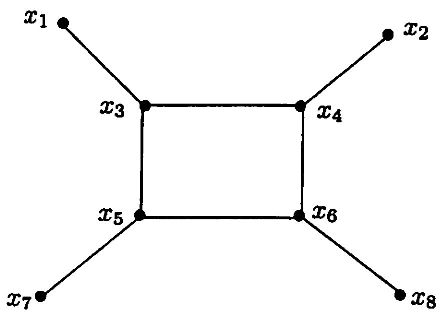

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Figure 2.1

Note 2.3. *The converse of Corollary 2.2 is not true. For the graph G given in Figure 2.1, $k = 4$ and $g_x(G) = p - k$ or $p - k - 1$ for any vertex x in G. However, it is not a connected block graph.*

Corollary 2.4. *Let T be a tree with number of pendent vertices t. Then $g_x(T) = t - 1$ or t according as x is a pendent or non-pendent vertex.*

Proof. This follows from Corollary 2.2. □

Theorem 2.5. *Let G be a connected graph. Then $G = K_1 + \cup m_j K_j$ if and only if $g_x(G) = p - 1$ or $p - 2$ for any vertex x in G.*

Proof. Let $G = K_1 + \cup m_j K_j$. Then G has at most one cut vertex. Suppose G has no cut vertex. Then $G = K_p$, hence by Theorem 1.8, $g_x(G) = p-1$ for every vertex x in G. Suppose G has exactly one cut vertex. Then all the remaining vertices are simplicial and hence by Theorem 2.1, $g_x(G) = p - 1$ or $p - 2$ for any vertex x in G.

Conversely, suppose $g_x(G) = p - 1$ or $p - 2$ for any vertex x in G. If $p = 2$, then $G = K_2 = K_1 + K_1$. If $p \geq 3$, then by Theorem 1.2, there exists a vertex x, which is not a cut vertex of G. If G has two or more cut vertices, then by Theorem 1.5, $g_x(G) \leq p - 3$, which is a contradiction. Thus, the number of cut vertices k of G is at most one.

Case 1. If $k = 0$, then the graph G is a block. If $p = 3$, then $G = K_3 = K_1 + K_2$. If $p \geq 4$, we claim that G is complete. Suppose G is not complete. Then there exist two vertices x and y in G such that $d(x, y) \geq 2$. By Theorem 1.3, x and y lie on a common cycle and hence x and y lie on a smallest cycle $C : x, x_1, \ldots, y, \ldots, x_n, x$ of length at least 4. Then x, x_1 and x_n do not belong to the g_x-set of G and hence $g_x(G) \leq p - 3$, which is a contradiction to the assumption. Hence G is the complete graph K_p and so $G = K_1 + K_{p-1}$.

Case 2. If $k = 1$, let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + \cup m_j K_1$, where $\sum m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. It is enough to prove that every block of G is complete. Suppose there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u, v) \geq 2$. Then by Theorem 1.3, both u and v lie on a common cycle and hence u and v lie on a smallest cycle of length at least 4. Then as in case 1 $g_u(G) \leq p - 3$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \cup m_j K_j$, where K_1 is the vertex x and $\sum m_j \geq 2$. □

Theorem 2.6. *Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$ if and only if $g_x(G) = p - 1$ or $p - 2$ for any vertex x in G.*

Proof. The proof is contained in Theorem 2.5. □

Theorem 2.7. *Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then the following are equivalent:*

(i) $g(G) = p - 1$.

(ii) $G = K_1 + \cup m_j K_j$, *where* $\sum m_j \geq 2$.

(iii) $g_x(G) = p - 1$ *or* $p - 2$ *for any vertex* x *in* G.

Proof. This follows from Theorem 1.4 and Theorem 2.6. □

Now, Theorems 1.8 and 2.5 lead to the natural question whether there exists a graph G for which $g_x(G) = p - 2$ for every vertex x in G.

Theorem 2.8. *There is no graph G of order p with $g_x(G) = p - 2$ for every vertex x in G.*

Proof. Suppose there exists a graph G with $g_x(G) = p - 2$ for every vertex x in G. Let x be any vertex of G. Let S_x be the g_x-set of G so that $g_x(G) = |S_x| = p - 2$. Since $x \notin S_x$ and $g_x(G) = p - 2$, there exists exactly one vertex y such that $y \notin S_x$. Hence y lies on the geodesic x, y, w for some $w \in S_x$.

