The Properties of Middle Digraphs*

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Abstract: Let D be a strongly connected digraph with order at least two, $M(D)$ denote the middle digraph of D, $\kappa(D)$ and $\lambda(D)$ denote the connectivity and arc-connectivity of D, respectively. In this paper we study super-arc-connected and super-connected middle digraphs and the spectral of middle digraphs.

Key words: Middle digraph; Super-arc-connected; Super-connected; Spectral

1 Introduction

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [2]. Let $D = (V(D), A(D))$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, we denote the indegree, the outdegree of v, the minimum indegrees and outdegrees in D by $d_D^-(v)$, $d_D^+(v)$, $\delta^-(D)$ and $\delta^+(D)$, respectively. We denote the minimum degree of D by $\delta(D) = min\{\delta^-(D), \delta^+(D)\}$. $\overrightarrow{K_n}$ denotes the complete digraph of order n.

Let $D = (V(D), A(D))$ be a digraph, $|V(D)| = n$, $|A(D)| = m$, $V(D) = \{v_1, v_2, \cdots, v_n\}$. In 1960, Harary and Norman [4] introduced the concept of the line digraph. For a digraph D, the *line digraph* of D, denoted by $L(D)$, is the digraph with vertex set $V(L(D)) = \{a_{ij} | a_{ij} = (v_i, v_j)$ is an

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arc in $D\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D)$ if and only if $v_j = v_s$ in D. In 1977, Zamfirescu [8] introduced the concept of the middle digraph. For a digraph D, the *middle digraph* of D, denoted by $M(D)$, is the digraph with vertex set $V(M(D)) = V(D) \cup A(D)$, there is an arc $(a, b) \in A(M(D))$ from vertex a to vertex b in $M(D)$ if and only if one of the following cases holds: (1). If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D. (2). If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D. (3). If $a \in A(D)$ and $b \in A(D)$, then the head of arc a is the tail of arc b in D. The middle digraph has been discussed in [7, 8, 9].

In fact, the middle digraph $M(D)$ can be viewed as $V(M(D)) = V(D) \cup V(L(D))$ and $A(M(D)) = A(L(D)) \cup A(D, L(D))$, where $A(D, L(D))$ denotes the arcs with one end in $V(D)$ and the other end in $V(L(D))$. For each vertex $v \in V(D)$, $d^+_{M(D)}(v) = d^+_D(v)$, since there are $d^+_D(v)$ out-arcs from v to vertices in $V(L(D))$. For each vertex $a_{ij} \in V(L(D))$, $d^+_{M(D)}(a_{ij}) = d^+_D(v_j) + 1$, since $d^+_{L(D)}(a_{ij}) = d^+_D(v_j)$ and there are exactly one out-arc (a_{ij}, v_j) from a_{ij} to vertices in $V(D)$. Similarly, $d^-_{M(D)}(v) = d^-_D(v)$, $d^-_{M(D)}(a_{ij}) = d^-_D(v_i) + 1$.

An *arc-cut* of a strongly connected digraph D with order at least two is a set of arcs whose remove makes D no longer strongly connected. The *arc-connectivity* $\lambda(D)$ is the minimum cardinality of an arc-cut over all arc-cuts of D. The inequality $\lambda(D) \leq \delta(D)$ is wellknown. We call a digraph D *maximally arc-connected*, for short *max-λ*, if $\lambda(D) = \delta(D)$. For a vertex $v \in V(D)$, denote by $N^+_D(v)$ the set of out-neighbors of v, $N^-_D(v)$ the set of in-neighbors of v, $E^+_D(v)$ the set of out-arcs of v, $E^-_D(v)$ the set of in-arcs of v. A strongly connected digraph D is *super-arc-connected*, for short *super-λ*, if every minimum arc-cut is either $E^+_D(v)$ or $E^-_D(v)$ for some vertex v. The *connectivity* $\kappa(D)$, *max-κ*, *super-κ* are defined similarly.

