(a, d)-Edge-Antimagic Total Labelings Of Cycle

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ABSTRACT. An (a, d)-edge-antimagic total labeling for a graph $G(V, E)$ as an injective mapping f from $V \cup E$ onto the set $\{1, 2, \cdots, |V| + |E|\}$ such that the set $\{f(v) + \sum f(uv) | uv \in E\}$ where v ranges over all of V is $\{a, a+d, a+2d, \cdots, a+(|V|-1)d\}$. Simanjuntak et al conjecture: 1. C_{2n} has a $(2n+3, 4)$-or a $(2n+4, 4)$- edge-antimagic total labeling; 2. cycles have no (a, d)-edge-antimagic total labelings with $d > 5$. In this paper, these conjectures are shown that they are true.

1. INTRODUCTION

Graph labelings were first introduced by Rosa [9] in the late 1960s, most graph labeling methods trace their origin to one introduced by Rosa [9] in 1967, or one given by Graham and Sloane [7] in 1980. With labeled graphs serving as useful models for a broad range of applications such as: coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management (see [3], [4] and [11] for details), many new concepts were introduced by experts. Hartsfield and Ringel [8] introduced antimagic graphs in 1990, the concept of an(a, d)-antimagic labelings was introduced by Bodendiek and Wagner [5] in 1993, Bača et al [1] introduced the notion of a(a, d)-vertex-antimagic total labeling in 2000, Simanjuntak et al [10] define an (a, d)-edge-antimagic vertex labeling and an (a, d)-edge-antimagic total labeling. The problem of deciding whether a given graph is antimagic is very difficult. Joseph A. Gallian introduced in [6] that Simanjuntak et al conjecture: 1. paths have no (a, d)-edge-antimagic vertex labeling with $d > 2$; 2. C_{2n} has a $(2n+3, 4)$-or a $(2n+4, 4)$-edge-antimagic total labeling; 3. C_{2n+1} has a $(n+4, 5)$-or a $(n+5, 5)$-edge-antimagic total labeling; 4.

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cycles have no (a,d)-edge-antimagic total labelings with $d > 5$. Bača et al [2] proved that paths have no (a,d)-edge-antimagic vertex labeling with $d > 2$. In this paper, we show that C_{2n} has a $(2n+4,4)$-edge-antimagic total labeling; cycles have no (a,d)-edge-antimagic total labelings with $d > 5$.

An (a,d)-edge-antimagic total labeling for a graph $G(V,E)$ as an injective mapping f from $V \cup E$ onto the set $\{1, 2, \cdots, |V| + |E|\}$ such that the set $\{f(v) + \sum f(uv) | uv \in E\}$ where v ranges over all of V is $\{a, a+d, a+2d, \cdots, a+(|V|-1)d\}$.

Suppose the sets $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_n\}$, therefore $A + B = \{a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n\}$.

The vertices of C_n are denoted in proper order: $x_1, x_2, \cdots, x_n$.

2. MAIN RESULTS

Theorem 1 . *C_{2n} has a $(2n+4,4)$-edge-antimagic total labeling.*

Lemma 1. *If n is even, C_{2n} has a $(2n+4,4)$ − edge − antimagic total labeling.*

Proof. when $n > 2$, we define the vertices and the edges of C_{2n} in the following way: Suppose

$A_1 = \{x_n x_1, x_1, x_1 x_2\}$,

$A_2 = \{x_2, x_2 x_3, x_3\}$,

$A_3 = \{x_3 x_4, x_4, x_4 x_5\}$,

$\cdots \quad \cdots \quad \cdots$

$A_{n-1} = \{x_{\frac{3n}{2}-3} x_{\frac{3n}{2}-2}, x_{\frac{3n}{2}-2}, x_{\frac{3n}{2}-2} x_{\frac{3n}{2}-1}\}$. Let

$f(A_1) = \{2n+1, 1, 2\}$,

$f(A_{k+1}) = f(A_k) + \{2, 2, , 2\}, \quad k = 1, 2, \cdots, n-2;$

we get $3(n-1)$ different numbers, and

$\cup_{k=1}^{n-1} f(A_k) = \{2n+1, 1, 2\} \cup \{2n+3, 3, 4\} \cup \cdots \cup \{4n-3, 2n-3, 2n-2\} = \{1, 2, \cdots, 2n-2\} \cup \{2n+1, 2n+3, \cdots, 4n-3\}$; so

$$f(x_1) + f(x_1 x_{2n}) + f(x_1 x_2) = 2n+4,$$

$$f(x_2) + f(x_1 x_2) + f(x_2 x_3) = 2n+8,$$

$$\cdots \quad \cdots \quad \cdots$$

$$f(x_{\frac{3n}{2}-2}) + f(x_{\frac{3n}{2}-2} x_{\frac{3n}{2}-3}) + f(x_{\frac{3n}{2}-2} x_{\frac{3n}{2}-1}) = 8n-8.$$