Case 1. Suppose y is not a cut vertex of G. Then $G - \{y\}$ is connected and hence there is an x-w geodesic, say P, in $G - \{y\}$. Thus $C : P \cup (w, y, x)$ is a shortest cycle containing both the vertices x and w. Since y is the internal vertex of the x-w geodesic, the number of vertices in C is at least 4 and hence $g_x(G) \leq p - 3$, which is a contradiction to the assumption.

Case 2. Suppose y is a cut vertex of G. If $deg\ y = p - 1$, then by Theorem 1.7, $g_y(G) = p - 1$, which is a contradiction. If $deg\ y \leq p - 2$, then there exists a vertex u in G such that $d(u, y) \geq 2$. Since the vertex u, the internal vertices of u-y geodesic and the cut vertex y do not belong to the g_u-set of G, we have $g_u(G) \leq p - 3$, which is a contradiction to the assumption. Thus there is no graph G with $g_x(G) = p - 2$ for every vertex x in G. □

Note 2.9. *It follows from Theorem 2.8 that for each pair n, p of integers with $1 \leq n \leq p - 1$, there does not exist a graph G of order p with vertex geodomination number $g_x(G) = n$ for every vertex x in G.*

Theorem 2.10. *Let $p \geq 2$ be any integer.*

(i) *If p is even, then for any odd integer n with $1 \leq n \leq p - 1$, there exists a connected graph G with order p and $g_x(G) = n$ for any vertex x in G.*

(ii) *If p is odd, then for any even integer n with $1 \leq n \leq p-1$, there exists a connected graph G with order p and $g_x(G) = n$ for any vertex x in G.*

Proof. (i) Let p be even. For $p = 2, G = K_2$ has the desired properties. For $p = 4$, $G = C_4$ or K_4 has the desired properties according as $n = 1$ or $n = 3$. For $p \geq 6$, let $C : x_1, x_2, \ldots, x_p, x_1$ be an even cycle. Now we consider three cases.

Case 1. Let $n = 1$. Then $G = C$ has the desired properties.

Case 2. Let $3 \leq n \leq p-3$. Let $m = \frac{n+3}{2}$ and $k = p - \frac{n-1}{2}$. Then it is clear that $2 < m < k < p$. Let G be the graph obtained from C by joining every pair of vertices of $\{x_1, x_2, \ldots, x_m\}$ and also every pair of vertices of $\{x_k, x_{k+1}, \ldots, x_p, x_1\}$. The graph G is shown in Figure 2.2(i) for $p = 12$ and $n = 7$. Then $S = \{x_2, x_3, \ldots, x_{m-1}, x_{k+1}, x_{k+2}, \ldots, x_p\}$ is the set of all simplicial vertices of G with $|S| = n - 1$.

Subcase 2.1. Let $x \in S$. Without loss of generality we take $x = x_2$. Then x lies on the cycle $C' : x_1, x_2, x_m, x_{m+1}, \ldots, x_{k-1}, x_k, x_1$ of length $p - n + 2$, which is an odd integer greater than or equal to 5. Let y_1 and y_2 be the eccentric vertices of x in C'. It follows from the construction of G that y_1 and y_2 are also eccentric vertices of x in G. Hence every vertex on C' is x-geodominated by either y_1 or y_2. Further, all the remaining vertices of G which are not on C' are simplicial. Hence the g_x-set of G is $\{y_1, y_2\} \cup (S - \{x\})$ so that $g_x(G) = n$.

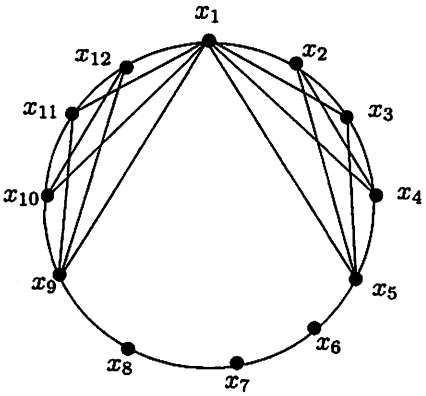

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Figure 2.2(i)

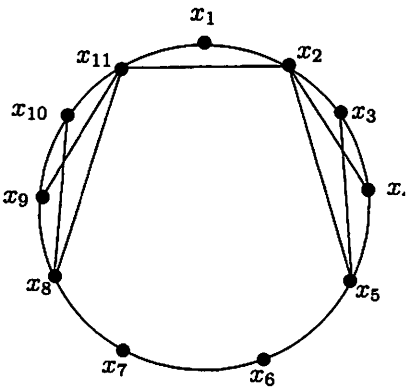

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Figure 2.2(ii)