In this paper, we study super-arc-connected and super-connected middle digraphs and the spectra of middle digraphs.

The following two lemmas will be used in our discussions.

Lemma 1.1. *[10] Let D be a digraph with order at least two. Then D is strongly connected if and only if the line digraph $L(D)$ is strongly connected.*

Lemma 1.2. *Let D be a stongly connected digraph, then $\kappa(D) \leq \lambda(D) = \kappa(L(D)) \leq \lambda(L(D))$.*

2 The spectra of middle digraphs

Let $D = (V(D), A(D))$ be a pseudodigraph, i.e. loops and multiple arcs may occur in $A(D)$. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $A(D) = \{e_1, e_2, \ldots, e_m\}$. Its out-incidence and in-incidence matrices (both are $(0,1)$-matrices), denoted by X_o and X_I respectively, are defined as follows.

$$X_o = (x_{ij}^o), \quad X_I = (x_{ij}^I),$$

where

$$x_{ij}^o = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } e_j; \\ 0 & \text{otherwise;} \end{cases}$$

$$x_{ij}^I = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j; \\ 0 & \text{otherwise.} \end{cases}$$

The following lemmas are needed for the proof of Theorem 2.4.

Lemma 2.1. *[5, 6, 11] Let M, M_L be the adjacency matrices of digraph D and its line digraph $L(D)$ respectively. Let X_o and X_I be the out- and in-incidence matrices of D, then*

$$M = X_o X_I^T, \quad M_L = X_I^T X_o.$$

Lemma 2.2. *[3]*

$$det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{cases} detA_{11} det(A_{22} - A_{21}A_{11}^{-1}A_{12}) & \text{if } A_{11} \text{ is invertible;} \\ detA_{22} det(A_{11} - A_{12}A_{22}^{-1}A_{21}) & \text{if } A_{22} \text{ is invertible .} \end{cases}$$

Lemma 2.3. *Let $\lambda_i (i = 1, 2, \ldots, n)$ be the eigenvalues of matrix A, let $f(x)$ be a polynomial. Then the eigenvalues of $f(A)$ are $f(\lambda_i)(i = 1, 2, \ldots, n)$.*

Theorem 2.4. *Let D be a pseudodigraph with n vertices and m arcs, let $M(D)$ be the middle digraph of D. Then*

$$P_{M(D)}(\lambda) = \lambda^{m-n} \prod_{i=1}^{n} (\lambda^2 - (\lambda + 1)\lambda_i),$$

where $\lambda_i (1 \le i \le n)$ are the eigenvalues of D.

Proof. By a suitable labeling the vertices of $M(D)$ we can see that the adjacency matrix of $M(D)$ is

$$M_{M(D)} = \begin{pmatrix} 0 & X_o \\ X_I^T & M_L \end{pmatrix} = \begin{pmatrix} 0 & X_o \\ X_I^T & X_I^T X_o \end{pmatrix}.$$

Therefore

$$\begin{aligned} P_{M(D)}(\lambda) &= |\lambda I_{m+n} - M_{M(D)}| \\ &= \begin{vmatrix} \lambda I_n & -X_o \\ -X_I^T & \lambda I_m - X_I^T X_o \end{vmatrix}. \end{aligned}$$

By adding the product of $-X_I^T$ and the first row of the block matrix to the second row, we have

$$P_{M(D)}(\lambda) = \begin{vmatrix} \lambda I_n & -X_o \\ -(\lambda+1)X_I^T & \lambda I_m \end{vmatrix}.$$

By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} P_{M(D)}(\lambda) &= |\lambda I_m||\lambda I_n - \frac{\lambda+1}{\lambda} X_o X_I^T| \\ &= \lambda^m |\lambda I_n - \frac{\lambda+1}{\lambda} M| \\ &= \lambda^{m-n} |\lambda^2 I_n - (\lambda+1)M| \\ &= \lambda^{m-n} \prod_{i=1}^{n} (\lambda^2 - (\lambda+1)\lambda_i). \end{aligned}$$

□

Corollary 2.5. *Let D be a pseudodigraph with n vertices and m arcs ($m \geq n$). Then $M(D)$ has $m - n$ zero eigenvalues, and the following $2n$ eigenvalues:*

$$\frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4\lambda_i}}{2} \qquad (i = 1, 2, \ldots, n)$$

where $\lambda_i (1 \leq i \leq n)$ are the eigenvalues of D.

