Number of Embeddings of Wheel Graphs on Surfaces of Small Genus

Yan Yang
Department of Mathematics, Beijing Jiaotong University, Beijing 100044,
P.R.China
E-mail: yanyang0206@126.com

Yanpei Liu
Department of Mathematics, Beijing Jiaotong University, Beijing 100044,
P.R.China
E-mail: ypliu@center.njtu.edu.cn

Abstract In this paper, we obtain the numbers of embeddings of wheel graphs on some orientable and nonorientable surfaces of small genera, mainly on torus, double torus, and nonorientable surfaces of genus 1, 2, 3 and 4. These are the first results for embeddings of wheel graphs on nonorientable surfaces as known up to now.

Keywords genus, embedding, associate surface

1. Introduction

In topology, a *surface* is a compact 2-dimensional manifold without boundary. In fact, it can be seen as what is obtained by identifying each pair of edges on a polygon of even edges pairwise. The sphere is written as $O_0 = aa^-$ where a^- is with the opposite direction of a on the boundary of the polygon. Thus, the projective plane, torus and Klein bottle are, respectively, aa, aba^-b^- and $aabb$. In general,

$$O_p = \prod_{i=1}^{p} a_i b_i a_i^- b_i^- \tag{1}$$

and

$$Q_q = \prod_{i=1}^{q} a_i a_i \tag{2}$$

denote, respectively, a surface of orientable genus p and a surface of nonorientable genus q. The surfaces given by (1), (2) are called standard forms[9] of the surfaces. Of course, O_0, Q_1, O_1, Q_2 are, respectively, the sphere, projective plane, torus and Klein bottle.

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Let $\mathcal{S}$ be the collection of surfaces and let AB be a surface. The following three operations[9] and their inverses do not change the orientability and genus of a surface.

Operation 1: $Aaa^{-}B \Leftrightarrow AB$ where $a \notin AB$

Operation 2: $AabBab \Leftrightarrow AcBc$ where $c \notin AB$

Operation 3: $AB \Leftrightarrow (Aa)(a^{-}B)$ where $AB \neq \emptyset$

Notice that A and B are both linear orders of letters and permitted to be empty. The parentheses stand for cyclic order when more than one cyclic orders occur, for distinguishing from one to another. In fact, what is determined under these operations is just a topological equivalence $\sim$ on $\mathcal{S}$.

The following relations[9] can be deduced by using Operations $1-3$.

Relation 1: $(AxByCx^{-}Dy^{-}) \sim ((ADCB)(xyx^{-}y^{-}))$

Relation 2: $(AxBx) \sim ((AB^{-})(xx))$

Relation 3: $(Axxyzy^{-}z^{-}) \sim ((A)(xx)(yy)(zz))$

In the there relations, A, B, C, and D are all linear orders of letters and permitted to be empty. $B^{-} = b_s \cdots b_3b_2b_1$ is also called the inverse of $B = b_1b_2b_3 \cdots b_s, s \geq 1$. Parentheses are always omitted when they are unnecessary to distinguish cyclic or linear order.

An *embedding* of a graph G into a surface S is a homeomorphism $h: G \to S$ of G into S such that every component of $S - h(G)$ is a 2-cell.

Given a graph, how many distinct embeddings does it have on each orientable surface . This problem was inaugurated by Gross and Furst[3]. Gross et al.[4] solved it for bouquets of circles; Furst et al.[2] for closed-end ladders and cobblestone paths; Kwak et al.[5] for dipoles; and Tesar[13] for Ringel ladders, etc. Chen et al.[1] generalized this problem to nonorientable surface, and calculated the total genus distribution of necklace, closed-end ladders and cobblestone paths; Kwak et al.[6] for bouquets of circles and dipoles. And Liu[8] gave the number of embeddings of a graph on the plane.

In the following, we will introduce the joint tree model of a graph embedding, this theory established in [9] by Liu, based on his initial work in 1979[7]. By using joint trees, Wan and Liu[15,16] calculated orientable embedding distributions for certain type of non-planar graphs.

Given a spanning tree of a graph G, the edge set E can be partitioned into E_T(tree edge) and $\overline{E}_T$(cotree edge), *i.e.*,$E = E_T + \overline{E}_T$. Let $\overline{E}_T = \{e_i | i = 1, 2, \cdots, \beta\}$, $\beta = \beta(G)$ be the Betti number of G. Distinguish all edges in $\overline{E}_T$ by letters. For each $e_i = (u[e_i], v[e_i]) \in \overline{E}_T$, splitting it into two semi-edges e_{u_i}, e_{v_i}, which incident with $u[e_i], v[e_i]$ respectively. The two semi-edges labelled with same letters as e_i. Write $G' = (V + V_1, E_T + E_1)$, where $V_1 = \{v_i, \bar{v}_i | 1 \leq i \leq \beta\}$ and $E_1 = \{(u[e_i], v_i), (v[e_i], \bar{v}_i) | 1 \leq i \leq \beta\}$. G' is a tree, denoted by $\hat{T}$. According to the rotation at each vertex, all lettered semi-edges of $\hat{T}$ form a polygon with β pair of edges.

Let $\delta = (\delta_1, \delta_2, \cdots, \delta_\beta)$ be a binary vector. Denote by $\hat{T}^\delta$ that edges $(u[e_i], v_i)$,

$(v[e_i], \bar{v}_i)$,$1 \le i \le \beta$, in $\hat{T}$ are labelled by same letter with indices: +(always omitted) or −, where $\delta_i = 0$ means that the two indices are the different; otherwise, same. Then, δ is called an assignment of indices on $\hat{T}$.

A *rotation* at a vertex v, denoted by σ_v, is a cyclic permutation of semi-edges incident with v. Let σ_G be a rotation system of G, then $\sigma_G = \prod_{v \in V(G)} \sigma_v$.

Given a rotation σ_G for a graph G, the joint tree of G is $\hat{T}^\delta_\sigma$. According to the rotation at each vertex of $\hat{T}^\delta_\sigma$, all lettered semi-edges with indices of them form a surface. We call it is a associate surface of G, denoted by $F(\hat{T}^\delta_\sigma)$.

Two associate surfaces of G are the same is meant that they have the same cyclic order with same δ. Otherwise, distinct.

Let $\mathcal{F}(G)$ be the set of all distinct associate surfaces of G , and Let $\mathcal{F}_p(G; i)$, $\mathcal{F}_q(G; j)$, $i \ge 0, j \ge 1$ be the set of all distinct associate surfaces of orientable genus i and the set of nonorientable genus j, respectively.

The lemmas following are all proved by Liu[10], and they are very useful in the following sections.

Lemma 1.1[10] *$F(\hat{T}^\delta_\sigma)$ is orientable if and only if $\delta = 0$.*

Lemma 1.2[10] *An orientable surface is a surface of orientable genus 0 if and only if there is no form as $AxByCx^{-1}Dy^{-1}$ in it.*

Lemma 1.3[10] *An orientable surface $AxByCx^{-1}Dy^{-1}$ is a surface of genus $k (k \ge 1)$ if and only if the surface $(ADCB)$ is orientable, and the genus of it is $k - 1$.*

Lemma 1.4[10] *A nonorientable surface $(AxBx)$ is a surface of nonorientable genus $k(k \ge 1)$ if and only if the surface (AB^{-1}) is nonorientable with genus $k - 1$, when k is even; the surface (AB^{-1}) is nonorientable with genus $k - 1$ or (AB^{-1}) is orientable with genus $\frac{k-1}{2}$, when k is odd.*

Lemma 1.5[10] *For any integer $i \ge 0$(or $j \ge 1$), the cardinality $|\mathcal{F}_p(G; i)|$(or $|\mathcal{F}_q(G; j)|$) is independent of the choice of tree T on G. Further, it is the number of distinct embeddings of G on surfaces of orientable genus i(or nonorientable genus j).*

Lemma 1.6[10] *There is a 1-to-1 correspondence between associate surfaces and embeddings of a graph.*

For $n \ge 3$, the wheel graph of n spokes is the graph W_n obtained from the cycle C_n by adding a new vertex and joining it to all vertices of C_n. In this paper, we get $|\mathcal{F}_p(W_n; 1)|$, $|\mathcal{F}_p(W_n; 2)|$ and $|\mathcal{F}_q(W_n; j)|, 1 \le j \le 4$. Those are the numbers of embeddings of W_n on torus, double torus, projective plane, Klein bottle and the surfaces of nonorientable genus 3 and 4. All of the results are in explicit expressions.

There are three main reasons why we do these research. Firstly, wheel graph have some special characters, which had been proved very useful in the research of 3-connected graph[14]. Ren[11] investigate the flexibility of wheel graphs on torus. Stahl[12] obtained some asymptotic estimates on the region distribution of wheel graph W_n on orientable surface, but the exact numbers of embeddings of W_n on each surface can't be obtained from their results. Secondly, there is no result about embedding of wheel graph on nonorientable surfaces as we known up to now. Thirdly, the number of embeddings on surfaces of higher genera is much depend on those of lower genera, so it will be helpful for the further research of embeddings on the surfaces of higher genera.

Let S be a set of surfaces, $g_p(S), \tilde{g}_q(S), p \geq 0, q \geq 1$ be the numbers of surfaces of the orientable genus p and the nonorientable genus q in S, respectively.

