SUPERGRAPHS AND GRAPHICAL COMPLEXITY OF PERMUTATION GROUPS

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ABSTRACT. This paper introduces the concepts of a *supergraph* and *graphical complexity* of a permutation group, intended as a tool for investigating the structure of concrete permutation groups. Basic results are established and some research problems suggested.

1. INTRODUCTION

We speak of concrete permutation groups in contrast with abstract groups they represent. Instead of abstract isomorphism we apply permutation isomorphism in our study: two permutation groups are considered as different unless they can be obtained from each other by renaming elements of the base sets. Obviously, the permutation groups viewed in this way give more insight into the the symmetry of an object than abstract groups, Figure 1 being here a striking example. Yet, the problems involving concrete permutation groups are usually much harder then their abstract group counterparts.

It is well-known that while every abstract group is isomorphic to the automorphism group of a graph, not every permutation group can be represented as equal to the automorphism group of a concrete graph. This paper is the result of our attempt to find a simple graphical representation for concrete permutation groups, which could form a base for an exhaustive classification. Attempts to do such classifications via k-ary relations go back to Wielandt [14], but the concept generally turned out to be hard (see also [6, 12] and [1, 2, 15] for further references). A trouble with k-ary relations is that they lack a structure that could be easily visualised. Supergraphs we introduce in this paper have superedges coming in a variety of types that have to be preserved by automorphism. This makes possible and often easy to see a solution of a problem and to construct proofs that can be subsequently reproduced in terms of relations. So, our hope is that

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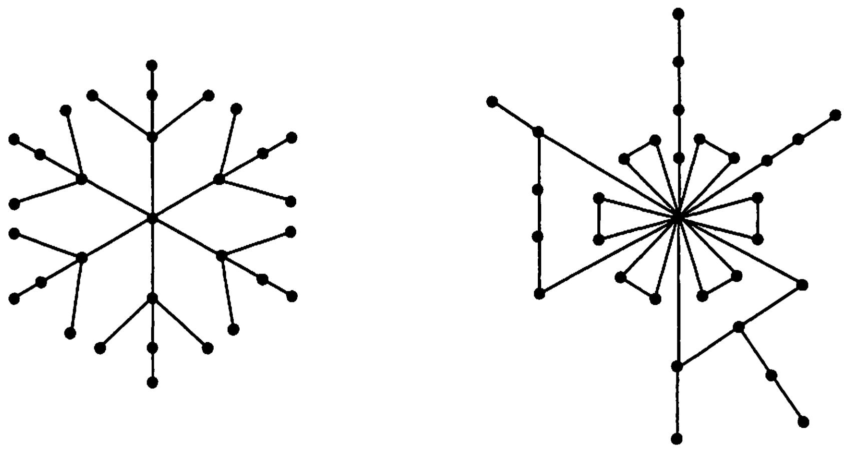

FIGURE 1. Graphs with automorphism groups isomorphic to $S_2 \, wr \, S_6$.

this uniform graphical representation may prove not only to be more accessible, and interesting as a new point of view, but also helpful in attacking old problems.

Formal definitions are given in Section 2, where we also discuss diagrams, and prove the basic result. In Section 3, we introduce the concept of *graphical complexity* based on the rank of a supergraph, and establish the complexity of the simplest classes of permutation groups. Here we apply an approach used already in [7, 4, 5], where various constructions of permutation groups are studied with regard to the growth of complexity. The most interesting open problem and further directions of research are suggested in conclusion.

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2. SUPERGRAPHS

A supergraph consists of vertices and superedges; each superedge, like an edge in a simple graph, joins two elements, and the only difference is that these elements may be also other superedges.

Formally, a *supergraph* is a pair (V, S) of sets V of *vertices* and S of *superedges* such that $S \subseteq E^i(V)$, for some $i \geq 1$, where $E^1(X)$ denotes the set of unordered pairs of X, and $E^i(V) = E^1(V \cup E^{i-1}(V))$ for $i > 1$. To fit the simple idea described above we require also that S is *hereditary* in the following sense: if $\{s, t\} \in S$, and s is not a vertex, then $s \in S$, as well.

We consider only finite simple supergraphs, i.e., with both V and S finite, and with no loops (each superedge is a two-element set). A superedge $\{s, t\} \in E^i(V) \setminus E^{i-1}(V)$ is said to be of *rank* i. The *rank of a supergraph* is the maximum of the ranks of its superedges; for empty supergraphs, those with $S = \emptyset$, the rank is zero.

