Direct Sum, Direct Product and Lexicographic Product of Lattices

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Abstract

This article discusses the geometricity of the direct sum, direct product and lexicographic products of two lattices, and compute their characteristic polynomials and classify their geometricity.

1 Introduction

We recall some terminology and definitions about finite posets and lattices. For more theory about finite posets and lattices, we would like to refer readers to [1, 9].

Let P denote a finite set. A *partial order* on P is a binary relation $\leq$ on P such that

(i) $\alpha \leq \alpha$ for any $\alpha \in P$.

(ii) $\alpha \leq \beta$ and $\beta \leq \alpha$ implies $\alpha = \beta$.

(iii) $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$.

By a *partial ordered set* (or *poset* for short), we mean a pair $(P, \leq)$ where P is a finite set and $\leq$ is a partial order on P. As usual, we write $\alpha < \beta$ whenever $\alpha \leq \beta$ and $\alpha \neq \beta$. By abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P be a poset and let R be a commutative ring with the identical element. A binary function $\mu(\alpha, \beta)$ on P with values in R is said to be the *Möbius function* of P if

$$\mu(\alpha,\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \not\leq \beta, \\ -\sum\limits_{\alpha \leq \gamma < \beta} \mu(\alpha,\gamma), & \text{if } \alpha < \beta. \end{cases}$$

For any two elements $\alpha, \beta \in P$, we say α *covers* β, denoted by $\beta < \cdot\, \alpha$, if $\beta < \alpha$ and there exists no $\gamma \in P$ such that $\beta < \gamma < \alpha$. If P has the minimum (resp. maximum) element, then we denote it by $\perp$ (resp. $\top$) and say that P is a poset with $\perp$ (resp.

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$\top$). Let P be a finite poset with $\perp$. By a *rank function* on P, we mean a function r from P to the set of all the nonnegative integers such that

(i) $r(\perp) = 0$.

(ii) $r(\alpha) = r(\beta) + 1$ whenever $\beta <\cdot\, \alpha$.

Let P be a finite poset with $\perp$ and $\top$. The polynomial

$$\chi(P, x) = \sum_{\alpha \in P} \mu(\perp, \alpha) x^{r(\top) - r(\alpha)}$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset P is said to be a *lattice* if both $\alpha \vee \beta := \sup\{\alpha, \beta\}$ and $\alpha \wedge \beta := \inf\{\alpha, \beta\}$ exist for any two elements $\alpha, \beta \in P$. Let P be a finite lattice with $\perp$. By an *atom* in P, we mean an element in P covering $\perp$. We say P is *atomic* if any element in $P \setminus \{\perp\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(\alpha \wedge \beta) + r(\alpha \vee \beta) \leq r(\alpha) + r(\beta), \forall\, \alpha, \beta \in P.$$

Let P be a lattice. A bijective map f from P to P is an *automorphism* of P if f is *join-preserving* and *meet-preserving*, that is, for all $\alpha, \beta \in P$,

$$f(\alpha \vee \beta) = f(\alpha) \vee f(\beta) \text{ and } f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta).$$

All the automorphisms of P form a group, called the *full automorphism group* of P, denoted by $\mathrm{Aut}(P)$.

For $i = 1, 2$, let L_i be a lattice with the minimum element $\perp_i$ and the maximum element $\top_i$. Suppose r_i denotes the rank function, μ_i denotes the Möbius function of L_i respectively.

For any two lattices L_1 and L_2 with $L_1 \cap L_2 = \{\top, \perp\}$ and $\top = \top_1 = \top_2, \perp = \perp_1 = \perp_2$, let $L = L_1 \cup L_2$. For $\alpha, \beta \in L$, define

$$\alpha \leq \beta \text{ if and only if } \alpha \leq \beta \text{ in } L_1 \text{ or } \alpha \leq \beta \text{ in } L_2.$$

Then L is a lattice with $\perp$ and $\top$, which is said to be the *direct sum* of L_1 and L_2, denoted by $L_1 \oplus L_2$.

For any two lattices L_1 and L_2, let $L = L_1 \times L_2$. For $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L$, define

$$\alpha \leq \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2.$$

Then L is a lattice, which is said to be the *direct product* of L_1 and L_2, denoted by $L_1 \otimes L_2$.

For any two lattices L_1 and L_2, let $L = L_1 \times L_2$. For $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L$, define

$$\alpha \leq \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ or } \alpha_1 = \beta_1 \textit{ and } \alpha_2 \leq \beta_2.$$

Then L is a lattice, which is said to be the *lexicographic product* of L_1 and L_2, denoted by $L_1[L_2]$.