2. The associate surfaces of W_n and some lemmas

In this section, we classify all of the associate surfaces of W_n into n types. Let $a_1, \ldots, a_{n-1}$ are letters with binary indices, A, B, C and D are all linear orders of these letters and permitted to be empty. We define 9 surface sets as follows:

$$T_1^{n-1} = \{a_1a_2\cdots a_{n-1}A\}, \text{ in which } |A| = n-1;$$

$$T_2^{n-1} = \{a_1a_2\cdots a_{n-1}AB\}, \text{ in which } |A| + |B| = n-1;$$

$$T_3^{n-1} = \{a_1a_2\cdots a_{n-1}ABC\}, \text{ in which } |A| + |B| + |C| = n-1;$$

$$T_4^{n-1} = \{a_1a_2\cdots a_{n-1}ABCD\}, \text{ in which } |A| + |B| + |C| + |D| = n-1;$$

$$T_5^{n-2} = \{a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B\}, \text{ in which } |A| + |B| = n-2, \ 1 \leq i \leq n-3;$$

$$T_6^{n-2} = \{a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}BC\}, \text{in which } |A|+|B|+|C| = n-2, \ 1 \leq i \leq n-3;$$

$$T_7^{n-4} = \{a_i\cdots a_1ABa_{i+1}\cdots a_{n-4}CD\}, \ T_8^{n-4} = \{a_i\cdots a_1ABCa_{i+1}\cdots a_{n-4}D\},$$

in which $|A| + |B| + |C| + |D| = n-4, \ 1 \leq i \leq n-5$;

$$T_9^{n-4} = \{a_1\cdots a_{i-1}Aa_i\cdots a_{j-1}Ba_j\cdots a_{n-4}C\}$$

in which $|A| + |B| + |C| + |D| = n-4, \ i \geq 2, \ j-i \geq 1$ and $n-j \geq 4$.

The numbers of surfaces of small genera in these sets are investigated, all the results that we obtain are in explicit expressions.

For wheel graph, the Betti number of W_n is $\beta(W_n) = |E| - |V| + 1 = n$. Let $a_0, \cdots, a_3$ be cotree edges of W_4, according to the way to get a joint tree of a graph mentioned in Section 1, and the rotations of $v_1, \cdots, v_4$, we draw 16 trees of W_4 in Fig.1. For each of the 16 trees, once the rotation of v_0 and $\delta = (\delta(a_0), \cdots, \delta(a_3))$ are given, we will get a joint tree of W_4. Because there are 3! different rotations of v_0, and 2^4 different binary vectors of δ. We will get $16 \times 3! \times 2^4$ different joint trees of W_4, which is equal to the number of embeddings of W_4 on all surfaces that it can be embedded.

According to the rotation σ of W_n and the indices assignment $\delta = (\delta_0, \cdots, \delta_{n-1})$ for n cotree edges, the associate surfaces of W_n can be classified into n types as follows:

$M_0 = \{a_0 a_1 \cdots a_{n-1} a_0^{\varepsilon_0} A\}$,$M_1 = \{a_1 a_0 a_2 \cdots a_{n-1} a_0^{\varepsilon_0} A\}, \cdots,$
$M_i = \{a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-1} a_0^{\varepsilon_0} A\}, \cdots,$
$M_{n-2} = \{a_{n-2} \cdots a_1 a_0 a_{n-1} a_0^{\varepsilon_0} A\}$, $M_{n-1} = \{a_{n-1} \cdots a_1 a_0 a_0^{\varepsilon_0} A\}$.
in which $a_1^{\varepsilon_1}, \cdots, a_{n-1}^{\varepsilon_{n-1}} \in A$, and $|A| = n-1$. For $0 \le i \le n-1$

$$\varepsilon_i = \begin{cases} +(\text{always omit}) & \text{when } \delta(a_i) = 1; \\ - & \text{when } \delta(a_i) = 0. \end{cases}$$

Since we have $\binom{n-1}{i}$ ways to choose $a_i, \cdots, a_1$ from $\{a_1, \cdots a_{n-1}\}$, and two ways to place $a_0, a_0^{\varepsilon_0}$, the number of type M_i is $2\binom{n-1}{i}, (0 \le i \le n-1)$.

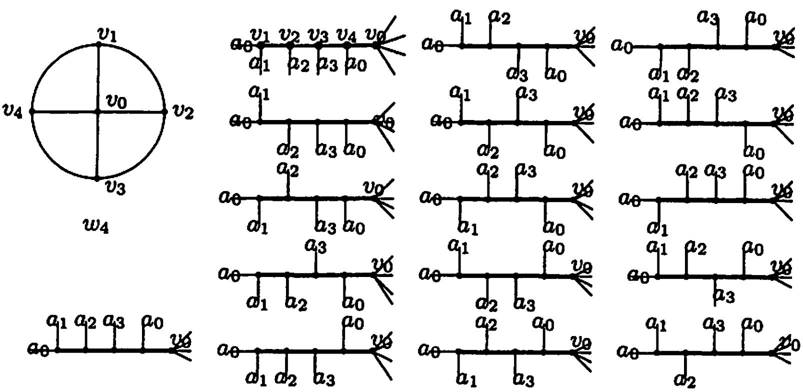

Fig. 1

Lemma 2.1 *The 4 sets of surfaces:*

$$T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A\}, \quad T_2^{n-1} = \{a_1 a_2 \cdots a_{n-1} AB\},$$

$$T_3^{n-1} = \{a_1 a_2 \cdots a_{n-1} ABC\}, \quad T_4^{n-1} = \{a_1 a_2 \cdots a_{n-1} ABCD\}$$

the numbers of planes in the 4 sets are

$$g_0(T_1^{n-1}) = 1,\ g_0(T_2^{n-1}) = n,\ g_0(T_3^{n-1}) = \frac{n(n+1)}{2},\ g_0(T_4^{n-1}) = \frac{n(n+1)(n+2)}{6},$$

respectively.

Proof According to Lemma 1.2 and Operation 1, there is only one surface in T_1^{n-1} is a plane, that is $a_1 a_2 \cdots a_{n-1} a_{n-1}^{-1} \cdots a_2^- a_1^-$. So as for the numbers of planes in $T_2^{n-1}, T_3^{n-1}, T_4^{n-1}$, they are equal to the numbers of non-negative integral solutions of the equations $x_1 + x_2 + \cdots + x_r = n-1$, $r = 2, 3, 4$, respectively. And the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_r = n-1$ is $\binom{r+n-2}{n-1}$. For $r = 2, 3, 4$, we can get $g_0(T_2^{n-1}) = \binom{2+n-2}{n-1}, g_0(T_3^{n-1}) = \binom{3+n-2}{n-1}, g_0(T_4^{n-1}) = \binom{4+n-2}{n-1}$, the lemma is obtained.

Lemma 2.2 *The 5 sets of surfaces:*

$$T_5^{n-2} = \{a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B\}, \qquad T_6^{n-2} = \{a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} BC\},$$

$$T_7^{n-4} = \{a_i \cdots a_1 AB a_{i+1} \cdots a_{n-4} CD\}, \quad T_8^{n-4} = \{a_i \cdots a_1 ABC a_{i+1} \cdots a_{n-4} D\},$$

$$T_9^{n-4} = \{a_1 \cdots a_{i-1} A a_i \cdots a_{j-1} B a_j \cdots a_{n-4} C\},$$

the numbers of planes in the 5 sets are

$$g_0(T_5^{n-2}) = (i+1)(n-i-1), \qquad g_0(T_6^{n-2}) = \frac{(i+1)(n-i-1)n}{2},$$

$$g_0(T_7^{n-4}) = \frac{1}{6}(i+1)(n-i-3)\big(n^2 - 3n + (n-4)i - i^2 + 2\big),$$

$$g_0(T_8^{n-4}) = \frac{1}{12}(i+1)(n-i-3)\big(2n^2 - (6+i)n + i^2 + 4i + 4\big),$$

and

$$g_0(T_9^{n-4}) = \frac{1}{2} i(j-i+1)(n-j-2)(n-2),$$

respectively.

Proof According to Lemmas 1.1, 1.2, T_5^{n-2} can be written as

$$\{a_i \cdots a_1 A_1 A_2 a_{i+1} \cdots a_{n-2} B_2 B_1\}$$

in which $a_i^-, \ldots, a_1^- \in A_1 + B_1$, $a_{i+1}^-, \ldots, a_{n-2}^- \in A_2 + B_2$. And

$$a_i \cdots a_1 A_1 A_2 a_{i+1} \cdots a_{n-2} B_2 B_1 \sim O_0$$

$$\Leftrightarrow B_1 a_i \cdots a_1 A_1 \sim O_0 \ \text{ and } \ A_2 a_{i+1} \cdots a_{n-2} B_2 \sim O_0.$$

From Lemma 2.1, we can get that

$$g_0(T_5^{n-2}) = g_0(T_2^i) g_0(T_2^{n-i-2}) = (i+1)(n-i-1).$$

By Lemmas 1.1, 1.2, T_6^{n-2} can be written as

$$\{a_i \cdots a_1 A_1 A_2 a_{i+1} \cdots a_{n-2} B_2 B_1 C_2 C_1\}$$

in which $a_i^-, \ldots, a_1^- \in A_1 + B_1 + C_1$, $a_{i+1}^-, \ldots, a_{n-2}^- \in A_2 + B_2 + C_2$ and $|B_1| \times |C_2| = 0$.
Case 1 $|B_1| = 0$.

$$a_i \cdots a_1 A_1 A_2 a_{i+1} \cdots a_{n-2} B_2 C_2 C_1 \sim O_0$$

$$\Leftrightarrow C_1 a_i \cdots a_1 A_1 \sim O_0 \ \text{ and } \ A_2 a_{i+1} \cdots a_{n-2} B_2 C_2 \sim O_0.$$

In this case, the number of planes in T_6^{n-2} is

$$g_0(T_2^i) g_0(T_3^{n-i-2}) = \frac{(i+1)(n-i-1)(n-i)}{2}.$$

Case 2 $|B_1| \neq 0$.
According to Lemma 1.2, $|C_2| = 0$, otherwise, the genus will be more than 0, and

$$a_i \cdots a_1 A_1 A_2 a_{i+1} \cdots a_{n-2} B_2 B_1 C_1 \sim O_0$$

$$\Leftrightarrow C_1 B_1 a_i \cdots a_1 A_1 \sim O_0 \text{ and } A_2 a_{i+1} \cdots a_{n-2} B_2 \sim O_0.$$

Notice that $|B_1| \neq 0$, in this case, the number of planes in T_6^{n-2} is

$$\begin{aligned}(g_0(T_3^i) - g_0(T_2^i))g_0(T_2^{n-i-2}) &= \Big(\frac{(i+1)(i+2)}{2} - (i+1)\Big)(n-i-1) \\ &= \frac{(n-i-1)(i+1)i}{2}\end{aligned}$$

Summarizing Cases 1,2, we get that

$$\begin{aligned}g_0(T_6^{n-2}) &= \frac{(i+1)(n-i-1)(n-i)}{2} + \frac{(n-i-1)(i+1)i}{2} \\ &= \frac{(i+1)(n-i-1)n}{2}.\end{aligned}$$

In a similar way, we can get that

$$g_0(T_7^{n-4}) = \frac{1}{6}(i+1)(n-i-3)\big(n^2 - 3n + (n-4)i - i^2 + 2\big),$$

$$g_0(T_8^{n-4}) = \frac{1}{12}(i+1)(n-i-3)\big(2n^2 - (6+i)n + i^2 + 4i + 4\big),$$

and

$$g_0(T_9^{n-4}) = \frac{1}{2}i(j-i+1)(n-j-2)(n-2),$$

respectively.