Subcase 2.2. Let $x \in V-S$. Then x lies on the cycle $C'' : x_1, x_m, x_{m+1}, x_{m+2}, \ldots, x_{k-1}, x_k, x_1$ of length $p-n+1$, which is an even integer greater than or equal to 4. Let y be the eccentric vertex of x in C''. It follows from the construction of G that y is also an eccentric vertex of x in G. Then as in Subcase 2.1, it can be seen that the g_x-set of G is $\{y\} \cup S$. Thus $g_x(G) = n$.

Case 3. Let $n = p-1$. Then by Theorem 1.8, $G = K_p$ gives the desired result.

(ii) Let p be odd. For $p = 3$, $G = K_3$ has the desired properties. For $p = 5$, $G = C_5$ or K_5 has the desired properties according as $n = 2$ or $n = 4$. For $p \geq 7$, let $C : x_1, x_2, \ldots, x_p, x_1$ be an odd cycle. Now we consider three cases.

Case 1. Let $n = 2$. Then $G = C$ has the desired properties.

Case 2. Let $4 \leq n \leq p-3$. Let $m = \frac{n+4}{2}$ and $k = p - \frac{n}{2}$. Then it is clear that $3 < m < k < p$. Let G be the graph obtained from C by joining every pair of vertices of $\{x_2, x_3, \ldots, x_m\}$ and every pair of vertices of $\{x_k, x_{k+1}, \ldots, x_p\}$ and also adding the edge x_2x_p. The graph G is shown in Figure 2.2(ii) for $p = 11$ and $n = 6$. Then $S = \{x_1, x_3, x_4, \ldots, x_{m-1}, x_{k+1}, x_{k+2}, \ldots, x_{p-1}\}$ is the set of all simplicial vertices of G with $|S| = n-1$.

Subcase 2.1. Let $x \in S$. Let $x = x_1$. Then x lies on the cycle $C' : x_1, x_2, x_m, x_{m+1}, \ldots, x_{k-1}, x_k, x_p, x_1$ of length $p-n+2$, which is an odd integer greater than or equal to 5. Let y_1 and y_2 be the eccentric vertices of x in C'. Then as in Subcase 2.1 of (i), it can be seen that the g_x-set of G is $\{y_1, y_2\} \cup (S - \{x\})$ so that $g_x(G) = n$.

Let $x = x_i (3 \leq i \leq m-1)$. Then x lies on the cycle $C'' : x, x_m, x_{m+1}, \ldots, x_{k-1}, x_k, x_p, x_2, x$ of length $p-n+2$, which is an odd integer greater than or equal to 5. Let y_i and z_i be the eccentric vertices of x in C''. Then as in the first part of this subcase the g_x-set of G is $\{y_i, z_i\} \cup (S - \{x\})$ so that $g_x(G) = n$. Similarly for $x = x_i (k+1 \leq i \leq p-1)$ it can be proved that $g_x(G) = n$. Thus for all x in S, $g_x(G) = n$.

Subcase 2.2. Let $x \in V-S$. Then x lies on the cycle $C''' : x_2, x_m, x_{m+1}, x_{m+2}, \ldots, x_{k-1}, x_k, x_p, x_2$ of length $p-n+1$, which is an even integer greater than or equal to 4. Let y be the eccentric vertex of x in C'''. Then as in Subcase 2.2 of (i), it can be seen that the g_x-set of G is $\{y\} \cup S$. Thus $g_x(G) = n$.

Case 3. Let $n = p-1$. Then by Theorem 1.8, $G = K_p$ gives the desired result. □

It is proved in Theorem 2.8 that there is no graph G of order p with $g_x(G) = p-2$ for every vertex x in G. This result and Theorem 2.10 suggest the following problem.

Problem 2.11. *Let $p \geq 5$ be any integer.*

(i) *If p is even and n is even with $1 \leq n \leq p-1$ and $n \neq p-2$, does there exist a connected graph G of order p and $g_x(G) = n$ for any vertex x in G?*

(ii) *If p is odd and n is odd with $1 \leq n \leq p-1$ and $n \neq p-2$, does there exist a connected graph G of order p and $g_x(G) = n$ for any vertex x in G?*

Now we proceed to construct regular as well as non-regular graphs G for which $g_x(G) = n$ for any vertex x in G, when an integer $n \geq 2$ is given.