Supergraphs have natural and simple representations, both graphical and symbolic. In diagrams vertices are drawn as circles, and superedges as lines joining suitable elements. A superedge joined with another superedge or a vertex has a filled circle on it, such that, all lines joining the superedge with other elements come out from this very circle. In such a way the picture consists only of lines and two kinds of circles (in a three-dimensional picture, without line crossings, filled circles could be removed). To give a precise definition we start from symbolic representation.

In this representation, to denote superedges, we use *words* consisting of symbols of vertices and brackets, in the natural way. Thus, (xy) denotes the superedge joining vertices x and y, and (st) denotes the superedge joining elements s and t. Then, for example, $(c(ab))$ is a superedge joining vertex c with edge (ab), and $((ab)(c(ab)))$ is a superedge joining superedge (ab) with the superedge $(c(ab))$. The rank of a superedge coincides with what is sometimes also called the *rank* of the corresponding word.

Note that this notation is not quite unique: the elements in brackets can be written in either order. Formally, the strings denoting superedges may be considered as *idempotent commutative groupoid* words (IC-*words*, in short), put in their normal form (since by definition we do not allow loops, we do not have substrings of the form ss). A superedge s is *contained* in a superedge t if the word corresponding to s is a subword of t (up to commutativity).

By definition, if s belongs to a supergraph S, then each superedge contained in s also belongs to S. Therefore, usually it is enough to deal only with *maximal superedges*, those not contained in other superedges. In particular, only maximal superedges are listed in symbolic representation. From this point of view, a supergraph is simply a set of noncomparable

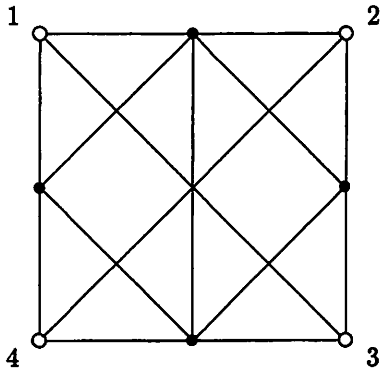

FIGURE 2. A supergraph K on 4 vertices representing the Klein 4-group.

IC-words, and from now on we adopt convention that in notation (V, S) the set S contains only maximal superedges.

Now, in order to draw a diagram of a supergraph (V, S), we first make a list of all superedges (which can be easily derived from S), and then we draw in turns: vertices, superedges of rank 1, superedges of rank 2, etc. To distinguish between usual line crossings and the starting points of superedges, for the latter we use filled circles. To distinguish between starting points of lines and points on lines (representing lines) we adopt the convention that the lines representing superedges joining two points are smooth segments of lines, while a line starting in a point has no smooth continuation through the point.

Example 1. Let $V = \{1, 2, 3, 4\}$ and

$$S = \{((12)(14)), ((12)(23)), ((12)(34)), ((23)(34)), ((34)(14)), (13), (24)\},$$

$K = (V, S)$ is a supergraph with six superedges of rank 1 (which are (12), (14), (13), (23), (24), (34)), and five superedges of rank 2 (listed in S). It is pictured in Figure 2.

While given a set of noncomparable IC-words we may easily draw a corresponding diagram (at least, in principle), the converse is not so obvious. It is even not quite obvious that the diagram determines the supergraph uniquely. To see this, we need to observe that the maximal superedges are distinguished by the fact that these are the only lines joining two points (starting in them) having no filled circle on themselves. Once we remove them from the diagram, and then remove filled circles of "degree 2" (those being no longer starting points for other superedges) we get a diagram of

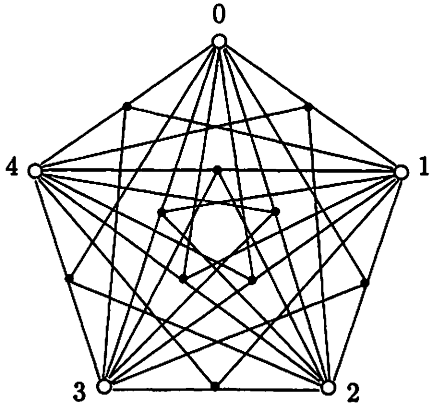

FIGURE 3. Supergraph with double transitive automorphism group $AGL_1(5)$.

rank less by one. Thus, using simple induction we may prove that the diagram determine uniquely the supergraph.