Lemma 2.3 *The number of torus in surface set* $T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A\}$ *is*

$$g_1(T_1^{n-1}) = \frac{(n-2)(n-1)n(n+1)}{24}.$$

Proof **Case 1** $A = a_{n-1}^{-1} A'$

$$a_1 a_2 \cdots a_{n-1} A \sim a_1 a_2 \cdots a_{n-2} a_{n-1} a_{n-1}^{-} A' \sim a_1 a_2 \cdots a_{n-2} A' = T_1^{n-2}$$

by Operation 1. In this case, the number of torus in T_1^{n-1} is equal to the number of torus in T_1^{n-2}.

Case 2 $A = A_1 a_j^{-} a_{n-1}^{-} A_2, j \in [1, n-2]$

$$\begin{aligned}a_1 a_2 \cdots a_{n-1} A &\sim a_1 a_2 \cdots a_j \cdots a_{n-1} A_1 a_j^{-} a_{n-1}^{-} A_2 \\ &\sim a_1 a_2 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 a_j a_{n-1} a_j^{-} a_{n-1}^{-},\end{aligned}$$

by Relation 1. And according to Lemma 1.3,

$$a_1a_2\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2a_ja_{n-1}a_j^-a_{n-1}^-\sim O_1$$

$$\Leftrightarrow a_1a_2\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2\sim O_0$$

Let $j-1=i$, then $a_1\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2=a_1\cdots a_iA_1a_{i+2}\cdots a_{n-2}A_2$

$$\sim a_1\cdots a_iA_1a_{i+1}\cdots a_{n-3}A_2=T_5^{n-3}.$$

From Lemma 2.2, in this case, the number of torus is $\sum_{j=1}^{n-2} j(n-1-j)=\frac{(n-2)(n-1)n}{6}$.

Summarizing Cases 1,2, $g_1(T_1^{n-1})=g_1(T_1^{n-2})+\frac{(n-2)(n-1)n}{6}$. Since it is easily to get that $g_1(T_1^2)=1$, *i.e.*, $a_1a_2A=a_1a_2a_1^-a_2^-$, the number of torus in T_1^{n-1} is

$$g_1(T_1^{n-1})=1+\sum_{i=4}^{n}\frac{(i-2)(i-1)i}{6}=\frac{(n-2)(n-1)n(n+1)}{24}.$$

Lemma 2.4 *The number of torus in surface set*

$$T_5^{n-2}=\{a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B\}$$

is

$$g_1(T_5^{n-2})=\frac{1}{24}(n-2)(n-1)(n-i-1)(i+1)\left(n^2-(2i+3)n+2i^2+4i\right).$$

Proof We can classify T_5^{n-2} to two sets $T_{5_1}^{n-2}$ and $T_{5_2}^{n-2}$, in which

$$T_{5_1}^{n-2}=\{a_i\cdots a_1A_1a_{n-2}^-A_2a_{i+1}\cdots a_{n-2}B\}$$

and

$$T_{5_2}^{n-2}=\{a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B_1a_{n-2}^-B_2\}.$$

Of course $|A_1|+|A_2|+|B|=n-3$ and $|A|+|B_1|+|B_2|=n-3$. We will discuss $T_{5_1}^{n-2}$ and $T_{5_2}^{n-2}$ respectively in the following.

In the set $T_{5_1}^{n-2}$, there are three possible ways to place a_{n-3}^-, so we can classify $T_{5_1}^{n-2}$ into $T_{5_{1,1}}^{n-2}$, $T_{5_{1,2}}^{n-2}$ and $T_{5_{1,3}}^{n-2}$, *i.e.*,

$$T_{5_{1,1}}^{n-2}=\{a_i\cdots a_1A_{11}a_{n-3}^{-1}A_{12}a_{n-2}^-A_2a_{i+1}\cdots a_{n-3}a_{n-2}B\}$$

$$T_{5_{1,2}}^{n-2}=\{a_i\cdots a_1A_1a_{n-2}^-A_2a_{i+1}\cdots a_{n-3}a_{n-2}B_1a_{n-3}^-B_2\}$$

$$T_{5_{1,3}}^{n-2}=\{a_i\cdots a_1A_1a_{n-2}^-A_{21}a_{n-3}^-A_{22}a_{i+1}\cdots a_{n-3}a_{n-2}B\}$$

From Relation 1,

$$a_i\cdots a_1A_{11}a_{n-3}^-A_{12}a_{n-2}^-A_2a_{i+1}\cdots a_{n-3}a_{n-2}B$$

$$\sim a_i\cdots a_1A_{11}A_2a_{i+1}\cdots a_{n-4}A_{12}Ba_{n-3}a_{n-2}a_{n-3}^-a_{n-2}^-$$

and

$$a_i \cdots a_1 A_1 a_{n-2}^- A_2 a_{i+1} \cdots a_{n-3} a_{n-2} B_1 a_{n-3}^- B_2$$
$$\sim a_i \cdots a_1 A_1 B_1 A_2 a_{i+1} \cdots a_{n-4} B_2 a_{n-3} a_{n-2} a_{n-3}^- a_{n-2}^-$$

By Lemma 1.3, we get

$$g_1(T_{5_{1,1}}^{n-2}) = g_0(T_7^{n-4}) = \frac{1}{6}(i+1)(n-i-3)\big(n^2-3n+(n-4)i-i^2+2\big),$$

and

$$g_1(T_{5_{1,2}}^{n-2}) = g_0(T_8^{n-4}) = \frac{1}{12}(i+1)(n-i-3)\big(2n^2-(6+i)n+i^2+4i+4\big).$$

Now we discuss $T_{5_{1,3}}^{n-2}$. If $|A_{21}| = 0$, according to Operation 2,

$$a_i \cdots a_1 A_1 a_{n-2}^- a_{n-3}^- a_{i+1} \cdots a_{n-3} a_{n-2} B$$
$$\sim a_i \cdots a_1 A_1 a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} B = T_{5_1}^{n-3}.$$

If $|A_{21}| \neq 0$, then $A_{21} = a_j A'_{21}, j \in [1, i]$ or $j \in [i+1, n-4]$.
1)when $j \in [1, i]$,

$$a_i \cdots a_j \cdots a_1 A_1 a_{n-2}^- a_j A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B$$
$$\sim a_i \cdots a_{j+1} A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{j-1} \cdots a_1 A_1 B a_j a_{n-2} a_j^- a_{n-2}^-$$

by Relation 1. And according to Lemmas 1.2, 1.3,

$$a_i \cdots a_{j+1} A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{j-1} \cdots a_1 A_1 B a_j a_{n-2} a_j^- a_{n-2}^- \sim O_1$$
$$\Leftrightarrow A_{22} a_{i+1} \cdots a_{n-4} \sim O_0 \quad \text{and} \quad a_{j-1} \cdots a_1 A_1 B a_i \cdots a_{j+1} A_2 \sim O_0.$$

2) when $j \in [i+1, n-4]$,
in the same way, we can get that

$$a_i \cdots a_1 A_1 a_{n-2}^- a_j A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_j \cdots a_{n-3} a_{n-2} B \sim O_1$$
$$\Leftrightarrow a_i \cdots a_1 A_1 B \sim O_0 \quad \text{and} \quad a_{j+1} \cdots a_{n-4} A_{22} a_{i+1} \cdots a_{j-1} A'_{21} \sim O_0.$$

By applying Lemmas 2.1, 2.2, we have

$$\begin{aligned} g_1(T_{5_{1,3}}^{n-2}) &= g_1(T_{5_1}^{n-3}) + \sum_{j=1}^{i} \frac{j(i+1-j)(i+1)}{2} \\ &\quad + \sum_{j=i+1}^{n-4} (i+1)(j-i)(n-j-3). \end{aligned}$$

Summarizing the above, we have

$$\begin{aligned} g_1(T_{5_1}^{n-2}) &= g_1(T_{5_{1,1}}^{n-2}) + g_1(T_{5_{1,2}}^{n-2}) + g_1(T_{5_{1,3}}^{n-2}) \\ &= g_1(T_{5_1}^{n-3}) + \frac{1}{12}(n-2)(i+1)\big(4i^2 + i(23-9n) + 6(6-5n+n^2)\big). \end{aligned}$$

In order to get $g_1(T_{5_1}^{n-2})$, we have to know $g_1(T_{5_1}^{i+1})$, $T_{5_1}^{i+1} = \{a_i \cdots a_1 A_1 a_{i+1}^- A_2 a_{i+1} B\}$. Consider set $T_{5_1}^{i+1}$. If $|A_2| = 0$, then

$$a_i \cdots a_1 A_1 a_{i+1}^- a_{i+1} B \sim a_i \cdots a_1 A_1 B = T_2^i$$

If $|A_2| \neq 0$, then $A_2 = a_j^- A_2'$, $j \in [1, i]$ and A_2' can be empty.

$$a_i \cdots a_j \cdots a_1 A_1 a_{i+1}^- a_j^- A_2' a_{i+1} B \sim a_i \cdots a_{j+1} A_2' a_{j-1} \cdots a_1 A_1 B a_j a_{i+1} a_j^- a_{i+1}^-$$