Theorem 2.12. *For any integer $n \geq 2$ there exist connected regular as well as non-regular graphs G for which $g_x(G) = n$ for every vertex x in G.*

Proof. The complete graph K_{n+1} is regular and by Theorem 1.8, $g_x(K_{n+1}) = n$ for every vertex x in K_{n+1}.

Now to construct a non-regular graph we proceed as follows: Let $p = n + 3$. Then $n = p - 3$. If n is odd, then the graph constructed in case 2 of the proof of Theorem 2.10(i) satisfies the required properties. Similarly, if n is even, then the graph constructed in case 2 of the proof of Theorem 2.10(ii) satisfies the required properties. In both cases, it is clear from the construction that G is non-regular. The graphs in both cases are given in Figure 2.3(i) and Figure 2.3(ii) when $n = 7$ and $n = 8$ respectively. □

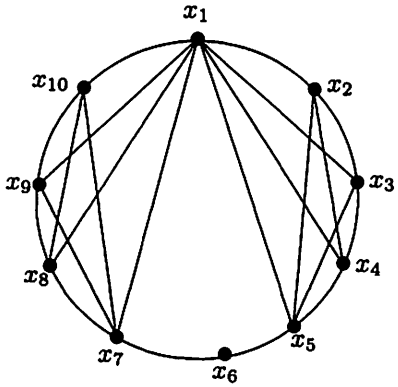

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Figure 2.3(i)

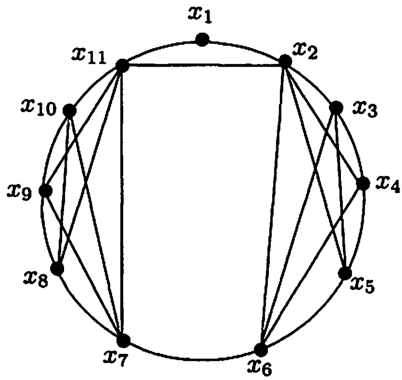

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Figure 2.3(ii)

3. Bounds for the Vertex Geodomination Number of a graph

We have seen that if G is a connected graph of order $p \geq 2$, then $1 \leq g_x(G) \leq p - 1$ for any vertex x in G. Also we have for a vertex x in

G, $g_x(G) = 1$ if and only if there is a unique eccentric vertex y of x such that every vertex of G is on an x-y geodesic and a graph G is complete if and only if $g_x(G) = p - 1$ for every vertex x in G. Now it is proved in Theorem 2.5 that for a connected graph G, $G = K_1 + \cup m_j K_j$ if and only if $g_x(G) = p - 1$ or $p - 2$ for any vertex x in G. Also it is proved in Theorem 2.8 that there is no graph G of order p with $g_x(G) = p - 2$ for every vertex x in G. In the following theorem we give an improved upper bound for the vertex geodomination number of a graph in terms of its order and diameter.

Theorem 3.1. *If G is a connected graph of order p and diameter d, then $g_x(G) \leq p - d + 1$ for any vertex x in G.*

Proof. If $G = K_p$, then $g_x(G) = p - 1 = p - d$ for every vertex x in G. So, let $G \neq K_p$. Let u and v be two vertices of G such that $d(u, v) = d$ and let $u = v_0, v_1, \ldots, v_d = v$ be a u-v geodesic of length d. Now let $S = V(G) - \{v_1, v_2, \ldots, v_{d-1}\}$. If $x = v_i (1 \leq i \leq d-1)$, then clearly S is an x-geodominating set of G so that $g_x(G) \leq |S| = p-d+1$. If $x = v_i (i = 0, d)$, then $S - \{x\}$ is an x-geodominating set of G so that $g_x(G) \leq |S| - 1 = p - d$.

Let $x \neq v_i (0 \leq i \leq d)$. Let P and Q be x-v_0 and x-v_d geodesics respectively. Let y be the last vertex common to both P and Q. Let P_1 be the y-v_0 geodesic on P and let Q_1 be the y-v_d geodesic on Q. Let $T = (V(G) - [V(P_1) \cup V(Q_1)]) \cup \{v_0, v_d\}$. Then it is clear that T is an x-geodominating set of G and so

$$\begin{aligned} g_x(G) &\leq p - [d(y, v_0) + d(y, v_d) + 1] + 2 \\ &\leq p - [d(v_0, v_d) + 1] + 2, \text{ by triangle inequality} \\ &= p - d + 1 \end{aligned}$$