The *automorphism group* Aut(S) of a supergraph $S = (V, S)$ may be defined and viewed into two ways: 1) as the set of all permutations on V preserving the diagram (for every superedge $s \in S$ its image under the action induced by a permutation of V is again in S), and 2) as the symmetry group of the set of words representing maximal superedges (i.e. the set of all permutations preserving the set of these words). So, investigating automorphism groups of supergraphs we may work both with graphical and symbolic representation. This makes supergraphs easier objects to study then k-ary relations, and may be even a way to solve analogous problems on the symmetry groups of relations.

For example, in the study of the automorphism groups of graphs, a very useful invariant is the degree of a vertex. For supergraphs we may also consider analogously the *degree of a superedge* defined as the number of outcoming other superedges, and even the degrees with respect to types of superedges (these where used successfully in [5]).

Example 1(continued). In Figure 2 every vertex has degree 3, but among the superedges of rank 2, there are two superedges of degree 3, two of degree 2, and the remaining two of degree 0. It follows that every automorphism preserves or exchange the pairs (1,2) with (3,4), (1,3) with (2,4), and (1,4) with (2,3). It is not difficult to see that each of this exchangings is

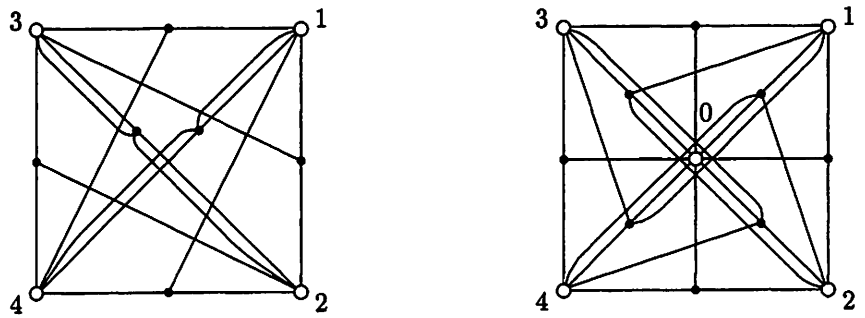

FIGURE 4. Decomposition of the supergraph of Figure 3.

actually possible, and that the full automorphism group is just the Klein 4-group with its regular action.

Example 2. In Figure 3, a supergraph is given whose automorphism group is the one-dimensional linear affine group $AGL_1(5)$ over the five-element field. It may be instructive to check that this is indeed the case. First observe that this supergraph has 20 superedges of rank 2 (the starting points are filled circles), and (the remaining) 10 superedges of rank 1. As it is well known (cf. [3]), $AGL_1(5)$ can be presented as the permutation group generated by two cycles: $(0,1,2,3,4)$ and $(1,2,4,3)$ (the first is translation by 1, and the second is multiplication by 2). From Figure 3 one sees immediately that the first cycle preserves the supergraph structure. In order to see that the second cycle does, as well, we decompose the supergraph into two parts and draw them separately in a way that makes the statement, again, obvious: see Figure 4. The only thing to check is that the supergraphs in Figures 3 and 4 have exactly the same superedges. Now, from Figure 3 we also see that transposition $(2,4)$ does not preserve the structure of the supergraph (the image of the superedge $((40)1)$ under this transposition is $((20)1)$, and the latter does not belong to the supergraph). Therefore, since $AGL_1(5)$ is a maximal subgroup of S_5, the automorphism group of the supergraph given in Figures 3 and 4 is precisely $AGL_1(5)$.

Of course, the practical use of diagrams, as in case of graphs (or even more), is restricted only to small supergraphs. What however counts here is that the very idea of existence of graphical representation may be very useful in arguments (arguments in [5] are very illuminating with this respect).

Before we state our first result we fix some notational conventions. By a group we mean always a permutation group, unless otherwise stated. We say often briefly that a graph (digraph, colored graph, or other mathematical object) *represents* a permutation group $G = (X, G)$ meaning that the automorphism (symmetry) group of this object is equal (up to permutation isomorphism) to G (i.e., the elements of the base sets may be identified in such a way that the automorphisms coincide with the permutations in G). The automorphism (or symmetry) group of an object S is denoted generally $\mathrm{Aut}(S)$. If the base set is clear from the context we use short notation $S = (X, S)$, for a supergraph on a set X of vertices, and similarly, $G = (X, G)$ for a permutation group on the same set X. We also write informally $x, s \in S$ both for vertices and superedges. We use the notation $w = w(x_1, x_2, \ldots, x_m)$ to denote a word of in m letters $x_1, x_2, \ldots, x_m$. Following common practice, writing down concrete words, the external pair of brackets is usually omitted. Below, $\log x$ stands for $\log_2 x$, and we put $\log x = 0$ for $x = 0$.