According to lemmas 1.3, 2.1, 2.2 and 2.3, we have

$$\begin{aligned} g_1(T_{5_1}^{i+1}) &= (i+1) g_1(T_1^i) + \sum_{j=1}^{i} \frac{j(i+1-j)(i+1)}{2} \\ &= (i+1)\frac{(i-1)i(i+1)(i+2)}{24} + \frac{i(i+1)^2(i+2)}{12} \\ &= \frac{i(i+1)^3(i+2)}{24}. \end{aligned}$$

So

$$\begin{aligned} g_1(T_{5_1}^{n-2}) &= g_1(T_{5_{1,1}}^{n-2}) + g_1(T_{5_{1,2}}^{n-2}) + g_1(T_{5_{1,3}}^{n-2}) \\ &= \frac{i(i+1)^3(i+2)}{24} \\ &\quad + \sum_{j=i+4}^{n} \frac{1}{12}(j-2)(i+1)\big(4i^2 + i(23-9j) + 6(6-5j+j^2)\big) \\ &= \frac{1}{24}(n-2)(n-1)(i+1)\big(3n^2 - (13+6i)n + 4i^2 + 14i + 12\big). \end{aligned}$$

In a similar way, we can get

$$g_1(T_{5_2}^{n-2}) = \frac{1}{24}(i+1)(n-i-2)(n-2)(n-1)\big(n^2 - (2i+5)n + 2i^2 + 6i + 6\big).$$

Hence

$$\begin{aligned} g_1(T_5^{n-2}) &= g_1(T_{5_1}^{n-2}) + g_1(T_{5_2}^{n-2}) \\ &= \frac{1}{24}(n-2)(n-1)(n-i-1)(i+1)\big(n^2 - (2i+3)n + 2i^2 + 4i\big). \end{aligned}$$

Lemma 2.5 *The number of double torus in surface set $T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A\}$ is*

$$g_2(T_1^{n-1}) = \frac{1}{5760}(n-4)(n-3)(n-2)(n-1)n(n+1)(3n^2 - n - 6).$$

Proof Similar argument with the proof of Lemma 2.3.

Case 1 $A = a_{n-1}^- A'$. $a_1 a_2 \cdots a_{n-2} a_{n-1} a_{n-1}^- A' \sim a_1 a_2 \cdots a_{n-2} A' = T_1^{n-2}$.

Case 2 $A = A_1 a_j^- a_{n-1}^- A_2, j \in [1, n-2]$.

$a_1 a_2 \cdots a_j \cdots a_{n-1} A_1 a_j^- a_{n-1}^- A_2 \sim a_1 a_2 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 a_j a_{n-1} a_j^- a_{n-1}^-$.

In this case, for $1 \le j \le n-2$, with the application of Lemmas 1.3, 2.4, we get that the number of double torus is

$$\begin{aligned}
& \frac{1}{24}(n-3)(n-2) \\
& \times \sum_{j=1}^{n-2} (n-j-1)j((n-1)^2 - (2(j-1)+3)(n-1) + 2(j-1)^2 + 4(j-1)) \\
= \quad & \frac{1}{720}(n-4)(n-3)(n-2)^2(n-1)n(3n+1).
\end{aligned}$$

Summarizing Cases 1 and 2

$$g_2(T_1^{n-1}) = g_2(T_1^{n-2}) + \frac{1}{720}(n-4)(n-3)(n-2)^2(n-1)n(3n+1).$$

Since it is easy to get that $g_2(T_1^4) = 8$, $T_1^4 = \{a_1 \cdots a_4 A\}$, the number of double torus in T_1^{n-1} is

$$\begin{aligned}
g_2(T_1^{n-1}) &= 8 + \sum_{i=6}^{n} \frac{1}{720}(i-4)(i-3)(i-2)^2(i-1)i(3i+1) \\
&= \frac{(n-4)(n-3)(n-2)(n-1)n(n+1)(3n^2-n-6)}{5760}.
\end{aligned}$$

Lemma 2.6 *Let S be a nonorientable surface, if there is a form as $AxByCx^-Dy$ in S, then the genus of S will be not less than 3; if there is a form as $AxByCx^-Dy$ in S, then the genus of S will be not less than 2.*

Proof If the form as $AxByCx^-Dy^-$ exists in S, by Relation 1, $AxByCx^-Dy^- \sim ADCBxyx^-y^-$, and there is at least one pair of semi-edges z, z^ε with same indices, for S is nonorientable. By Relation 1-3, we can get $S \sim A'zzxyx^-y^- \sim A'zzxxyy$. So the genus will be not less than 3.

If the form as $AxByCx^-Dy$ exists, by using Relation 2 twice, we get that $AxByCx^-Dy \sim AxBD^-xC^-yy \sim ADB^-C^-xxyy$, so the genus of S will be not less than 2. Thus the proof is complete.

Lemma 2.7 *The numbers of projective planes and Klein bottles in the surface set $T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A\}$ are*

$$\tilde{g}_1(T_1^{n-1}) = \frac{(n-1)n}{2} \quad \text{and} \quad \tilde{g}_2(T_1^{n-1}) = \frac{(n-2)(n-1)n^2}{6},$$

respectively.

Proof We can classify T_1^{n-1} into two sets $T_{1_1}^{n-1}$ and $T_{1_2}^{n-1}$, according to $\delta(a_{n-1}) = 1$ or 0, *i.e.*,

$T_{1_1}^{n-1} = \{a_1a_2\cdots a_{n-1}A_1a_{n-1}A_2\}$ and $T_{1_2}^{n-1} = \{a_1a_2\cdots a_{n-1}A_1a_{n-1}^{-}A_2\}$. We will discuss the two sets respectively in the following.

1) $T_{1_1}^{n-1} = \{a_1a_2\cdots a_{n-1}A_1a_{n-1}A_2\}$

$$a_1a_2\cdots a_{n-1}A_1a_{n-1}A_2 \sim a_1a_2\cdots a_{n-2}A_1^{-}A_2a_{n-1}a_{n-1}$$

Applying Lemmas 1.4, 2.1, it is easy to get that

$$\tilde{g}_1(T_{1_1}^{n-1}) = n-1 \quad \text{and} \quad \tilde{g}_2(T_{1_1}^{n-1}) = (n-1)\tilde{g}_1(T_1^{n-2}).$$

2) $T_{1_2}^{n-1}\{a_1a_2\cdots a_{n-1}A_1a_{n-1}^{-}A_2\}$.

We can classify $T_{1_2}^{n-1}$ into $T_{1_{2,1}}^{n-1}, T_{1_{2,2}}^{n-1}$, and $T_{1_{2,3}}^{n-1}$, in which

$$T_{1_{2,1}}^{n-1} = \{a_1a_2\cdots a_{n-1}a_{n-1}^{-}A_1\},$$

$$T_{1_{2,2}}^{n-1} = \{a_1\cdots a_j\cdots a_{n-1}A_1a_j^{-}a_{n-1}^{-}A_2\}(1 \le j \le n-2),$$

and

$$T_{1_{2,3}}^{n-1} = \{a_1\cdots a_j\cdots a_{n-1}A_1a_ja_{n-1}^{-}A_2\}(1 \le j \le n-2).$$

From Operation 1, $a_1a_2\cdots a_{n-2}a_{n-1}a_{n-1}^{-}A_1 \sim a_1a_2\cdots a_{n-2}A = T_1^{n-2}$, we can get that

$$\tilde{g}_1(T_{1_{2,1}}^{n-1}) = \tilde{g}_1(T_1^{n-2}), \quad \tilde{g}_2(T_{1_{2,1}}^{n-1}) = \tilde{g}_2(T_1^{n-2}).$$

From Relation 1,

$$a_1\cdots a_j\cdots a_{n-1}A_1a_j^{-}a_{n-1}^{-}A_2 \sim a_1\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2a_ja_{n-1}a_j^{-}a_{n-1}^{-},$$

then according to Lemma 2.6, we know that

$$\tilde{g}_1(T_{1_{2,2}}^{n-1}) = 0, \quad \tilde{g}_2(T_{1_{2,2}}^{n-1}) = 0.$$

By applying Relation 2 twice, we get that

$$a_1\cdots a_j\cdots a_{n-1}A_1a_ja_{n-1}^{-}A_2 \sim a_1\cdots a_{j-1}A_1^{-}a_{n-1}^{-}\cdots a_{j+1}^{-}a_{n-1}^{-}A_2a_ja_j$$

$$\sim a_1\cdots a_{j-1}A_1^{-}a_{j+1}\cdots a_{n-2}A_2a_{n-1}a_{n-1}a_ja_j.$$

By Lemmas 1.4, 2.2, we have that

$$\tilde{g}_1(T_{1_{2,3}}^{n-1}) = 0, \quad \tilde{g}_2(T_{1_{2,3}}^{n-1}) = \sum_{j=1}^{n-2} j(n-j-1) = \frac{(n-2)(n-1)n}{6}.$$

Summarizing the above,

$\tilde{g}_1(T_1^{n-1}) = (n-1) + \tilde{g}_1(T_1^{n-2})$, since $\tilde{g}_1(T_1^1) = 1$, we get that

$$\tilde{g}_1(T_1^{n-1}) = 1 + \sum_{j=3}^{n}(j-1) = \frac{(n-1)n}{2},$$

and

$$\begin{aligned}\tilde{g}_2(T_1^{n-1}) &= (n-1)\frac{(n-2)(n-1)}{2}+\tilde{g}_2(T_1^{n-2})+\frac{(n-2)(n-1)n}{6}\\ &= \frac{1}{6}(4n^3-15n^2+17n-6)+\tilde{g}_2(T_1^{n-2}).\end{aligned}$$

Since it is easy to calculate that $\tilde{g}_2(T_1^2)=3$, we can get that

$$\tilde{g}_2(T_1^{n-1})=\frac{1}{6}\sum_{j=4}^{n}(4j^3-15j^2+17j-6)+3=\frac{1}{6}(n-2)(n-1)n^2.$$

