Sums of Products of Binomial Coefficients

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Abstract

We given a two parameter generalization of identities of Carlitz and Gould involving products of binomial coefficients. The generalization involves Jacobi polynomials.

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1 Introduction

Let $[x^n](f(x))$ denote the coefficient of x^n in $f(x)$. An old problem of Dixon from 1891 [3] is to study the sums

$$S_n(p,x) := \sum_{k=0}^{n} \binom{n}{k}^p x^k, \quad n = 1, 2, \dots . \tag{1.1}$$

The following special values of $S_n(p,x)$ are known

$$S_n(2,1) = \binom{2n}{n}, \quad S_{2n}(2,-1) = (-1)^n \binom{2n}{n}, \quad S_{2n}(3,-1) = (-1)^n \binom{2n}{n}\binom{3n}{n}.$$

Carlitz [2] showed that

$$\sum_{k=0}^{n} \binom{n}{k}^3 = [x^n](1-x^2)^n P_n\left(\frac{1+x}{1-x}\right), \tag{1.2}$$

$$\sum_{k=0}^{n} \binom{n}{k}^4 = [x^n](1-x)^{2n} \left[P_n\left(\frac{1+x}{1-x}\right)\right]^2, \tag{1.3}$$

where P_n is a Legendre polynomial of degree n. Gould [6] proved the following analogous identities

$$(1.4)\quad \sum_{k=0}^{n}(-1)^k \binom{n}{k}^3 = [x^n](1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right),$$

$$(1.5)\qquad \sum_{k=0}^{n} \binom{n}{k}^5 = [x^n](1-x)^n P_n\left(\frac{1+x}{1-x}\right) \sum_{k=0}^{n} \binom{n}{k}^3 x^k.$$

The purpose of this note is to extend these results to the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$,

$$(1.6)\; P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} \left(\frac{x-1}{2}\right)^k,$$

[12, (4.21.2)] with the shifted factorial $(A)_n$ defined as

$$(1.7)\qquad (A)_0 = 1, \quad (A)_n = \prod_{j=1}^{n}(A+j-1) = \frac{\Gamma(A+n)}{\Gamma(A)}.$$

The Legendre polynomials are the special case

$$(1.8)\qquad P_n(x) = P_n^{(0,0)}(x).$$

In terms of gamma functions the binomial coefficient is

$$(1.9)\qquad \binom{A}{k} = \frac{\Gamma(A+1)}{k!\,\Gamma(A-k+1)}$$

Our main results are Theorems 1 and 2 below as well as (1.14). The proofs will be given in §2. Section 2 also contains several remarks about identities.

Theorem 1.1. *We have*

$$[x^n]\,(1-x)^n(1+tx)^\lambda\, P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right)$$

$$(1.10)\qquad = \frac{(\beta+1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k}\binom{\lambda}{k}\binom{n+\alpha}{k} \frac{k!\,t^k}{(\beta+1)_k}$$

$$= \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{\lambda}{k}\binom{n+\beta}{n-k} t^k.$$

The equality of the second and third lines follow from (1.9).

It is clear that the special case $\alpha = \beta = 0, \lambda = n, t = 1$ of Theorem 1 is (1.2) while $\alpha = \beta = 0, \lambda = n, t = -1$ is (1.4). It is clear that the case $\beta = 0$ is

$$\begin{aligned}[x^n]\,(1-x)^n(1+tx)^\lambda\, P_n^{(\alpha,0)}\left(\frac{1+x}{1-x}\right) \\ = \sum_{k=0}^{n} \binom{n}{k}\binom{\lambda}{k}\binom{n+\alpha}{k} t^k,\end{aligned} \tag{1.11}$$

which is a two parameter generalization of (1.2).

Theorem 1.2. *The following identity holds*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\binom{n+\beta}{n-k} \\ = [x^n]\ (1-x)^{2n}\left[P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right)\right]^2,\end{aligned} \tag{1.12}$$

Here again Theorem 2 is a two parameter extension of (1.3).

It is very curious to see that (1.12) is a convolution of the summands in (1.10) with $\lambda = 0$.

As we shall see in §2 the proofs are based on

$$(1-x)^n P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k} x^k. \tag{1.13}$$

This shows how one can generalize (1.5). Indeed

$$\begin{aligned}[x^n]\left((1-x)^n P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right)\sum_{j=0}^{n} a_j x^j\right) \\ = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k} a_{n-k}.\end{aligned} \tag{1.14}$$

By specializing the coefficients a_k one can generate many binomial coefficient identities each of which resembles (1.5).

2 Proofs and Remarks

The proofs use the following representation of Jacobi polynomials

$$(2.1)\ P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}.$$

which is (4.3.2) in [12].

Proof of Theorem 1.1. Apply (2.1) to see that the left-hand side of (1.10) equals

$$[x^n] \left(\sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} x^k (1+tx)^\lambda \right).$$

Expand $(1+tx)^\lambda$ by the binomial theorem and evaluate the coefficient of x^n. This establishes Theorem 1. □

Theorem 1.2 follows similarly from (2.1).

Remarks:

1. Simons [10] proved

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1+x)^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} x^k,$$

which he called acurious identity. Later Gould [7] pointed out that this is merely a restatement of the well known property $P_n(-x) = (-1)^n P_n(x)$ of Legendre polynomials with x replcaed by $1+2x$. This latter identity is a special case of the more general identity

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

which when rewritten using (1.6) with x replcaed by $1+2x$ becomes a binomial coefficient indentity.

2. In 1891 Dixon evaluated the sum $S_n(3,-1)$ in [3]. This, however, is a very special case of Dixon's later and more general theorem [4]

$$(2.2)\quad \begin{aligned} &{}_3F_2(a,b,c;1+a-b,1+a-c;1) \\ &= \frac{\Gamma(1+a/2)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma 1-b-c+a/2)}{\Gamma(1+a)\Gamma(1-b+a/2)\Gamma(1-c+a/2)\Gamma(1+a-b-c)}, \end{aligned}$$

[11, (III.8)]. Indeed $S_n(3,-1)$ is the special case $a = b = c = -n$. A more general terminating case is

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(a)_n\,(b)_n\,(-1)^k}{(1+a-b)_n(1+a+n)_k} = \frac{(1+a)_n(1-n+a/2)_n}{(1+a/2)_n(1+a-b)_n},$$

[11, (2.3.3.6)].

3. One may wonder whether Theorems 1.1-1.2 can be extended to a more general class of orthogonal polynomials. The Askey Scheme [9] has the Wilson and Racah polynomials at the top and the other orthogonal polynomials are special or limiting cases of them. In spite of the fact that the Jacobi polynomials are a very special limiting case of the Wilson polynomials, the Jacobi polynomials are neverthless the most general polynomials for which our method of proof works. The reason is that the polynomials which are at any level in the Askey Scheme higher than the Jacobi polynomials have the polynomial variable as a parameter(s) in the hypergeometric function representation. This complicates the evaluation of the respective polynomial at $(1+x)/(1-x)$.

4. In terms of q-analogues, it is not clear how to use the composition of $x \to (x+1)/(1-x)$ with q-function. There are q-analogues of the Legendre and Jacobi polynomials [1], [8], [5] but it is not clear how to find q-analogues of our results.

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