3 Super-arc-connected middle digraphs

In this section, we consider strict digraph D (digraph having no loops and no parallel arcs are allowed). If $|V(D)| = 1$, then $M(D)$ is an isolated vertex. So we consider the case that $|V(D)| \geq 2$.

Theorem 3.1. *Let D be a digraph with order at least two. Then $M(D)$ is strongly connected if and only if D is strongly connected.*

Proof. If D is strongly connected, then $L(D)$ is strongly connected by Lemma 1.1. According to the definition of middle digraph, we know that the vertices of $V(D)$ have both in-arcs and out-arcs in $M(D)$. Hence, $M(D)$ is strongly connected. If D is not strongly connected, then there exist two subsets X_1 and X_2 in $V(D)$ such that $X_1 \cup X_2 = V(D)$ and there are no arcs from X_1 to X_2 in D. It is easy to see that, in $M(D)$, there are no directed paths from vertices in X_1 to vertices in X_2. □

Theorem 3.2. *Let D be a strongly connected digraph with order at least two. Then $\lambda(M(D)) \geq \min\{\delta(D), 2\lambda(D)\}$.*

Proof. Clearly, $\delta(M(D)) = \delta(L(D)) = \delta(D) \geq 1$. Let S be a minimum arc-cut of $M(D)$, then there exists a nonempty proper vertex subset $X \subseteq V(M(D))$ such that there is no arc from X to $\overline{X}$ in $M(D) - S$, where $\overline{X} = V(M(D))\backslash X$.

We consider three cases.

Case 1. $X \subseteq V(D)$.

If $|X| = 1$, then $|S| \geq \delta(M(D))$. If $n \geq |X| \geq 2$, since $D - S$ is no longer strongly connected, and every vertex $v \in X$ has $d_D^+(v) \geq \delta(D)$ out-neighbors in $V(L(D))$, we have

$$|S| \geq |X| \cdot \delta(D) > \delta(D).$$

Case 2. $X \subseteq V(L(D))$.

If $|X| = 1$, then $|S| \geq \delta(M(D))$. If $\delta(D) \geq |X| \geq 2$, since every vertex $a \in X$ has at least $\delta(L(D)) - (|X| - 1) = \delta(D) - |X| + 1$ out-neighbor in $V(L(D))\backslash X$ and exactly one out-neighbors in V(D), we have

$$|S| \geq |X|(\delta(L(D)) - |X| + 1) + |X| > \delta(D).$$

If $m \geq |X| > \delta(D)$, then

$$|S| \geq |X| > \delta(D).$$

Case 3. $X \cap V(L(D)) \neq \emptyset$ and $X \cap V(D) \neq \emptyset$.

We may suppose that $V(D) \nsubseteq X$ and $V(L(D)) \nsubseteq X$. In fact, in the case that $V(D) \subseteq X$ or $V(L(D)) \subseteq X$, by considering $\overline{X}$, the proof is analogous to the proof of Case 1 or Case 2. For each arc (v_i, v_j) from $X \cap V(D)$ to $\overline{X} \cap V(D)$ in D, if the corresponding vertex $a_{ij} \in \overline{X} \cap V(L(D))$, then (v_i, a_{ij}) is an arc from X to $\overline{X}$ in $M(D)$; if the corresponding vertex $a_{ij} \in X \cap V(L(D))$, then (a_{ij}, v_j) is an arc from X to $\overline{X}$ in $M(D)$. Hence,

$$|S| \geq \lambda(D) + \lambda(L(D)) \geq 2\lambda(D).$$

We thus conclude that $\lambda(M(D)) \geq \min\{\delta(D), 2\lambda(D)\}$. □

Remark 3.3. *If the digraph D has the following properties, then $\lambda(M(D)) = 2\lambda(D)$.*

(i). $\lambda(D) = \lambda(L(D))$.