Lemma 2.8 *The number of projective planes in surface set* $T_5^{n-2}=\{a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B\}$ *is*

$$\tilde{g}_1(T_5^{n-2})=\frac{1}{2}(i+1)(n-i-1)\big(n^2-(i+3)n+i^2+2i+2\big).$$

Proof **Case 1** $\delta(a_{n-2})=1$

Subcase 1.1 $a_{n-2}\in A$

$$\begin{aligned}a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B &= a_i\cdots a_1A_1a_{n-2}A_2a_{i+1}\cdots a_{n-2}B\\ &\sim a_i\cdots a_1A_1a_{n-3}^-\cdots a_{i+1}^-A_2^-Ba_{n-2}a_{n-2}\end{aligned}$$

Subcase 1.2 $a_{n-2}\in B$

$$\begin{aligned}a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B &= a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B_1a_{n-2}B_2\\ &\sim a_i\cdots a_1Aa_{i+1}\cdots a_{n-3}B_1^-B_2a_{n-2}a_{n-2}\end{aligned}$$

According to Lemmas 1.4, 2.2, the number of projective planes in Case 1 is

$$2g_0(T_6^{n-3})=(i+1)(n-i-2)(n-1).$$

Case 2 $\delta(a_{n-2})=0$

Subcase 2.1 $a_{n-2}^-\in A$

$$a_i\cdots a_1Aa_{i+1}\cdots a_{n-2}B\sim a_i\cdots a_1A_1a_{n-2}^-A_2a_{i+1}\cdots a_{n-3}a_{n-2}B$$

According to Lemmas 1.4, 2.6,

$$a_i\cdots a_1A_1a_{n-2}^-A_2a_{i+1}\cdots a_{n-3}a_{n-2}B\sim Q_1\Leftrightarrow$$

$$Ba_i\cdots a_1A_1\sim O_0 \text{ and } A_2a_{i+1}\cdots a_{n-3}\sim Q_1;$$

or

$$A_2a_{i+1}\cdots a_{n-3}\sim O_0 \text{ and } Ba_i\cdots a_1A_1\sim Q_1.$$

From Lemmas 2.1, 2.7, the number of projective planes in Subcase 2.1 is

$$(i+1)\frac{(n-i-3)(n-i-2)}{2}+(i+1)\frac{i(i+1)}{2}.$$

Subcase 2.2 $a_{n-2}^{-} \in B$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B = a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} a_{n-2} B_1 a_{n-2}^{-} B_2$$

If $|B_1| \neq 0$, according to Lemma 2.6, the nonorientable genus of this case will be more than one, so $|B_1| = 0$. By Operation 1,

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} a_{n-2} B_1 a_{n-2}^{-} B_2 \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B_2 = T_5^{n-3}$$

So the number of projective planes in Case 2 is

$$\frac{(i+1)(n-i-3)(n-i-2)}{2} + \frac{i(i+1)^2}{2} + \tilde{g}_1(T_5^{n-3}).$$

Summarizing Cases 1, 2, we have

$$\tilde{g}_1(T_5^{n-2}) = \frac{1}{2}(i+1)\big(3n^2 - (11+4i)n + 2(5+4i+i^2)\big) + \tilde{g}_1(T_5^{n-3}).$$

With a similar argument, we can get that $\tilde{g}_1(T_5^{i+1}) = (i+1)(i+2) + i(i+1)^2$. Hence

$$\begin{aligned}\tilde{g}_1(T_5^{n-1}) &= \frac{(1+i)}{2} \sum_{j=i+4}^{n} \big(3j^2 - (11+4i)j + 2(5+4i+i^2)\big) + \tilde{g}_1(T_5^{i+1}) \\ &= \frac{1}{2}(i+1)(n-i-1)\big(n^2 - (i+3)n + i^2 + 2i + 2\big).\end{aligned}$$

Lemma 2.9 *The number of nonorientable surfaces of genus* 3 *in surface set* $T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A\}$ *is*

$$\tilde{g}_3(T_1^{n-1}) = \frac{1}{720}(n-3)(n-2)(n-1)n(41n^2 - 9n - 20).$$

Proof The argument is similar to that of Lemma 2.7, and we adopt the same notations as that in Lemma 2.7. According to Lemmas 1.4, 2.1, 2,3 and 2.7, we have

$$\begin{aligned}\tilde{g}_3(T_{1_1}^{n-1}) &= (n-1)\tilde{g}_2(T_1^{n-2}) + (n-1)g_1(T_1^{n-2}) \\ &= \frac{(n-3)(n-2)(n-1)^2(5n-4)}{24}.\end{aligned}$$

It is easy to get that $\tilde{g}_3(T_{1_{2,1}}^{n-1}) = \tilde{g}_3(T_1^{n-2})$.
As for sets $T_{1_{2,2}}^{n-1}$, and $T_{1_{2,3}}^{n-1}$, by Lemma 1.4, we know that

$$a_1 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 a_j a_{n-1} a_j^{-} a_{n-1}^{-} \sim Q_3 \Leftrightarrow$$

$$a_1 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 \sim Q_1$$

and

$$a_1 \cdots a_{j-1} A_1^- a_{j+1} \cdots a_{n-2} A_2 a_{n-1} a_{n-1} a_j a_j \sim Q_3 \Leftrightarrow$$
$$a_1 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 \sim Q_1$$

So with the application of Lemma 2.8, we get that

$$\begin{aligned}
& \tilde{g}_3(T_{12,2}^{n-1}) = \tilde{g}_3(T_{12,3}^{n-1}) \\
= & \frac{1}{2}\sum_{j=1}^{n-2} j(n-j-1)\left((n-1)^2 - (j+2)(n-1) + (j-1)^2 + 2(j-1) + 2\right) \\
= & \frac{1}{30}(n-3)(n-2)(n-1)n(2n-3).
\end{aligned}$$

Summarizing the above, we have

$$\tilde{g}_3(T_1^{n-1}) = \frac{(n-3)(n-2)(n-1)(41n^2 - 69n + 20)}{120} + \tilde{g}_3(T_1^{n-2}).$$

Since it is easy to calculate that $\tilde{g}_3(T_1^3) = 20$, we can get that

$$\begin{aligned}
\tilde{g}_3(T_1^{n-1}) &= \frac{1}{120}\sum_{j=5}^{n}\Big((j-3)(j-2)(j-1)(41j^2 - 69j + 20)\Big) + 20 \\
&= \frac{1}{720}(n-3)(n-2)(n-1)n(41n^2 - 9n - 20).
\end{aligned}$$

Lemma 2.10 *The number of projective planes in the surface set* $T_6^{n-3} = \{a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} BC\}$ *is*

$$\tilde{g}_1(T_6^{n-3}) = \frac{1}{4}(i+1)(n-i-2)(n-1)\left(n^2 - (5+i)n + i^3 + 3i + 6\right).$$

Proof **Case 1** $\delta(a_{n-3}) = 1$

Subcase 1.1 $a_{n-3} \in A$

$$a_i \cdots a_1 A_1 a_{n-3} A_2 a_{i+1} \cdots a_{n-3} BC \sim a_i \cdots a_1 A_1 a_{n-4}^- \cdots a_{i+1}^- A_2^- BC a_{n-3} a_{n-3}$$

Subcase 1.2 $a_{n-3} \in B$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B_1 a_{n-3} B_2 C \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-4} B_1^- B_2 C a_{n-3} a_{n-3}$$

Subcase 1.3 $a_{n-3} \in C$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B C_1 a_{n-3} C_2 \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-4} C_1^- B^- C_2 a_{n-3} a_{n-3}$$

According to Lemma 1.4, the number of projective planes in Case 1 is $3g_0(T_8^{n-4})$.

Case 2 $\delta(a_{n-3}) = 0$

Subcase 2.1 $a_{n-3}^{-} \in A$, *i.e.*, $a_i \cdots a_1 A_1 a_{n-3}^{-} A_2 a_{i+1} \cdots a_{n-3} BC$.
By Lemmas 2.6, 1.4, we get that $a_{i+1}^{\varepsilon_{i+1}}, \ldots, a_{n-4}^{\varepsilon_{n-4}} \in A_2$ and $a_i^{\varepsilon_i}, \ldots, a_1^{\varepsilon_1} \in A_1 + B + C$. So

$$a_i \cdots a_1 A_1 a_{n-3}^{-} A_2 a_{i+1} \cdots a_{n-3} BC \sim Q_1 \Leftrightarrow$$

$$a_i \cdots a_1 A_1 BC \sim Q_1 \quad \text{and} \quad a_{i+1} \cdots a_{n-4} A_2 \sim O_0$$

or

$$a_i \cdots a_1 A_1 BC \sim O_0 \quad \text{and} \quad a_{i+1} \cdots a_{n-4} A_2 \sim Q_1$$

From Lemmas 2.1, 2.7, the number of projective planes in Subcase 2.1 is

$$\frac{i(i+1)}{2}\frac{(i+1)(i+2)}{2} + \frac{(i+1)(i+2)}{2}\frac{(n-i-4)(n-i-3)}{2}$$

$$= \frac{(i+1)(i+2)\big(12 + 2i(4+i) - 7n - 2in + n^2\big)}{4}.$$

Subcase 2.2 $a_{n-3}^{-} \in B$, *i.e.*, $a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B_1 a_{n-3}^{-} B_2 C$.
By Lemma 2.6, we get that $|B_1| = 0$.

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} a_{n-3}^{-} B_2 C \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-4} B_2 C$$

So the number of projective planes in Subcase 2.2 is $\tilde{g}_1(T_6^{n-4})$.

Subcase 2.3 $a_{n-3}^{-} \in C$, *i.e.*, $a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B C_1 a_{n-3}^{-} C_2$.
By Lemma 2.6, we get that $|B| = |C_1| = 0$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} a_{n-3}^{-} B_2 \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-4} B_2$$

So the number of projective planes in Subcase 2.3 is $\tilde{g}_1(T_5^{n-4})$.