Theorem 1. *For every permutation group $G = (X, G)$ on an n-element set X, there exists a supergraph $S = (X, S)$ such that $\mathrm{Aut}(S) = G$, and the rank of S is less than $2 + \log(n-2)$.*

Proof. It is well-known (and easy to see) that each finite permutation group $G = (X, G)$ is the automorphism group of a relational structure (X, R), where R is an $(n-1)$-ary relation on X [3, p. 43]. Indeed, it is enough to fix an $(n-1)$-tuple $(x_1 x_2 \ldots x_{n-1})$ of distinct elements of X and take R to be the least set containing $(x_1 x_2 \ldots x_{n-1})$ and closed on permutations in G (under its natural action on k-tuples).

So, all we need is to show that for every relational structure (X, R), with a single relation R, there exists a supergraph (X, S) such that $\mathrm{Aut}(S) = \mathrm{Aut}(R)$. This is, in fact, a problem of encoding relations into supergraphs. The construction below will be useful also later.

Let us consider words of the form $x_1(x_2(\ldots x_{m-1}(x_m x_{m+1})\ldots))$, which we denote briefly $x_1 x_2 \ldots x_{m-1}(x_m x_{m+1})$ (we leave the last pair of brackets, because $x_m x_{m+1}$ can be written in the reverse order). Then, there is an obvious one-to-one correspondence between m-tuples $(x_1, x_2, \ldots, x_m)$ of elements X and words of the form $x_1 x_2 \ldots x_{m-1}(x_m x_1)$. It is also obvious that the supergraph S consisting of all superedges corresponding in such a way to m-tuples in a relational structure (X, R) has the same automorphism group as (X, R).

This natural correspondence yields the bound $n-1$ for the rank of the supergraph. To get better bound we must find a more compact encoding. What we need is to assign to every m an asymmetric IC-word $w = w(x_1, x_2, \ldots, x_m)$ (i.e., one with the property that each permutation of variables yields a different IC-word; for example, $x(yz) = x(zy)$ is not asymmetric). Above, we have used uniformly the word $w = x_1 x_2 \ldots x_{m-1}(x_m x_1)$ for all m. Now we look for words of lower rank.

We prove that for all $k \geq 3$ and $n \geq 4$ satisfying $n \leq 2^{k-1} + 1$ there exists an asymmetric IC-word $w(x_1, x_2, \ldots, x_n)$ in n letters of rank k.

The proof is by induction on k. For $k = 3$, the words $((xy)(zt))(x(zu))$ in 5 letters, and $((xy)(zy))(x(zu))$ in 4 letters, are as required.

For induction step, define $u(x_1, x_2) = (x_1 x_2)$, and

$$u(x_1, \ldots, x_{2k}) = u(x_1, \ldots, x_k)u(x_{k+1}, \ldots, x_{2k}).$$

Suppose that the claim above is true for a fixed k, and let $w(x_1, x_2, \ldots, x_m)$ be an asymmetric IC-word in $m = 2^{k-1} + 1$ letters and of rank k. Then, consider the word

$$u(x_1, y_1, \ldots, x_{m-1}, y_{m-1})w(x_1, x_2, \ldots, x_{m-1}, z).$$

This word is, by assumption, of rank $k+1$, in $2(m-1)+1 = 2^k + 1$ letters, and it can be easily seen to be asymmetric. Hence, it proves the required statement for $n = 2^k + 1$. For $2^{k-1} + 1 < n < 2^k + 1$ it is enough to put $y_1 = \ldots = y_r$ with $r = 2^k + 2 - n$. A key observation is that, in any case, the word u on the left hand side is not asymmetric itself, in contrast with the word $w(x_1, x_2, \ldots, x_{m-1}z)$ on the right hand side, and therefore the resulting word is always asymmetric. This completes the induction proof.

Now, having the claim proved, it is routine to see that given $n \geq 4$, the minimal rank of an asymmetric word in n letters does not exceed $\lceil 1 + \log(n-1) \rceil$. Taking into account that according to the argument at the beginning of the proof, it is enough to apply asymmetric words in $n-1$ letters, the rank of a supergraph S we construct (using these words as corresponding to m-tuples) does not exceed $\lceil 1 + \log(n-2) \rceil$.

