Domination in lexicographic product graphs*

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Abstract

In this paper, we consider the total domination number, the restrained domination number, the total restrained domination number and the connected domination number of lexicographic product graphs.

Keywords: Lexicographic product; Total domination number; Restrained domination number; Connected domination number

1 Introduction

Throughout this article, a graph $G = (V, E)$ always means a finite undirected graph without loops and multiple edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. For a vertex $v \in V$, $N(v)$ denotes the set of vertices adjacent to v. For a subset S of V, $N(S)$ denotes the set of all vertices adjacent to some vertex in S and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. P_n denotes a path with n vertices. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to some vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a $\gamma(G)$-set of G if $|S| = \gamma(G)$. Let D and U be two vertex sets of V, U is called a *monitor set* of D if $D \subseteq N[U]$. The monitor number

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of D, denoted by $\iota(D)$, is the minimum cardinality of a monitor set of D. Set $\iota(G) = \min\{\iota(D) :$ D is a $\gamma(G)$-set of G$\}$.

A set $S \subseteq V$ is a *total dominating set* (**TDS**) if every vertex in V is adjacent to some vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a **TDS** of G. A **TDS** S is called a $\gamma_t(G)$-set of G if $|S| = \gamma_t(G)$. Clearly, $\gamma(G) \leq \gamma_t(G)$. A set $S \subseteq V$ is a *restrained dominating set* (**RDS**) if every vertex not in S is adjacent to some vertex in S and to some vertex in $V - S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** S is called a $\gamma_r(G)$-set of G if $|S| = \gamma_r(G)$. Clearly, $\gamma(G) \leq \gamma_r(G)$. A set $S \subseteq V$ is a *total restrained dominating set* (**TRDS**) if every vertex in $V - S$ is adjacent to some vertex in S and to some vertex in $V - S$, and every vertex in S is adjacent to another vertex in S. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** S is called a $\gamma_{tr}(G)$-set of G if $|S| = \gamma_{tr}(G)$. Clearly, $\gamma(G) \leq \gamma_{tr}(G)$. A dominating set of G is called a *connected domination set* (**CDS**) if the induced subgraph $\langle S \rangle$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a **CDS** of G. A **CDS** S is called a $\gamma_c(G)$-set of G if $|S| = \gamma_c(G)$. Clearly, $\gamma(G) \leq \gamma_c(G)$. Please refer to the related studies[1]-[5].

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, where $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ and $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. The *lexicographic product* $G_1[G_2]$ of G_1 and G_2 has vertex set $V_1 \times V_2$ and $(x_i, y_j)(x_{i'}, y_{j'}) \in E(G_1[G_2])$ if and only if either $x_i x_{i'} \in E_1$, or $x_i = x_{i'}$ and $y_j y_{j'} \in E_2$. The subgraph $G_2^{x_i}$ is the graph with vertex set $\{(x_i, y_j) | j = 1, 2, \ldots, n_2\}$ and edge set $\{(x_i, y_j)(x_i, y_{j'}) | y_j y_{j'} \in E_2\}$. Clearly, $G_2^{x_i}$ is isomorphic to graph G_2 for $i = 1, 2, \ldots, n_1$. From the definition of lexicographic product, it is easy to see that $G_1[G_2]$ can be obtained from G_1 by replacing each vertex of G_1 with a copy of G_2, in such a way that for every edge (x_i, x_j) in G_1, contains all possible edges from $G_2^{x_i}$ to $G_2^{x_j}$.

In this paper, we will consider the total domination number, the restrained domination number, the total restrained domination number and the connected domination number of lexicographic product graphs.

2 Main results

Clearly, For any two graphs G_1 and G_2, if G_1 is an isolated vertex, then $G_1[G_2] \cong G_2$, if G_2 is an isolated vertex, then $G_1[G_2] \cong G_1$. Hence we consider that G_1 and G_2 are two graphs with at least two vertices.

First, we consider the domination number of $G_1[G_2]$.

Theorem 2.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with at least two vertices. If $\gamma(G_2) = 1$, then $\gamma(G_1[G_2]) = \gamma(G_1)$.*

Proof. Clearly, $\gamma(G_1[G_2]) \geq \gamma(G_1)$. Now we prove that $\gamma(G_1[G_2]) \leq \gamma(G_1)$. Let D_1 be a $\gamma(G_1)$-set of G_1, and let $D_2 = \{y_1\}$ be a $\gamma(G_2)$-set of G_2. Set $D = D_1 \times \{y_1\} \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1$. If $y = y_1$, then $(x, y) \in D$. If $y \neq y_1$, then $yy_1 \in E_2$ for $\gamma(G_2) = 1$. Thus, $(x, y)(x, y_1) \in E(G_1[G_2])$ and $(x, y_1) \in D$.