Example 1.1 Let F_n be a set of all positive factors of n. For any $s,t \in F_n$, define $s \leq t$ if and only if $s|t$. Then F_n is a lattice. Let $L_1 = F_3 \cup \{33\}$. For any $s,t \in L_1$, define $s \leq t$ if and only if $s|t$. Then L_1 is a lattice. Let $L_2 = F_{11} \cup \{33\}$ For any $s,t \in L_2$, define $s \leq t$ if and only if $s|t$. Then L_2 is a lattice. Note that $F_{33} = L_1 \oplus L_2$.

Example 1.2 Let X be a set with n elements, and let L be the set of all subsets of X. It is well known that, partially ordered by ordinary inclusion L is a geometric lattice. Let $P = \{0,1\}$, define $0 \leq 1$, then P is a lattice. Note that L is isomorphic to $\underbrace{P \otimes \cdots \otimes P}_{n}$.

In a series of papers ([4, 5, 6, 7, 8, 10, 11]), Huo, Liu and Wan et al. constructed lattices from orbits of subspaces under finite classical groups, computed their characteristic polynomials and discussed their geometricity. Very recently, lattices associated with distance-regular graphs have been discussed by Gao et. al [3, 12]. In this paper, we classify the geometricity of the direct sum, direct product and lexicographic products of two lattices, compute their characteristic polynomials.

2 The geometricity

In this section, we discusses the geometricity of direct sum, direct product and lexicographic products of two lattices.

Theorem 2.1 *Let L_1 and L_2 be two lattices with at least four vertices and the same rank. If $L_1 \cap L_2 = \{\top, \bot\}$ and $\top = \top_1 = \top_2, \bot = \bot_1 = \bot_2$, then $L = L_1 \oplus L_2$ is geometric if and only if both L_1 and L_2 are geometric and $r(\top) \leq 2$.*

Proof. Suppose both L_1 and L_2 are geometric lattices. Then the atomic set of L is consists of all the atomics of L_1 and L_2. For any $\alpha \in L$, there exist atomics $\gamma_1, \gamma_2, \ldots, \gamma_m$ of L_1 or atomics $\delta_1, \delta_2, \ldots, \delta_n$ of L_2 such that

$$\alpha = \begin{cases} \bigvee_{i=1}^{m} \gamma_i, & \text{if } \alpha \in L_1, \\ \bigvee_{i=1}^{n} \delta_i, & \text{if } \alpha \in L_2; \end{cases}$$

consequently L is atomic.

For any $\alpha \in L$, define

$$r(\alpha) = \begin{cases} r_1(\alpha), & \text{if } \alpha \in L_1, \\ r_2(\alpha), & \text{if } \alpha \in L_2. \end{cases}$$

Then r is the rank function of L.

Pick any $\alpha, \beta \in L$, if $\alpha, \beta \in L_i$, then we have

$$r(\alpha \wedge \beta) + r(\alpha \vee \beta) \leq r(\alpha) + r(\beta);$$

if $\alpha \in L_1, \beta \in L_2$ or $\alpha \in L_2, \beta \in L_1$, by $r(\top) \leq 2$ we have

$$r(\alpha \wedge \beta) + r(\alpha \vee \beta) = r(\top) + r(\bot) \leq r(\alpha) + r(\beta).$$

Hence L is a geometric lattice.

Conversely suppose $L_1 \oplus L_2$ is geometric. It is routine to check that both L_1 and L_2 are geometric. If $r(\top) > 2$, pick two atoms $\alpha \in L_1 \setminus \{\bot\}, \beta \in L_2 \setminus \{\bot\}$. Then

$$r(\alpha \wedge \beta) + r(\alpha \vee \beta) = r(\top) + r(\bot) > 2 = r(\alpha) + r(\beta).$$

Hence L is not a geometric lattice, a contradiction. □

Theorem 2.2 *Let L_1 and L_2 be two lattices. Then $L = L_1 \otimes L_2$ is a geometric lattice if and only if both L_1 and L_2 are geometric lattices.*

Proof. Suppose both L_1 and L_2 are geometric. Then the atomic set of L is $\{(\gamma, \bot_2) \mid r_1(\gamma) = 1\} \cup \{(\bot_1, \delta)| \mid r_2(\delta) = 1\}$. For any $(\alpha_1, \alpha_2) \in L$, there exist atomics $\gamma_1, \gamma_2, \ldots, \gamma_m$ of L_1 and atomics $\delta_1, \delta_2, \ldots, \delta_n$ of L_2 such that

$$\alpha_1 = \bigvee_{i=1}^{m} \gamma_i \text{ and } \alpha_2 = \bigvee_{i=1}^{n} \delta_i;$$

consequently $(\alpha_1, \alpha_2) = (\bigvee_{i=1}^{m} (\gamma_i, \bot_2)) \bigvee (\bigvee_{i=1}^{n} (\bot_1, \delta_i))$, i.e., L is atomic.