Summarizing Cases 1, 2, by Lemmas 2.2, 2.8, we have

$$\tilde{g}_1(T_6^{n-3}) = \tilde{g}_1(T_6^{n-4}) + \frac{(i+1)}{4}\big(- (i^3 + 9i^2 + 36i + 60) + 2(37 + 15i + 2i^2)n - 6(5+i)n^2 + 4n^3\big).$$

With a similar argument, we can get that $\tilde{g}_1(T_6^{i+1}) = \dfrac{(i+1)(i+3)(i^2+2i+2)}{2}$.

Hence, we can get that

$$\tilde{g}_1(T_6^{n-3}) = \frac{(i+1)(i+3)(i^2+2i+2)}{2} + \tfrac{(i+1)}{4} \sum_{j=i+5}^{n} \big(- (i^3 + 9i^2 + 36i + 60) + 2(37 + 15i + 2i^2)j - 6(5+i)j^2 + 4j^3\big)$$

$$= \frac{1}{4}(i+1)(n-i-2)(n-1)\big(n^2 - (5+i)n + i^3 + 3i + 6\big).$$

Lemma 2.11 *The number of Klein bottles in surface set* $T_5^{n-2} = \{a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B\}$ *is*

$$\begin{aligned} \tilde{g}_2(T_5^{n-2}) &= \frac{(1+i)(n-i-1)}{6} \times \Big(n^4 - (7+2i)n^3 \\ &+ (17 + 12i + 3i^2)n^2 - (17 + 23i + 12i^2 + 2i^3)n + (1+i)^2(6 + 2i + i^2)\Big) \end{aligned}$$

Proof **Case 1** $\delta(a_{n-2}) = 1$

Subcase 1.1 $a_{n-2} \in A$

$$\begin{aligned} a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B &\sim a_i \cdots a_1 A_1 a_{n-2} A_2 a_{i+1} \cdots a_{n-2} B \\ &\sim a_i \cdots a_1 A_1 a_{n-3}^- \cdots a_{i+1}^- A_2^- B a_{n-2} a_{n-2} \end{aligned}$$

Subcase 1.2 $a_{n-2} \in B$

$$\begin{aligned} a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B &\sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B_1 a_{n-2} B_2 \\ &\sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} B_1^- B_2 a_{n-2} a_{n-2} \end{aligned}$$

According to Lemma 1.4, the number of Klein bottles in Case 1 is $2\tilde{g}_1(T_6^{n-3})$.

Case 2 $\delta(a_{n-2}) = 0$

Subcase 2.1 $a_{n-2}^- \in A$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B \sim a_i \cdots a_1 A_1 a_{n-2}^- A_2 a_{i+1} \cdots a_{n-3} a_{n-2} B = T_{5_1}^{n-2}$$

Subcase 2.1.1 $a_{n-3}^{\varepsilon_{n-3}} \in B$

Notice that $\delta(a_{n-3}) = 1$, otherwise the nonorientable genus of this case will more than 2, from Lemma 2.6. So this case should be $a_{n-3} \in B$. By using Relation 2 twice, we have

$$\begin{aligned} & a_i \cdots a_1 A_1 a_{n-2}^- A_2 a_{i+1} \cdots a_{n-3} a_{n-2} B_1 a_{n-3} B_2 \\ \sim\ & a_i \cdots a_1 A_1 B_1 a_{n-4}^- \cdots a_{i+1}^- A_2^- B_2 a_{n-2} a_{n-2} a_{n-3} a_{n-3} \end{aligned}$$

According to Lemma 1.4, the number of Klein bottles in Subcase 2.1.1 is $g_0(T_7^{n-4})$.

Subcase 2.1.2 $a_{n-3}^{\varepsilon_{n-3}} \in A_1$

For the same reason with that in Subcase 2.1.1, $\delta(a_{n-3}) = 1$, and this case should be $a_{n-3} \in A_1$.

$$\begin{aligned} & a_i \cdots a_1 A_{11} a_{n-3} A_{12} a_{n-2}^- A_2 a_{i+1} \cdots a_{n-3} B_1 a_{n-3} B_2 \\ \sim\ & a_i \cdots a_1 A_{11} a_{n-4}^- \cdots a_{i+1}^- A_2^- A_{12} B a_{n-2} a_{n-2} a_{n-3} a_{n-3} \end{aligned}$$

According to Lemma 1.4, the number of Klein bottles in Subcase 2.1.2 is $g_0(T_8^{n-4})$.

Subcase 2.1.3 $a_{n-3}^{\varepsilon_{n-3}} \in A_2$

a) When $\delta(a_{n-3}) = 0$, *i.e.*, $a_i \cdots a_1 A_1 a_{n-2}^- A_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B$.

If $|A_{21}| = 0$, then by Operation 2,

$$\begin{aligned} & a_i \cdots a_1 A_1 a_{n-2}^- a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B \\ \sim\ & a_i \cdots a_1 A_1 a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} B = T_{5_1}^{n-3} \end{aligned}$$

If $|A_{21}| \neq 0$, then $A_{21} = a_j^{\varepsilon_j} A'_{21}, j \in [1, i]$ or $j \in [i+1, n-4]$.

From Lemma 2.6, the value of $\delta(a_j)$ should be 1, otherwise the nonorientable genus will be more than 2. So $A_{21} = a_j A'_{21}, j \in [1, i]$ or $j \in [i+1, n-4]$.

when $A_{21} = a_j A'_{21}, j \in [1, i]$, by using Relation 2 twice, we have

$$\begin{aligned} & a_i \cdots a_j \cdots a_1 A_1 a_{n-2}^- a_j A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B \\ \sim \quad & a_i \cdots a_{j+1} a_{n-2} A_1^- a_1^- \cdots a_{j-1}^- A'_{21} a_{n-3}^- A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B a_j a_j \\ \sim \quad & a_i \cdots a_{j+1} a_{n-3}^- \cdots a_{i+1}^- A_{22}^- a_{n-3} A_{21}'^- a_{j-1} \cdots a_1 A_1 B a_{n-2} a_{n-2} a_j a_j \end{aligned}$$

According to Lemmas 1.4, 1.2, we have that

$$a_i \cdots a_{j+1} a_{n-3}^- \cdots a_{i+1}^- A_{22}^- a_{n-3} A_{21}^- a_{j-1} \cdots a_1 A_1 B a_{n-2} a_{n-2} a_j a_j \sim Q_2 \Leftrightarrow$$

$$a_{n-4}^- \cdots a_{i+1}^- A_{22}^- \sim O_0 \quad \text{and} \quad a_i \cdots a_{j+1} A_{21}'^- a_{j-1} \cdots a_1 A_1 B \sim O_0$$

By Lemma 2.2, the number of Klein bottles is $\sum_{j=1}^{i} \frac{j(i+1-j)(i+1)}{2}$, when $A_{21} = a_j A'_{21}, j \in [1, i]$.

With a similar argument, we can get that the number of Klein bottles is $\sum_{j=i+1}^{n-4} (i+1)(j-i)(n-j-3)$, when $A_{21} = a_j A'_{21}, j \in [i+1, n-4]$.

b) When $\delta(a_{n-3}) = 1$, *i.e.*, $a_i \cdots a_1 A_1 a_{n-2}^- A_{21} a_{n-3} A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B$.

By Relation 2,

$$\begin{aligned} & a_i \cdots a_1 A_1 a_{n-2}^- A_{21} a_{n-3} A_{22} a_{i+1} \cdots a_{n-3} a_{n-2} B \\ \sim \quad & a_i \cdots a_1 A_1 a_{n-2}^- A_{21} a_{n-4}^- \cdots a_{i+1}^- A_{22}^- a_{n-2} B a_{n-3} a_{n-3} \end{aligned}$$

According to Lemmas 1.4, 2.6, we get that

$$a_i \cdots a_1 A_1 a_{n-2}^- A_{21} a_{n-4}^- \cdots a_{i+1}^- A_{22}^- a_{n-2} B a_{n-3} a_{n-3} \sim Q_2 \Leftrightarrow$$

$$B a_i \cdots a_1 A_1 \sim Q_1 \quad \text{and} \quad A_{21} a_{n-4}^- \cdots a_{i+1}^- A_{22}^- \sim O_0$$

or

$$B a_i \cdots a_1 A_1 \sim O_0 \quad \text{and} \quad A_{21} a_{n-4}^- \cdots a_{i+1}^- A_{22}^- \sim Q_1$$

By Lemmas 2.1, 2.7, the number of Klein bottles is

$$\begin{aligned} & \frac{i(i+1)^2(n-3-i)}{2} + \frac{(i+1)(n-i-4)(n-i-3)^2}{2} \\ = \quad & \frac{(i+1)(n-i-3)\big(n^2 - 2in - 7n + 2i(4+i) + 12\big)}{2}. \end{aligned}$$

Summarizing the above,

$$\begin{aligned} \tilde{g}_2(T_{5_1}^{n-2}) \quad = \quad & g_0(T_7^{n-4}) + g_0(T_8^{n-4}) + \tilde{g}_2(T_{5_1}^{n-3}) \\ & + \sum_{j=1}^{i} \frac{j(i+1-j)(i+1)}{2} + \sum_{j=i+1}^{n-4} (i+1)(j-i)(n-j-3) \\ & + \frac{(i+1)(n-i-3)\big(n^2 - 2in - 7n + 2i(4+i) + 12\big)}{2} \\ = \quad & \tilde{g}_2(T_{5_1}^{n-3}) + \frac{1}{12}(i+1)\Big(6(2n^3 - 17n^2 + 49n - 48) \\ & -i(27n^2 - 167n + 262) - i^2(92 - 28n) - 12i^3\Big). \end{aligned}$$

With a similar argument, we have $\tilde{g}_2(T_{5_1}^{i+1}) = \frac{i^2(i+1)^2(1+2i)}{12}$, then we can get that

$$\begin{aligned}\tilde{g}_2(T_{5_1}^{n-2}) &= \frac{(i+1)}{12} \times \Big(3n^4 - (28+9i)n^3 + (99+70i+14i^2)n^2 \\ &\quad -(158+183i+78i^2+12i^3)n + 2(48+79i+53i^2+18i^3+3i^4)\Big).\end{aligned}$$

Subcase 2.2 $a_{n-2}^- \in B$

$$a_i \cdots a_1 A a_{i+1} \cdots a_{n-2} B \sim a_i \cdots a_1 A a_{i+1} \cdots a_{n-3} a_{n-2} B_1 a_{n-2}^- B_2 = T_{5_2}^{n-2}$$

With a similar argument in Subcase 2.1, we can get that
$\tilde{g}_2(T_{5_2}^{n-2}) = \frac{(n-i-2)(i+1)}{12} \times \Big(2(45+45i+24i^2+6i^3+i^4)$
$- (141+96i+34i^2+4i^3)n + 2(41+17i+3i^2)n^2 - (21+4i)n^3 + 2n^4\Big)$.
The number of Klein bottles in Case 2 is $\tilde{g}_2(T_{5_1}^{n-2}) + \tilde{g}_2(T_{5_2}^{n-2})$.