(ii). Let S be a minimum arc-cut of D and there exist two vertex set X and $\bar{X}$ such that there are no arcs from X to $\bar{X}$ in $D - S$, where $\bar{X} = V(D) - X$.

(iii). The arc set from $A(D[X]) \cup [\bar{X}, X] \cup S$ to $A(D[\bar{X}])$ is a minimum arc-cut of $L(D)$, where $[\bar{X}, X]$, $D[X]$ and $D[\bar{X}]$ denote the arc set from $\bar{X}$ to X, the vertex-induced subdigraph of X and $\bar{X}$, respectively.

Corollary 3.4. *Let D be a strongly connected digraph with order at least two. If $2\lambda(D) \geq \delta(D)$, then $M(D)$ is max-λ.*

Theorem 3.5. *Let D be a strongly connected digraph with order at least two. If $2\lambda(D) > \delta(D)$, then $M(D)$ is super-λ.*

Proof. From the proof of Theorem 3.2, only when $|X| = 1$, or in Case 3, the equality $\lambda(M(D)) = \delta(M(D))$ may hold. If $|X| = 1$, then we are done. In Case 3, the equality cannot hold when $2\lambda(D) > \delta(D)$. Hence if $2\lambda(D) > \delta(D)$, then $M(D)$ is super-λ. □

4 Super-connected middle digraphs

Theorem 4.1. *Let D be a strongly connected digraph with order at least two. Then $\kappa(M(D)) = \lambda(D)$.*

Proof. By the definition of middle digraph, in order to destroy the connectivity of $M(D)$, we must destroy all the out- (in-) arc set of some vertex in $V(D)$, or destroy the connectivity of $L(D)$ firstly. Thus, $\kappa(M(D)) \geq \kappa(L(D)) = \lambda(D)$. On the other hand, if S is a vertex-cut of $L(D)$, then S is a vertex-cut of $M(D)$. In fact, the corresponding arc set S' of S is an arc-cut in D. Therefore there exists a nonempty proper vertex subset $X \subseteq V(L(D))$ such that there is no arc from X to $\overline{X}$ in $L(D) - S$, and there exists a nonempty proper vertex subset $X' \subseteq V(D)$ such that there is no arc from X' to $\overline{X}'$ in $D - S'$, and where $\overline{X} = V(L(D))\backslash(X \cup S)$, $\overline{X}' = V(D)\backslash X'$. It is easy to see that there is no arcs from $X \cup X'$ to $\overline{X} \cup \overline{X}'$ in $M(D) - S$. Hence, $\kappa(M(D)) = \lambda(D)$. □

By Theorem 4.1, we have the following consequences.

Corollary 4.2. *Let D be a strongly connected digraph with order at least two. Then $M(D)$ is max-κ if and only if D is max-λ.*

Corollary 4.3. *Let D be a strongly connected digraph with order at least two. Then $M(D)$ is super-κ if and only if D is super-λ.*

Proof. Clearly, if $M(D)$ is super-κ, then D is super-λ. On the other hand, if D is super-λ, then $L(D)$ is super-κ, For any vertex-cut $S = \{a_{ij_1}, a_{ij_2}, \ldots, a_{ij_d}\} = N^+(a_{ti})$ of $L(D)$, we find that $S = \{a_{ij_1}, a_{ij_2}, \ldots, a_{ij_d}\} = N^+(v_i)$ in $M(D)$. Thus $M(D)$ is super-κ. □

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