For any $(\alpha_1, \alpha_2) \in L$, define

$$r(\alpha_1, \alpha_2) = r_1(\alpha_1) + r_2(\alpha_2).$$

Then r is the rank function of L.

For any two elements $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L$, we have

$$\begin{aligned} & r(\alpha \vee \beta) + r(\alpha \wedge \beta) \\ = \; & r((\alpha_1 \vee \beta_1, \alpha_2 \vee \beta_2)) + r((\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2)) \\ = \; & \sum_{i=1}^{2} (r_i(\alpha_i \vee \beta_i) + r_i(\alpha_i \wedge \beta_i)) \\ \leq \; & \sum_{i=1}^{2} (r_i(\alpha_i) + r_i(\beta_i)) \\ = \; & r(\alpha) + r(\beta). \end{aligned}$$

Hence L is geometric.

Conversely suppose L is geometric. It is clear that both L_1 and L_2 are atomic lattices. For any $\alpha_1, \beta_1 \in L_1$, we have

$$\begin{aligned} & r_1(\alpha_1 \vee \beta_1) + r_1(\alpha_1 \wedge \beta_1) \\ = \; & r((\alpha_1 \vee \beta_1, \bot_2)) + r((\alpha_1 \wedge \beta_1, \bot_2)) \\ = \; & r((\alpha_1, \bot_2) \vee (\beta_1, \bot_2)) + r((\alpha_1, \bot_2) \wedge (\beta_1, \bot_2)) \\ \leq \; & r((\alpha_1, \bot_2)) + r((\beta_1, \bot_2)) \\ = \; & r_1(\alpha_1) + r_1(\beta_1). \end{aligned}$$

Hence L_1 is a geometric lattice. Similarly L_2 is a geometric lattice. □

For $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L_1[L_2]$, we have

$$\alpha \vee \beta = \begin{cases} (\alpha_1, \alpha_2 \vee \beta_2), & \text{if } \alpha_1 = \beta_1, \\ (\alpha_1, \alpha_2), & \text{if } \beta_1 < \alpha_1, \\ (\beta_1, \beta_2), & \text{if } \alpha_1 < \beta_1, \\ (\alpha_1 \vee \beta_1, \bot_2), & \text{otherwise.} \end{cases}$$

Theorem 2.3 *For any two geometric lattices L_1 and L_2 with at least two vertices, $L_1[L_2]$ is not a geometric lattice.*

Proof. Since the atomic set of $L = L_1[L_2]$ is $\{(\gamma, \perp_2) \mid r_1(\gamma) = 1\} \cup \{(\perp_1, \delta) \mid r_2(\delta) = 1\}$, for any $(\alpha, \top_2) \in L$ and any atomics $\gamma_1, \gamma_2, \ldots, \gamma_m$ of L_1 and atomics $\delta_1, \delta_2, \ldots, \delta_n$ of L_2, we have

$$(\bigvee_{i=1}^{m}(\gamma_i, \perp_2)) \bigvee (\bigvee_{i=1}^{n}(\perp_1, \delta_j)) = (\bigvee_{i=1}^{m} \gamma_i, \perp_2) \neq (\alpha, \top_2).$$

It follows that L is not atomic; and so the theorem holds. □

3 Characteristic polynomials

In this section, we compute the characteristic polynomials of the direct sum, direct product and lexicographic products of two lattices. Moreover, we always assume that L_1 and L_2 be two lattices with rank functions.