It remains to consider cases $n = 2$ and 3. For $n = 2$ there are two groups: S_2 represented by a graph, and I_2 (trivial) represented by a directed graph. For $n = 3$, there are four groups (up to permutation isomorphism) that can be denoted S_3, S_2, C_3, I_3. Again they are represented either by a graph or a directed graph. Since, directed edges can be represented in supergraphs as superedges $x(xy)$, in any case, the rank of a supergraph is less or equal to 2, and the result follows. □

Note that the proof above contains a method that given a k-ary relation representing a permutation group makes possible to produce a supergraph representing this group.

3. Graphical complexity

In general, by a graphical complexity of a permutation group P we mean the degree of complexity of a graphical structure whose symmetry group (automorphism group) is P. In this paper, the result above suggests to define the *graphical complexity* of G as follows. By $\mathrm{gc}(G)$ we denote the least k such that supergraph $S = (X, S)$ of rank k represents G. Theorem 1 shows that gc(G) is well-defined, and gc(G) $< 2+\log(|X|-2)$. Since there is a finite number of supergraphs of a given rank on X, we see that gc(G) is, in principle, *computable.*

The trivial graphs without edges represent the full symmetric groups S_n, and therefore gc(S_n) $= 0$. The groups with gc(G) ≤ 1 are exactly the automorphism groups of graphs. It is also not difficult to see, using the encoding from the proof above, that for automorphism groups of directed graphs we have gc(G) ≤ 2. Taking into account the representation of $AGL_1(5)$ in Figure 3, one also easily infer that gc($AGL_1(5)$) $= 2$. A natural question arising in this connection, whether there are supergraphs with higher complexity, is answered by the next result.

Theorem 2. *For every $k \geq 1$ there exists a permutation group G such that* gc(G_k) $= k$.

Proof. The most widely used example of groups that are difficult to represent as a symmetry groups of given objects are the alternating groups A_n. From the previous theorem we have gc(A_n) $< 2 + \log(n-2)$, and we will see that this bound is close to the actual value. At this moment, we are not able to establish the exact value, but what we need here is only a suitable lower bound.

It should be obvious that we will be done when we establish the following two facts:

(i) gc(A_n) $> \log(n-1)$.

(ii) gc(A_{n+1}) $\leq$ gc(A_n) $+ 1$

First let us note that if Aut(S) $= A_n$ then S has a superedge involving more than $n-2$ vertices. Indeed, if the number of vertices (letters) involved in an superedge s' is $n-2$ or less, and say $s' = s(x_1, \ldots, x_{n-2})$, then for every permutation $\tau \in S_n$ there exists a permutation $\sigma \in A_n$ such that, under the induced action of S_n on superedges, $\tau(s') = \sigma(s')$. Hence, the set S' of all superedges $s \in S$ such that $s = \sigma(s')$ for some $\sigma \in S_n$ is invariant under any permutation of vertices. It follows that removing from S all the superedges that are in S' leaves us with a supergraph S'' whose automorphism group is the same as that of S. Hence, we may assume that each superedge in S involves at least $n-1$ vertices. Now, since the maximal number of vertices involved in an superedge of rank k is 2^{k-1}, (i) easily follows.

Going one step further we infer that it is necessary that S has an superedge (in $\geq n-1$ letters) such that no odd permutation in S_n fixes the superedge. Otherwise, the argument in the previous paragraph works. Conversely, if there exists such an superedge then it is not difficult to see that its orbit under the action of A_n forms a supergraph S with Aut(S) $= A_n$.

Now, to prove (ii), suppose that gc(A_n) $= k$. Then, there exists an superedge $s = s(x_1, \ldots, x_m)$ of rank k, with $m \leq n$, such that no odd

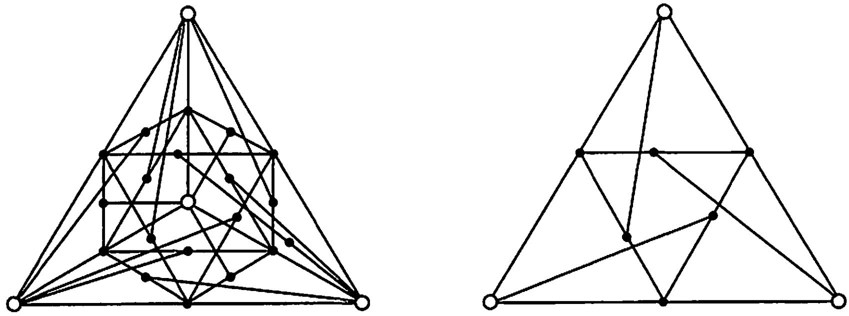

FIGURE 5. Supergraph with automorphism group A_4 and its „face".