Thus $g_x(G) \leq p - d + 1$ for any vertex x in G. □

Theorem 3.2. *There exists no graph G for which $g_x(G) = p - d + 1$ for every vertex x in G.*

Proof. The proof is contained in Theorem 3.1. □

Theorem 3.3. *If G is a connected graph of order p and diameter d, then $g_x(G) \leq p - d$ for at least two vertices of G.*

Proof. The proof is contained in Theorem 3.1. □

Remark 3.4. For the complete graph $G = K_p$, $g_x(G) = p - d$ for every vertex x in G. Further, Theorem 2.8 shows that there is no graph of order p and diameter $d = 2$ for which $g_x(G) = p - d$ for every vertex x in G. This suggests the following problem.

Problem 3.5. *Characterize graphs G of order p with diameter $d \geq 3$ for which $g_x(G) = p - d$ for every vertex x in G.*

Theorem 3.6. *For every non-trivial tree T, $g_x(T) = p - d$ or $p - d + 1$ for any vertex x in T if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, \ldots, v_d = v$ be a diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{d-1}$. Then $d - 1 + l + k = p$. By Corollary 2.4, $g_x(T) = k$ or $k - 1$ for any vertex x in T and so $g_x(T) = p - d - l + 1$ or $p - d - l$ for any vertex x in T. Hence $g_x(T) = p - d + 1$ or $p - d$ for any vertex x in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar. □

For every connected graph G, $rad\ G \leq\ diam\ G \leq 2\ rad\ G$. Ostrand [7] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the vertex geodomination number can be prescribed.

Theorem 3.7. *For positive integers r, d and $n \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $g_x(G) = n$ or $n - 1$ for any vertex x in G.*

Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_{n+1}$. Then by Theorem 1.8, $g_x(G) = n$ for any vertex x in G. If $d = 2$, let $G = K_{1,n}$. Then by Corollary 2.4, $g_x(G) = n$ or $n - 1$ for any vertex x in G. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:

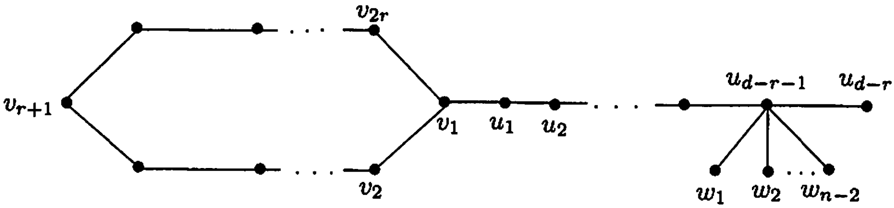
Case 1. Suppose $r = d$. For $n = 2$, let $G = C_{2r+1}$. Then $r = d$ and $g_x(G) = 2$ for any vertex x in G. Now, let $n \geq 3$. We split into two subcases.

Subcase 1. Suppose n is odd. Let $p = n + 2r - 1$. Then p is even and $3 \leq n \leq p - 3$. Hence by Theorem 2.10(i), there exists a connected graph G of order p and $g_x(G) = n$ for any vertex x in G. The required graph is the one constructed in the proof of Theorem 2.10(i) (Case 2). For this graph it is easily verified that the eccentricity of each vertex of G is r so that $rad\ G =\ diam\ G = r$.

Subcase 2. Suppose n is even. Let $p = n + 2r - 1$. Then p is odd and $4 \leq n \leq p - 3$. Hence by Theorem 2.10(ii), there exists a connected graph G of order p and $g_x(G) = n$ for any vertex x in G. The required graph is the one constructed in the proof of Theorem 2.10(ii) (Case 2). For this graph it is easily verified that the eccentricity of each vertex of G is r so that $rad\ G =\ diam\ G = r$.

Case 2. Suppose $r < d \leq 2r$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let

H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. If $n = 2$, then let $G = H$. Clearly, $g_x(G) = 1$ or 2 according as $x \in \{v_{r+1}, u_{d-r}\}$ or $x \in \{u_1, u_2, \ldots, u_{d-r-1}, v_1, v_2, \ldots, v_r, v_{r+2}, \ldots, v_{2r}\}$. Thus $g_x(G) = 1$ or 2 for any vertex x in G. If $n \geq 3$, then we add $n-2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H by joining each vertex $w_i (1 \leq i \leq n-2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 3.1.