Theorem 3.1 *Let $L_1 \cap L_2 = \{\top, \perp\}$ with $\top = \top_1 = \top_2, \perp = \perp_1 = \perp_2, r_1 = r_2$. Then*

$$\chi(L_1 \oplus L_2, x) = \chi(L_1, x) + \chi(L_2, x).$$

Proof. Since

$$\mu(\alpha, \beta) = \begin{cases} \mu_1(\alpha, \beta), & \text{if } \alpha, \beta \in L_1, \\ \mu_2(\alpha, \beta), & \text{if } \alpha, \beta \in L_2, \\ 0, & \text{otherwise.} \end{cases}$$

is the Möbius function of $L = L_1 \oplus L_2$, we obtain

$$\begin{aligned} & \chi(L_1 \oplus L_2, x) \\ = & \sum_{\beta \in L} \mu(\perp, \beta) x^{r(\top) - r(\beta)} \\ = & \sum_{\beta \in L_1} \mu_1(\perp, \beta) x^{r(\top) - r(\beta)} + \sum_{\beta \in L_2} \mu_2(\perp, \beta) x^{r(\top) - r(\beta)} \\ = & \chi(L_1, x) + \chi(L_2, x), \end{aligned}$$

as desired. □

Theorem 3.2 *Let L_1 and L_2 be two lattices. Then*

$$\chi(L_1 \otimes L_2, x) = \chi(L_1, x)\chi(L_2, x).$$

Proof. Since the Möbius function of $L = L_1 \otimes L_2$ is $\mu(\alpha,\beta) = \mu_1(\alpha_1,\beta_1)\mu_2(\alpha_2,\beta_2)$, we obtain

$$\begin{aligned}
& \chi(L_1 \otimes L_2, x) \\
= & \sum_{\beta\in L} \mu(\perp,\beta)x^{r(\top)-r(\beta)} \\
= & \sum_{\beta\in L} \mu_1(\perp_1,\beta_1)\mu_2(\perp_2,\beta_2)x^{r_1(\top_1)+r_2(\top_2)-r_1(\beta_1)-r_2(\beta_2)} \\
= & \sum_{\beta\in L} \mu_1(\perp_1,\beta_1)x^{r_1(\top_1)-r_1(\beta_1)}\mu_2(\perp_2,\beta_2)x^{r_2(\top_2)-r_2(\beta_2)} \\
= & \sum_{\beta_1\in L_1} [\mu_1(\perp_1,\beta_1)x^{r_1(\top_1)-r_1(\beta_1)} \cdot \sum_{\beta_2\in L_2} \mu_2(\perp_2,\beta_2)x^{r_2(\top_2)-r_2(\beta_2)}] \\
= & \sum_{\beta_1\in L_1} \mu_1(\perp_1,\beta_1)x^{r_1(\top_1)-r_1(\beta_1)} \cdot \sum_{\beta_2\in L_2} \mu_2(\perp_2,\beta_2)x^{r_2(\top_2)-r_2(\beta_2)} \\
= & \chi(L_1,x)\chi(L_2,x),
\end{aligned}$$

as desired. □

Theorem 3.3 *Let L_1 and L_2 be two lattices. Then*

$$\chi(L_1[L_2],x) = \sum_{\beta_1\in L_1} \chi^{\beta_1} \cdot \sum_{\beta_2\in L_2} \chi^{\beta_2}x^{r_1(\top_1)r_2(\top_2)-r_1(\beta_1)r_2(\beta_2))},$$

where $\chi^{\beta_1} = \mu_1(\perp_1,\beta_1)x^{r_1(\top_1)-r_1(\beta_1)}, \chi^{\beta_2} = \mu_2(\perp_2,\beta_2)x^{r_2(\top_2)-r_2(\beta_2)}$.

Proof. Since the Möbius function of $L = L_1[L_2]$ is $\mu(\alpha,\beta) = \mu_1(\alpha_1,\beta_1)\mu_2(\alpha_2,\beta_2)$, we obtain

$$\begin{aligned}
& \chi(L,x) \\
= & \sum_{\beta\in L} \mu(\perp,\beta)x^{r(\top)-r(\beta)} \\
= & \sum_{\beta\in L} \mu_1(\perp_1,\beta_1)\mu_2(\perp_2,\beta_2)x^{(r_1(\top_1)+1)(r_2(\top_2)+1)-(r_1(\beta_1)+1)(r_2(\beta_2)+1)} \\
= & \sum_{\beta\in L} \mu_1(\perp_1,\beta_1)x^{r_1(\top_1)-r_1(\beta_1)}\mu_2(\perp_2,\beta_2)x^{r_2(\top_2)-r_2(\beta_2)}x^{r_1(\top_1)r_2(\top_2)-r_1(\beta_1)r_2(\beta_2))} \\
= & \sum_{\beta_1\in L_1} \chi^{\beta_1} \cdot \sum_{\beta_2\in L_2} \chi^{\beta_2}x^{r_1(\top_1)r_2(\top_2)-r_1(\beta_1)r_2(\beta_2))},
\end{aligned}$$

as desired. □

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