permutation in S_n fixes it. It follows that the superedge $s' = (x_{n+1}s)$ has the same property for permutations in S_{n+1}. Indeed, if $\tau \in S_{n+1}$ were an odd permutation fixing s', then $\tau(x_{n+1}) = x_{n+1}$, and τ' obtained from τ by restricting its domain to $\{x_1, \dots, x_n\}$, would be odd and would fix s. It follows that the orbit of s' under the action of A_{n+1} forms a supergraph S' with $\text{Aut}(S') = A_{n+1}$. Therefore, $\text{gc}(A_{n+1}) \leq k + 1$, as required. □

In Figure 5 a supergraph is given whose automorphism group is A_4. It can be viewed in three dimensions as a tetrahedron whose face is pictured beside. This observation makes it immediately clear that any 3-cycle preserves the structure of the supergraph, while a 2-cycle does not. It follows that the automorphism group is the alternating group. (I am grateful to Peter Jipsen for this observation; in fact, the decomposition in Figure 4 also follows this idea). It follows that $\text{gc}(A_4) = 2$

Note that for each Mathieu group M, $\text{gc}(M) \geq 3$, since they are 4- and 5-transitive (in such a case, if $((xy)(zt))$ is a superedge of a supergraph representing M, then each word of this type must be a superedge). We also remark that using A_n one can construct various groups G with $\text{gc}(G)$ arbitrarily large. It is an open problem whether there are such groups not involving A_n in any way (see Section 4).

One easily establishes the graphical complexity of the simplest permutation groups on an n-element set X. The full symmetric group S_n, the trivial group I_n, the reflection group R_n (generated by a reflection permutation $\rho(i) = n + 1 - i$), the cyclic group C_n generated by an n-cycle, and the dihedral group D_n are all represented by graphs or directed graphs (cf. e.g. [4]), and therefore their graphical complexity does not exceed 2. For regular permutation groups the same is proved in [13] (most of the cases were established in papers on graphical regular representation; see [1, 2]).

In order to establish the complexity of more complex groups we adopt the approach introduced in [7] and look for closeness of the complexity classes on the most natural constructions. As we will see the classes based on supergraphs are much more "compatible" with these constructions, then classes considered so far.

Recall that by the *direct sum* $G_1 \oplus G_2$ of permutation groups $G_1 = (X_1, G_1)$ and $G_2 = (X_2, G_2)$ we mean the direct product of groups G_1 and G_2 acting on the *disjoint union* of sets X_1 and X_2 in the natural way (thus the product is determined up to permutation isomorphism). In the literature, this construction is more often called the *direct product*, but we prefer the first name (following [9]); this name is certainly more appropriate from the point of view of concrete permutation groups (to distinguish it from the direct product with product action, which leads to different permutation groups). It was established in [10] that the direct sum of two automorphism groups of simple graphs is the automorphism group of a simple graph, unless both the groups are permutation isomorphic, transitive, and *unique* (in the sense that there is a unique graph representing the group, up to graph isomorphism; see also [11] for a very interesting development of this topic). A similar result were established in [4] for automorphism groups of edge-colored graphs; exception are edge-colored counterparts of unique graphs. On the other hand it was also observed in [4] that the class of automorphism groups of directed graphs is closed on the direct sum.

The reason is somehow explained in the result below. It can be interpreted so that the complexity of the construction of the direct sum itself is 2: once superedges of rank 2 are accessible, the class is closed on the direct product.

Theorem 3. *For every pair of permutation groups G_1 and G_2, the complexity* $\mathrm{gc}(G_1 \oplus G_2) \leq \max\{\mathrm{gc}(G_1), \mathrm{gc}(G_2), 2\}$.

Proof. Let S_1 and S_2 be supergraphs representing G_1 and G_2 on disjoint sets of vertices X_1 and X_2, with $|X_1| = n$ and $|X_2| = m$, respectively. We form a supergraph S joining each vertex $x \in X_1$ with any vertex $y \in X_2$ by a directed edge $\langle x, y \rangle$, i.e, a superedge $x(xy)$. Then, obviously, $\mathrm{Aut}(S) \supseteq G_1 \oplus G_2$. It should be obvious that due to using directed edges we have also $\mathrm{Aut}(S) \subseteq S_n \oplus S_m$, and then, it is easy to see that also $\mathrm{Aut}(S) \subseteq G_1 \oplus G_2$ holds. Obviously, the graphical complexity of S satisfies the required conditions. □

This result suggest that also the problem of complexity of groups generated by a single permutation (i.e. those that are cyclic as abstract groups)