For the remaining vertices and edges,let

$$f(x_{\frac{3n}{2}-1}) = 4n-4,$$

$$f(x_{k+1}) = f(x_k) - 4 \quad k = \tfrac{3n}{2}-1, \tfrac{3n}{2}, \cdots, 2n-3,$$

$$f(x_{2n-1}) = 2n-1, \quad f(x_{2n}) = 4n,$$

$$f(x_{\frac{3n}{2}-1} x_{\frac{3n}{2}}) = 2n+2,$$

$$f(x_k x_{k+1}) = f(x_{k-1} x_k) + 4 \quad k = \tfrac{3n}{2}, \tfrac{3n}{2}+1, \cdots, 2n-2,$$

$$f(x_{2n-1} x_{2n}) = 4n-1;$$

thus, we obtain

$\cup_{k=\frac{3n}{2}-1}^{2n} f(x_k) = \{4n-4, 4n-8, \cdots, 2n-4, 2n, 2n-1, 4n\}$,

there are $\frac{n}{2}+2$ different numbers;

$\cup_{k=\frac{3n}{2}}^{2n} f(x_{k-1} x_k) = \{2n+2, 2n+6, \cdots, 4n-2, 4n-1\}$,

there are $\frac{n}{2}+1$ different numbers, these $n+3$ numbers are different, and these differ with the above $3(n-1)$ numbers too.

$$f(x_{\frac{3n}{2}-1}) + f(x_{\frac{3n}{2}-1} x_{\frac{3n}{2}-2}) + f(x_{\frac{3n}{2}-1} x_{\frac{3n}{2}}) = 8n-4,$$

$$f(x_{\frac{3n}{2}}) + f(x_{\frac{3n}{2}-1} x_{\frac{3n}{2}}) + f(x_{\frac{3n}{2}} x_{\frac{3n}{2}+1}) = 8n,$$

$$\cdots \quad \cdots \quad \cdots$$

$$f(x_{2n}) + f(x_{2n-1} x_{2n}) + f(x_{2n} x_1) = 4n + 4n - 1 + 2n + 1 = 10n.$$

We obtain

$$f(V \cup E) = \{1, 2, \cdots, 4n\},$$

$$\{f(v) + \sum f(uv) | uv \in E\} = \{2n+4, 2n+8, \cdots, 10n\}.$$

Hence, C_{2n} $(n > 2)$ has a $(2n+4, 4)$-edge-antimagic total labeling; when $n = 2$, the vertices and the edges labelings of C_4 are:

$$f(x_1) = 1, \quad f(x_2) = 4, \quad f(x_3) = 3, \quad f(x_4) = 8,$$

$$f(x_1 x_2) = 2, \quad f(x_2 x_3) = 6, \quad f(x_3 x_4) = 7, \quad f(x_4 x_1) = 5.$$

Lemma 2. *If n is odd, C_{2n} has a $(2n+4,4)-edge-antimagic$ total labeling.*

Proof. when $n>3$, we define the vertex and the edge labeling of C_{2n} as follows:

$f(x_1)=1, \qquad f(x_2)=2n+3,$

$f(x_3)=5, \qquad f(x_4)=4,$

$f(x_1x_{2n})=2n+1, \quad f(x_1x_2)=2,$

$f(x_2x_3)=3, \qquad f(x_3x_4)=2n+4,$

$f(x_5)=2n,$

$f(x_{k+1})=f(x_k)-4 \qquad k=5,6,\cdots,\frac{n+7}{2};$

$f(x_4x_5)=8,$

$f(x_kx_{k+1})=f(x_{k-1}x_k)+4, \qquad k=5,6,\cdots,\frac{n+7}{2};$

so we have

$\cup_{k=1}^{\frac{n+7}{2}} f(x_k)=\{1,2n+3,5,4\}\cup\{2n2n-4,\cdots,10,6\},$

there are $\frac{n+7}{2}$ different numbers;

$f(x_1x_{2n})\cup_{k=1}^{\frac{n+7}{2}} f(x_kx_{k+1})=\{2n+1,2,3,2n+4\}\cup\{8,12,\cdots,$
$2n+6\};$

these are $\frac{n+7}{2}+1$ different numbers, and these with th above $\frac{n+7}{2}$ numbers are different too.