G

Figure 3.1

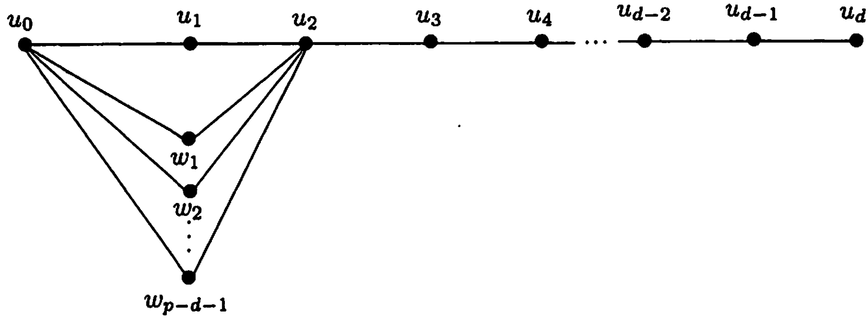
Now *rad* $G = r$, *diam* $G = d$ and G has $n-1$ end vertices. Clearly, $g_x(G) = n$ or $n-1$ according as $x \in \{u_1, u_2, \ldots, u_{d-r-1}, v_1, v_2, \ldots, v_r, v_{r+2}, \ldots, v_{2r}\}$ or $x \in \{v_{r+1}, u_{d-r}, w_1, w_2, \ldots, w_{n-2}\}$. Thus $g_x(G) = n$ or $n-1$ for any vertex x in G. □

In the following, we construct a graph of prescribed order, diameter and vertex geodomination number under suitable conditions.

Theorem 3.8. *If p, d and n are integers such that $3 \leq d \leq p-1$, $1 \leq n \leq p-1$, and $p-d-n+1 \geq 0$, then there exists a graph G of order p, diameter d and $g_x(G) = n$ for some vertex x in G.*

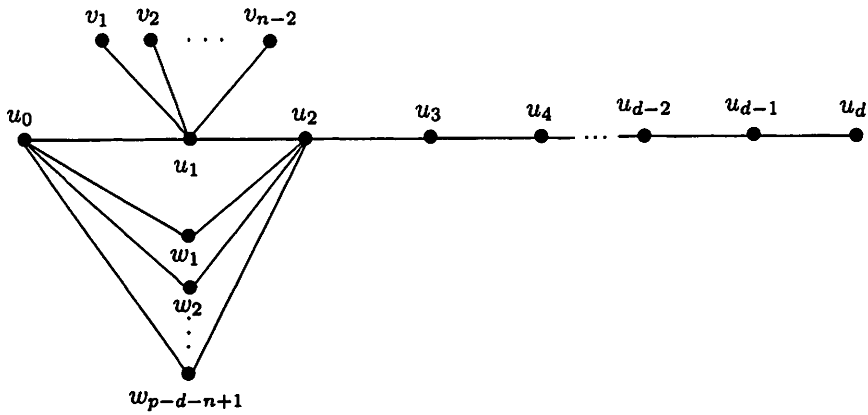
Proof. If $n = 1$ or 2, let $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-1}$ to P_{d+1} and join these to both u_0 and u_2, there by producing the graph G of Figure 3.2(i). Then G has order p and diameter d. For the vertex $x = u_0$, clearly $\{u_d\}$ is the g_x-set of G so that $g_x(G) = 1$. For the vertex $x = u_2$, clearly $\{u_0, u_d\}$ is the g_x-set of G so that $g_x(G) = 2$.

If $3 \leq n \leq p-1$, then add $n-2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ to P_{d+1} and join these to u_1, also add $p-d-n+1$ new vertices $w_1, w_2, \ldots, w_{p-d-n+1}$ to P_{d+1} and join these to both u_0 and u_2, there by producing the graph G of Figure 3.2(ii). Then G has order p and diameter d. Let $S_x = \{v_1, v_2, \ldots, v_{n-2}, u_d\}$ be the set of all simplicial vertices of G. Let $x = u_2$. By Theorem 1.5(i), $g_x(G) \geq |S_x| = n-1$. Clearly u_0 is not x-geodominated by any vertex in S_x and so $g_x(G) > n-1$. Let $T = S_x \cup \{u_0\}$. Then T is an x-geodominating set of G so that $g_x(G) = n$. Thus G has the desired properties. □



G

Figure 3.2(i)



G

Figure 3.2(ii)

Problem 3.9. *Under the conditions given in Theorem 3.8, does there exist a graph G of order p and diameter d such that $g_x(G) = n$ for every vertex x in G ?*

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