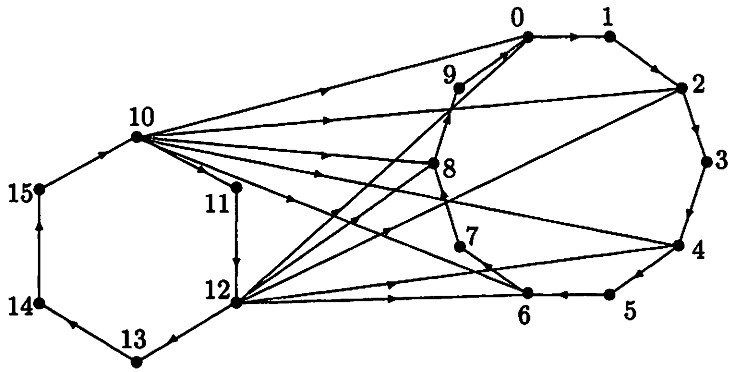

FIGURE 6. Drawing a graph whose automorphism group is generated by a single permutation, a product of two cycles.

should be easy. In fact, slight modification of the construction used above yields immediately

Theorem 4. *All permutation groups G generated by a single permutation have the complexity* $\mathrm{gc}(G) \leq 2$. *In fact, all these groups are automorphism groups of directed graphs.*

Proof. If $h = c_1 c_2 \ldots c_n$ is an arbitrary permutation given in the form of a product of disjoint cycles, then the group generated by h is a direct sum of groups generated, respectively, by cycles $c_1, c_2, \ldots, c_n$, up to some parallel actions caused by common divisors of the lengths of the cycles. Each such a group is represented by a directed cycle of suitable length. All we need to do is to join all these cycles by as few edges as possibly, just to ensure that the action in any of the cycles forces a suitable action in other cycles.

Let us denote the directed cycles corresponding to cyclic permutations $c_1, \ldots, c_n$ by $C^1, \ldots, C^n$. For each $i = 1, \ldots, n-1$ we join the cycles C^i and C^{i+1} by directed edges so that the automorphism group of the resulting directed graph is one generated by a single permutation $h' = c_i c_{i+1}$; it is enough to take one directed edge joining C^i with C^{i+1}, and add all the images of this edge under the permutation h'. (In Figure 6 the situation is pictured for $h' = (10, 11, \ldots, 15)(0, 1, \ldots, 9)$; to make the picture clear and show the pattern, edges outgoing from vertices 11,13,14, and 15 are omitted). This construction extends obviously on n cycles. □

The next natural construction to consider is that of *intersection of permutation groups*. The fact that the intersection can be handled by means of supergraphs and that the graphical complexity increases only by one in this case is perhaps the most interesting result of this paper.

Theorem 5. *For every finite set of permutation groups $G_i = (X, G_i)$ on a set X, the complexity* $\mathrm{gc}(\bigcap G_i) \leq 1 + \max\{\mathrm{gc}(G_i)\}$.

Proof. Let S_i be the supergraph representing G_i of the lowest possible rank, i.e., $\mathrm{Aut}(S_i) = G_i$, and the rank of S_i is $\mathrm{gc}(G_i)$. Denote $G = \bigcap G_i$ and $m = \max\{\mathrm{gc}(G_i)\}$. We wish to construct a supergraph S with $\mathrm{Aut}(S) = G$ and rank not exceeding $m + 1$. We cannot simply take $S = \bigcup S_i$, since at first, there can be comparable IC-words in $\bigcup S_i$, and at second, there may be new permutations preserving S that do not preserve all S_i individually.

To overcome this obstacle, our construction consists of a few steps.

First we show that we may assume that all S_i, as sets of IC-words, contain only words of rank m. Indeed, let S_i' be the set of superedges obtained from S_i by replacing each superedge $t \in S_i$ by the set of superedges $w_x = xx\ldots x(xt)$, for all $x \in X$, where the number of occurrences of x is chosen so that w_x is of rank m. Observe that, since the superedges in S_i are incomparable, this construction leads to pairwise distinct incomparable words of the same rank m. Moreover, for each i, $\mathrm{Aut}(S_i') = \mathrm{Aut}(S_i)$, and consequently, $\bigcap \mathrm{Aut}(S_i') = G$, as required.