Summarizing Cases 1,2, we get that

$$\begin{aligned}\tilde{g}_2(T_5^{n-2}) &= \frac{(1+i)(n-i-1)}{6} \times \Big(n^4 - (7+2i)n^3 + (17+12i+3i^2)n^2 \\ &\quad -(17+23i+12i^2+2i^3)n + (1+i)^2(6+2i+i^2)\Big).\end{aligned}$$

Lemma 2.12 *The number of nonorientable surfaces of genus* 4 *in surface set* $T_1^{n-1} = (a_1 a_2 \cdots a_{n-1} A)$ *is*

$$\tilde{g}_4(T_1^{n-1}) = \frac{(n-4)(n-3)(n-2)(n-1)n^2(61n^2-55n-74)}{5040}.$$

Proof The argument is similar to that of Lemma 2.7, and we adopt the same notations as that in Lemma 2.7.

According to Lemmas 1.4, 2.1, we have $\tilde{g}_4(T_{1_1}^{n-1}) = (n-1)\tilde{g}_3(T_1^{n-2})$.

It is easy to get that $\tilde{g}_4(T_{1_{2,1}}^{n-1}) = \tilde{g}_4(T_1^{n-2})$.

As for set $T_{1_{2,2}}^{n-1}$, by Lemma 1.4, we know that

$$a_1 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 a_j a_{n-1} a_j^- a_{n-1}^- \sim Q_4 \Leftrightarrow$$

$$a_1 \cdots a_{j-1} A_1 a_{j+1} \cdots a_{n-2} A_2 \sim Q_2.$$

With the application of Lemma 2.11, for $j \in [1, n-2]$, we can get

$$\tilde{g}_4(T_{1_{2,2}}^{n-1}) = \frac{(n-4)(n-3)(n-2)(n-1)n(5n-4)(9n-19)}{2520}.$$

As for set $T_{1_{2,3}}^{n-1}$, by Lemma 1.4, we know that

$$a_1 \cdots a_{j-1} A_1^- a_{j+1} \cdots a_{n-2} A_2 a_{n-1} a_{n-1} a_j a_j \sim Q_4 \Leftrightarrow$$

$$a_1\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2\sim Q_2 \text{ or } a_1\cdots a_{j-1}A_1a_{j+1}\cdots a_{n-2}A_2\sim O_1$$

With the application of Lemmas 2.11, 2.4, we get that

$$\tilde{g}_4(T_{1_{2,3}}^{n-1})=\frac{(n-4)(n-3)(n-2)(n-1)n(37n^2-99n+46)}{1680}.$$

Summarizing the above, we have

$$\begin{aligned}\tilde{g}_4(T_1^{n-1}) &= \tilde{g}_4(T_1^{n-2})\\ &+ \frac{(n-4)(n-3)(n-2)(n-1)(488n^3-1483n^2+1137n-210)}{5040}.\end{aligned}$$

With the same way, we can calculate $\tilde{g}_4(T_1^4)=140$. So we can get that

$$\tilde{g}_4(T_1^{n-1})=\frac{(n-4)(n-3)(n-2)(n-1)n^2(61n^2-55n-74)}{5040}.$$

3. The numbers of embeddings of W_n on torus and double torus

According to the classification of $\mathcal{F}(W_n)$ in section 2, we know that $|\mathcal{F}_p(W_n;k)| = 2\sum_{i=0}^{n-1}\binom{n-1}{i}g_k(M_i)$, $k\ge 0$. From [8], it is easy to get that $|\mathcal{F}_p(W_n;0)|=2$, *i.e.*, there are two distinct embeddings of W_n on the plane. In this section, we get $|\mathcal{F}_p(W_n;1)|$ and $|\mathcal{F}_p(W_n;2)|$, *i.e*, the numbers of distinct embeddings of W_n on torus and double torus.

Theorem 3.1 *The numbers of embeddings of W_n on torus and double torus are*

$$|\mathcal{F}_p(W_n;1)|=n(n-1)\Big(2^{n-2}+\frac{(n-2)(n+1)}{12}\Big)$$

and

$$\begin{aligned}|\mathcal{F}_p(W_n;2)| &= \frac{1}{192}(n-3)(n-2)(n-1)n\\ &\times\Big(2^n(n-1)(n+2)+\frac{(n-4)(n+1)(3n^2-n-6)}{15}\Big),\end{aligned}$$

respectively.

Proof According to the classification of $\mathcal{F}(W_n)$, we can get that $|\mathcal{F}_p(W_n;1)| = 2\sum_{i=0}^{n-1}\binom{n-1}{i}g_1(M_i)$, and $|\mathcal{F}_p(W_n;2)|=2\sum_{i=0}^{n-1}\binom{n-1}{i}g_2(M_i)$. In order to get $|\mathcal{F}_p(W_n;1)|$ and $|\mathcal{F}_p(W_n;2)|$, we will discuss $g_1(M_i)$ and $g_2(M_i)$, $0\le i\le n-1$. Because we discuss the embeddings on the orientable surfaces, from Lemma 1.1, the indices assignment $\delta=(\delta_0,\cdots,\delta_{n-1})=0$.

For $M_i=\{a_i\cdots a_1a_0a_{i+1}\cdots a_{n-1}a_0^-A\}, 0\le i\le n-2$,

$$\begin{aligned}a_i\cdots a_1a_0a_{i+1}\cdots a_{n-1}a_0^-A &\sim a_i\cdots a_1a_0a_{i+1}\cdots a_{n-2}a_{n-1}a_0^-A_1a_{n-1}^-A_2\\ &\sim a_i\cdots a_1A_1a_{i+1}\cdots a_{n-2}A_2a_0a_{n-1}a_0^-a_{n-1}^-\end{aligned}$$

According to Lemmas 1.3, 2.2 and 2.4, we can get that, for $0 \le i \le n-2$,

$$g_1(M_i) = g_0(T_5^{n-2}) = (i+1)(n-i-1),$$

$$\begin{aligned} g_2(M_i) &= g_1(T_5^{n-2}) \\ &= \frac{1}{24}(n-2)(n-1)(n-i-1)(i+1)(n^2-(2i+3)n+2i^2+4i). \end{aligned}$$

For $M_{n-1} = \{a_{n-1}\cdots a_1 a_0 a_0^- A\}$, $a_{n-1}\cdots a_1 a_0 a_0^- A \sim a_{n-1}\cdots a_1 A$, by Operation 1. According to Lemmas 2.3, 2.5, we can get that

$$g_1(M_{n-1}) = g_1(T_1^{n-1}) = \frac{(n-2)(n-1)n(n+1)}{24},$$

$$\begin{aligned} g_2(M_{n-1}) &= g_2(T_1^{n-1}) \\ &= \frac{1}{5760}(n-4)(n-3)(n-2)(n-1)n(n+1)(3n^2-n-6). \end{aligned}$$

So the number of embeddings of W_n on torus is

$$\begin{aligned} |\mathcal{F}_p(W_n;1)| &= 2\Big(\sum_{i=0}^{n-2}\binom{n-1}{i}g_1(M_i) + \binom{n-1}{n-1}g_1(M_{n-1})\Big) \\ &= 2^{n-2}n(n-1) + \frac{(n-2)(n-1)n(n+1)}{12}. \end{aligned}$$

And the number of embeddings of W_n on double torus is

$$\begin{aligned} |\mathcal{F}_p(W_n;2)| &= 2\Big(\sum_{i=0}^{n-2}\binom{n-1}{i}g_2(M_i) + \binom{n-1}{n-1}g_2(M_{n-1})\Big) \\ &= \frac{1}{192}(n-3)(n-2)(n-1)n \\ &\quad \times\Big(2^n(n-1)(n+2) + \frac{(n-4)(n+1)(3n^2-n-6)}{15}\Big). \end{aligned}$$

Thus the theorem is obtained.

Examples

According to theorem 3.1, one can calculate the numbers of embeddings of W_n on torus and double torus easily, give these numbers $|\mathcal{F}_p(W_n;1)|, |\mathcal{F}_p(W_n;2)|, 3 \le n \le 11$ as follows:

	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}
$\|\mathcal{F}_p(W_n;1)\|$	14	58	190	550	1484	3836	9636	23700	57310
$\|\mathcal{F}_p(W_n;2)\|$	0	36	576	4968	31178	160538	721602	2935842	11062326

4. The numbers of embeddings of W_n on nonorientable surfaces of genus 1,2,3,4

According to the classification of $\mathcal{F}(W_n)$, $|\mathcal{F}_q(W_n;k)| = 2\sum_{i=0}^{n-2}\binom{n-1}{i}\tilde{g}_k(M_i)(k \geq 1)$. In this section, we get $|\mathcal{F}_q(W_n;k)|$, $(1 \leq k \leq 4)$ *i.e.*, the numbers of embeddings of W_n on nonorientable surfaces of genus $1, 2, 3, 4$.