$f(x_1)+f(x_1x_{2n})+f(x_1x_2)=2n+4,$

$f(x_2)+f(x_1x_2)+f(x_2x_3)=2n+8,$

$\cdots \quad \cdots \quad \cdots$

$f(x_{\frac{n+7}{2}})+f(x_{\frac{n+7}{2}-1}x_{\frac{n+7}{2}})+f(x_{\frac{n+7}{2}}x_{\frac{n+7}{2}+1})=4n+14.$

For the remaining $3(n-3)+1$ vertices and edges, we construct label as follows: let

$f(x_{\frac{n+7}{2}+1})=2n+5,$ suppose

$A_1=\{x_{\frac{n+7}{2}+1}x_{\frac{n+7}{2}+2}, x_{\frac{n+7}{2}+2}, x_{\frac{n+7}{2}+2}x_{\frac{n+7}{2}+3}\},$

$A_2=\{x_{\frac{n+7}{2}+3}, x_{\frac{n+7}{2}+3}x_{\frac{n+7}{2}+4}, x_{\frac{n+7}{2}+4}\},$

$\cdots \quad \cdots \quad \cdots$

$A_{n-3} = \{x_{2n-1}, x_{2n-1}x_{2n}, x_{2n}\}$. Let

$f(A_1) = \{7, 2n+8, 2n+7\}$,

$f(A_{k+1}) = f(A_k) + \{2, 2, , 2\}, \quad k = 1, 2, \cdots, n-4;$

we obtain $3(n-3)$ numbers:

$f(A_1) = \{7, 2n+8, 2n+7\}$,

$f(A_2) = \{9, 2n+10, 2n+9\}$,

$\cdots \quad \cdots \quad \cdots$

$f(A_{n-3}) = \{2n-1, 4n, 4n-1\}$;

these numbers differ,as

$\cup_{k=1}^{n-3} f(A_k) = \{7, 9, \cdots, 2n-1\} \cup \{2n+7, 2n+8, \cdots, 4n\}$. And

$f(x_{\frac{n+7}{2}+1}) + f(x_{\frac{n+7}{2}} x_{\frac{n+7}{2}+1}) + f(x_{\frac{n+7}{2}+1} x_{\frac{n+7}{2}+2}) = 4n + 18,$

$f(x_{\frac{n+7}{2}+2}) + f(x_{\frac{n+7}{2}+1} x_{\frac{n+7}{2}+2}) + f(x_{\frac{n+7}{2}+2} x_{\frac{n+7}{2}+3}) = 4n + 22,$

$\cdots \quad \cdots \quad \cdots$

$f(x_{2n}) + f(x_{2n-1}x_{2n}) + f(x_{2n}x_1) = 10n.$

Therefore, we obtain

$f(V \cup E) = \{1, 2, \cdots, 4n\}$,

$\{f(v) + \sum f(uv) | uv \in E\} = \{2n+4, 2n+8, \cdots, 10n\}$.

Hence, C_{2n} $(n > 3)$ has a $(2n+4, 4)$-edge-antimagic total labeling; when $n = 3$, the vertices and the edges labelings of C_6 are:

$f(x_1) = 1, \quad f(x_2) = 9, \quad f(x_3) = 5,$

$f(x_4) = 4, \quad f(x_5) = 6, \quad f(x_6) = 11,$

$f(x_1x_2) = 2, \quad f(x_2x_3) = 3, \quad f(x_3x_4) = 10,$

$f(x_4x_5) = 8. \quad f(x_5x_6) = 12, \quad f(x_6x_1) = 7.$

Overall, C_{2n} has a $(2n+4, 4)$-edge-antimagic total labelings.

Theorem 2 . *Cycles have no (a, d)-edge-antimagic total labelings with $d > 5$.*

Proof. Suppose cycle has m vertices, the set

$$\{1, 2, \cdots, |V| + |E|\} = \{1, 2, \cdots, 2m\},$$

in this set, the maximum of the sum of arbitrary three numbers is $6m-3$, the minimum is 6; in the set $\{a, a+d, a+2d, \cdots, a+(|V|-1)d\}$, the maximum is $a+(m-1)d$. If the cycle has a (a,d)-edge-antimagic total labeling, then $a>5$; suppose $d>5$

$$a+(m-1)d \geq 6+(m-1)d \geq 6+(m-1)6 = 6m > 6m-3,$$

contradict. hence, cycles have no (a,d)-edge-antimagic total labelings with $d>5$.

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