Now we show that we may assume in addition that S_i, as sets of IC-words of rank m, are pairwise disjoint. Suppose that $I = S_k \cap S_j \neq \emptyset$ for some $k \neq j$. Then, obviously, I is closed under the action of $G = \bigcap G_i$, as both $S_k \setminus I$ and $S_j \setminus I$ are. Moreover, $G_k \supseteq \mathrm{Aut}(S_k \setminus I) \cap \mathrm{Aut}(I)$, and $G_j \supseteq \mathrm{Aut}(S_j \setminus I) \cap \mathrm{Aut}(I)$. It follows that if, in the family $\{G_i\}$, we replace G_k and G_j by the groups $\mathrm{Aut}(S_k \setminus I)$, $\mathrm{Aut}(S_j \setminus I)$, and $\mathrm{Aut}(I)$, then the intersection of all the groups in the family remains G. To keep the family $\{S_i\}$ as one consisting of supergraphs representing groups G_i, at the same time, we replace supergraphs S_k and S_j in the family $\{S_i\}$ by $S_k \setminus I$, $S_j \setminus I$, and I. Applying such a replacement until $I = S_k \cap S_j \neq \emptyset$ for some $k \neq j$, we finally obtain a family S_i of disjoint supergraphs representing a family G_i of groups whose intersection is $\bigcap G_i = G$, and such that all words in all S_i have the same rank m.

It may still happen that some superedges in different S_i and S_j have the same pattern (as IC-words), and taking the union may result in appearing new automorphisms. Therefore for the last step we make use of the possibility of increasing the rank as follows.

Let $S_1, S_2, \ldots, S_r$ denote now all the supergraphs in the recently obtained family. We define by induction, $S_1' = \{xt : x \in X \text{ and } t \in S_1$, and for $k > 1$, S_k' is the set consisting of all superedges (st) with $s \in S_{k-1}$ and $t \in S_k$. Then, the sets S_k', for $1 \leq k \leq r$, are pairwise disjoint and all consist of edges of rank $m + 1$. Moreover, an inductive argument shows that any automorphisms of the union $S' = \bigcup S_k'$ preserves S_k' for each

$k = 1, 2, \ldots, r$. It follows also that $\mathrm{Aut}(S') = \bigcap \mathrm{Aut}(S'_k) = G$, and the rank of S' is $m + 1$, as required. □

Obviously, the automorphism group of an edge-colored graph is the intersection of the automorphism groups of graphs given by particular colors. Such groups can be also characterized as invariance groups of families of binary symmetric relations, and in this connection they are called 2^*-*closed.* The automorphism groups of an edge-colored directed graphs are invariance groups of families of binary relations, and are called 2-*closed* ([3, 9, 14]). For these classes we have the following corollary.

Corollary 6. *For each* 2^*-*closed group* G *the complexity* $\mathrm{gc}(G) \leq 2$, *and for each* 2-*closed group* G *the complexity* $\mathrm{gc}(G) \leq 3$.

4. Final remarks

There are many open problems arising naturally in this connection. We point out here only one general question that seems most interesting, leads to many particular problems, and is strictly connected with other characteristics, by which one tries to compare the "complexity" of permutation groups.

In [4, 5] we consider the classes $GR(k)$ of permutation groups represented by k-colored graphs (more precisely, by k-edge-colored complete graphs). The main open problem here is whether for every $k > 1$ there are permutations groups in $GR(k)$ which are not in $GR(k-1)$. We have an example of a group which in this sense requires $k = 6$ colors. The situation is analogous for the classes $DGR(k)$ of permutation groups represented by k-colored digraphs. In contrast, by Theorem 2, the situation seems quite different for the classes $SGR(k)$ of permutation groups represented by supergraphs of rank k. Yet, also for these classes, one can formulate an analogue of the problem above.

Indeed, observe that the construction in the proof of Theorem 2 is based on alternating groups A_n. The fact that they are $(n-2)$-transitive causes that one needs at least $(n-1)$-tuples or corresponding superedges of high rank to represent such groups. Now, a question arises whether this is the only reason for this phenomenon, i.e. whether every permutation group of high graphical complexity must involve somehow an alternating group, or on the contrary, there exist other groups with arbitrarily large graphical complexity. A similar question may be asked for arity of representing relations: does there exist a constant k such that every permutation group not involving an alternating group can be represented as the symmetry group of a k-ary relation? Of course, to solve the problem one must also specify what means here "not involving". It seems that solving the first problem, which seems more accessible, may help to solve the second. Any answer to these questions, yes or no, would be of a great value, giving better insight into possible combinatorial complexity of permutation groups. A good starting point maybe an attempt to apply the results of [12] on k-closures of primitive groups, but it is certainly not straightforward.

For other open problems, more discussion and related bibliography the reader is referred to [8].

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