Theorem 4.1 *The numbers of embeddings of W_n on nonorientable surfaces of genus $1, 2, 3$, and 4 are*

$$|\mathcal{F}_q(W_n;1)| = 2^n + (n-1)n,$$

$$|\mathcal{F}_q(W_n;2)| = n(n-1)\Big(3 \times 2^{n-2} + \frac{n(n-2)}{3}\Big),$$

$$|\mathcal{F}_q(W_n;3)| = \frac{(n-2)(n-1)n\Big(2^n 15(19n-1) + 2(n-3)(41n^2-9n-20)\Big)}{720}$$

and

$$|\mathcal{F}_q(W_n;4)| = \frac{(n-3)(n-2)(n-1)n}{10080} \times\Big(2^n 7(157n^2-48n-85) + 4(n-4)n(61n^2-55n-74)\Big),$$

respectively.

Proof According to the classification of $\mathcal{F}(W_n)$, we can get that $|\mathcal{F}_q(W_n;k)| = 2\sum_{i=0}^{n-1}\binom{n-1}{i}\tilde{g}_k(M_i)$, In order to get $|\mathcal{F}_q(W_n;k)|$, $1 \leq k \leq 4$, we will discuss $\tilde{g}_k(M_i)$ in which $1 \leq k \leq 4, 0 \leq i \leq n-1$. Because we discuss the embeddings on the nonorientable surfaces, from Lemma 1.1, the indices assignment $\delta = (\delta_0, \cdots, \delta_{n-1}) \neq 0$.

Case 1 $\delta(a_0) = 1$

In this case, for $M_i = \{a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-1} a_0 A\}(0 \leq i \leq n-1)$, by Relation 2, we have

$$a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-1} a_0 A \sim a_i \cdots a_1 a_{n-1} \cdots a_{i+1} A a_0 a_0$$

Case 2 $\delta(a_0) = 0$

Subcase 2.1 $M_i = \{a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-2} a_{n-1} a_0^- A_1 a_{n-1} A_2\}, (0 \leq i \leq n-2)$. By Relation 2, we have

$$\begin{aligned} & a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-2} a_{n-1} a_0^- A_1 a_{n-1} A_2 \\ \sim\ & a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-2} A_1^- a_0 A_2 a_{n-1} a_{n-1} \\ \sim\ & a_i \cdots a_1 A_1 a_{n-2}^- \cdots a_{i+1}^- A_2 a_0 a_0 a_{n-1} a_{n-1} \end{aligned}$$

Subcase 2.2 $M_i = \{a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-2} a_{n-1} a_0^- A_1 a_{n-1}^- A_2\}, (0 \leq i \leq n-2)$. By Relation 1, we have

$$\begin{aligned} & a_i \cdots a_1 a_0 a_{i+1} \cdots a_{n-2} a_{n-1} a_0^- A_1 a_{n-1}^- A_2 \\ \sim\ & a_i \cdots a_1 A_1 a_{i+1} \cdots a_{n-2} A_2 a_0 a_{n-1} a_0^- a_{n-1}^- \end{aligned}$$

Subcase 2.3 $M_{n-1} = \{a_{n-1}\cdots a_1 a_0 a_0^{-1} A\}$
By Operation 3, we have

$$a_{n-1}\cdots a_1 a_0 a_0^{-1} A \sim a_{n-1}\cdots a_1 A.$$

The following arguments based on Lemma 1.4, and some Lemmas in Section 2 are applied.

a) For the number of embeddings of W_n on projective plane.
The number of Case 1 is

$$\sum_{i=0}^{n-1}\binom{n-1}{i} g_0(T_1^{n-1}) = 2^{n-1}.$$

There is no embedding on projective plane in Subcases 2.1 and 2.2. And the number in Subcase 2.3 is

$$\binom{n-1}{n-1}\tilde{g}_1(T_1^{n-1}) = \frac{(n-1)n}{2}.$$

So the number embeddings of W_n on projective plane is

$$|\mathcal{F}_q(W_n;1)| = 2\Big(2^{n-1} + \frac{(n-1)n}{2}\Big).$$

b) For the number of embeddings of W_n on Klein bottle.
The number of Case 1 is $\sum\limits_{i=0}^{n-1}\binom{n-1}{i}\tilde{g}_1(T_1^{n-1}) = \sum\limits_{i=0}^{n-1}\binom{n-1}{i}\frac{(n-1)n}{2} = 2^{n-1}\frac{(n-1)n}{2}$.
The number of Subcase 2.1 is $\sum\limits_{i=0}^{n-2}\binom{n-1}{i} g_1(T_5^{n-2}) = 2^{n-3}(n-1)n$.
There is no embedding on Klein bottle in Subcase 2.2.
And the number in Subcase 2.3 is $\binom{n-1}{n-1}\tilde{g}_2(T_1^{n-1}) = \frac{(n-1)n^2(n-2)}{6}$.
So the number of embeddings of W_n on Klein bottle is

$$|\mathcal{F}_q(W_n;2)| = n(n-1)\Big(3\times 2^{n-2} + \frac{n(n-2)}{3}\Big).$$

c) For the number of embeddings of W_n on nonorientable surface of genus 3.
The number of Case 1 is

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\Big(\tilde{g}_2(T_1^{n-1}) + g_1(T_1^{n-1})\Big) = 2^{n-4}\frac{(n-2)(n-1)n(5n+1)}{3}.$$

The number of Subcase 2.1 is $\sum\limits_{i=0}^{n-2}\binom{n-1}{i}\tilde{g}_1(T_5^{n-2}) = 2^{n-6}(n-2)(n-1)n(3n-1)$.
The number of Subcase2.2 is same to Subcase 2.1.
The number of Subcase2.3 is

$$\binom{n-1}{n-1}\tilde{g}_3(T_1^{n-1}) = \frac{1}{720}(n-3)(n-2)(n-1)n(41n^2-9n-20).$$

So the number of embeddings of W_n on the nonorientable surface of genus 3 is

$$|\mathcal{F}_q(W_n;3)| = \frac{(n-2)(n-1)n\Big(2^n 15(19n-1) + 2(n-3)(41n^2 - 9n - 20)\Big)}{720}.$$

d) For the number of embeddings of W_n on nonorientable surface of genus 4.

The number of Case 1 is $\sum\limits_{i=0}^{n-1} \binom{n-1}{i} \tilde{g}_3(T_1^{n-1})$.

The number of Subcase 2.1 is $\sum\limits_{i=0}^{n-2} \binom{n-1}{i} \Big(\tilde{g}_2(T_5^{n-2}) + g_1(T_5^{n-2})\Big)$.

The number of Subcase 2.2 is $\sum\limits_{i=0}^{n-2} \binom{n-1}{i} \tilde{g}_2(T_5^{n-2})$.

The number of Subcase 2.3 is $\binom{n-1}{n-1} \tilde{g}_4(T_1^{n-1})$.

So the number of embeddings of W_n on the nonorientable surface of genus 4 is
$|\mathcal{F}_q(W_n;4)|$

$$= \frac{(n-3)(n-2)(n-1)n\Big(2^n 7(157n^2 - 48n - 85) + 4(n-4)n(61n^2 - 55n - 74\Big)}{10080}.$$

Thus the theorem is obtained.

Examples

According to theorem 4.1, one can calculate the numbers of embeddings of W_n on nonorientable surfaces of genus 1,2,3 and 4 easily, give these numbers $|\mathcal{F}_q(W_n;1)|$, $|\mathcal{F}_q(W_n;2)|$, $|\mathcal{F}_q(W_n;3)|$, $|\mathcal{F}_q(W_n;4)|, 3 \le n \le 10$ as follows:

	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}
$\lvert\mathcal{F}_q(W_n;1)\rvert$	14	28	52	94	170	312	584	1114
$\lvert\mathcal{F}_q(W_n;2)\rvert$	42	176	580	1680	4522	11648	29160	71520
$\lvert\mathcal{F}_q(W_n;3)\rvert$	56	640	4080	19482	78414	282408	940968	2958900
$\lvert\mathcal{F}_q(W_n;4)\rvert$	0	596	9880	87536	560686	2933248	13353528	54900960

5. Application

Let $D_n (n \ge 2)$, is a dipole graph with 2 vertices, n multiple edges. Kwak and Lee obtained the genus polynomials of dipoles on orientable surfaces in [5], and got total embedding polynomials of dipoles in [6], but we can't get the numbers of embeddings on nonorientable surfaces from their results easily. In this section, by applying some lemmas in Section 2, we can get the numbers of embedding of D_n on nonorientable surface of genus $1, 2, 3$ and 4 in explicit expressions.

Theorem 5.1 *The numbers of embedding of dipole graph D_n on nonorientable surfaces of genus $1, 2, 3,$ and 4 are*

$$|\mathcal{F}_q(D_n;1)| = (n-1)!\frac{(n-1)n}{2},$$

$$|\mathcal{F}_q(D_n;2)| = (n-1)!\frac{(n-2)(n-1)n^2}{6},$$

$$|\mathcal{F}_q(D_n;3)| = (n-1)!\frac{1}{720}(n-3)(n-2)(n-1)n(41n^2-9n-20)$$

and

$$|\mathcal{F}_q(D_n;4)| = (n-1)!\frac{(n-4)(n-3)(n-2)(n-1)n^2(61n^2-55n-74)}{5040},$$

respectively.

Proof When we fix the rotation of one vertex of them, say v_1, then the associate surfaces of D_n can be written as $\{a_1a_2\cdots a_{n-1}A\} = T_1^{n-1}$, and there are $(n-1)!$ different rotations of v_1. Therefore, we have that

$$|\mathcal{F}_q(D_n;1)| = (n-1)!\tilde{g}_1(T_1^{n-1}), \quad |\mathcal{F}_q(D_n;2)| = (n-1)!\tilde{g}_2(T_1^{n-1})$$

$$|\mathcal{F}_q(D_n;3)| = (n-1)!\tilde{g}_3(T_1^{n-1}), \quad |\mathcal{F}_q(D_n;4)| = (n-1)!\tilde{g}_4(T_1^{n-1})$$

By Lemmas 2.7, 2.9 and 2.12, the theorem is obtained.

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