Case 2. $x \notin D_1$. There exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$. Thus, $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$.

Therefore, every vertex in $V(G_1[G_2]) - D$ is adjacent to some vertex in D, D is a dominating set of $G_1[G_2]$. Hence $\gamma(G_1[G_2]) \leq |D| = |D_1| = \gamma(G_1)$. From above we have $\gamma(G_1[G_2]) = \gamma(G_1)$. □

Theorem 2.2. *Let $G_1 = (V_1, E_1)$ be a graph with no isolated vertex, and let $G_2 = (V_2, E_2)$ be a graph with $\gamma(G_2) \geq 2$. Then $\gamma(G_1) \leq \gamma(G_1[G_2]) \leq \gamma(G_1) + \iota(G_1)$.*

Proof. Clearly, $\gamma(G_1[G_2]) \geq \gamma(G_1)$. Now we prove that $\gamma(G_1[G_2]) \leq \gamma(G_1) + \iota(G_1)$. Let D_1 be a $\gamma(G_1)$-set such that there exists a minimum monitor set U_1 of D_1 with $|U_1| = \iota(D_1) = \iota(G_1)$, and $|U_1 \cap D_1|$ is as small as possible. Take two vertices $y_1, y_2 \in G_2$ and set $D = (D_1 \times \{y_1\}) \cup (U_1 \times \{y_2\}) \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1$. If $y = y_1$, then $(x, y) \in D$. We consider the case that $y \neq y_1$. If there exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$, then $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. Otherwise, there exists a vertex $x_j \in U_1$ such that $xx_j \in E_1$, since G_1 has no isolated vertex and $|U_1 \cap D_1|$ is as small as possible. Thus, $(x, y)(x_j, y_2) \in E(G_1[G_2])$ and $(x_j, y_2) \in D$.

Case 2. $x \notin D_1$. There exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$. Thus, $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$.

Therefore, every vertex in $V(G_1[G_2]) - D$ is adjacent to some vertex in D, D is a dominating set of $G_1[G_2]$. Hence $\gamma(G_1[G_2]) \leq |D| = |D_1| + |U_1| = \gamma(G_1) + \iota(G_1)$.

□

Remark: The lower bound and upper bound in Theorem 2.2 are sharp. Clearly, $\gamma(P_4) = 2, \gamma(P_6) = 2, \iota(P_4) = 1, \iota(P_6) = 2$, we have $\gamma(P_4[P_4]) =$

$\gamma(P_4) = 2$, $\gamma(P_6[P_4]) = \gamma(P_6) + \iota(P_6) = 4$. Thus, a domination number of $P_4[P_4]$ achieves the lower bound (see Fig.1) and a domination number of $P_6[P_4]$ achieves the lower bound (see Fig.2).

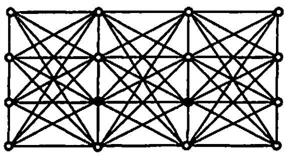

Fig.1. A dominating set of $P_4[P_4]$;

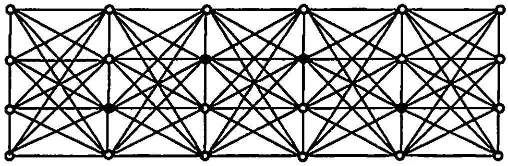

Fig.2. A dominating set of $P_6[P_4]$

We study the total domination number of $G_1[G_2]$ in the following theorem.

Theorem 2.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with at least two vertices. We have $\gamma_t(G_1[G_2]) = \gamma_t(G_1)$.*

Proof. Clearly, $\gamma_t(G_1[G_2]) \geq \gamma_t(G_1)$. Now we prove that $\gamma_t(G_1[G_2]) \leq \gamma_t(G_1)$. Let D_1 be a $\gamma_t(G_1)$-set of G_1. Set $D = D_1 \times \{y_1\} \subseteq V_1 \times V_2 \subseteq V(G_1[G_2])$ for some vertex $y_1 \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$. Then there exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$. Therefore $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. Thus, every vertex in $V(G_1[G_2])$ is adjacent to some vertex in D, D is a total dominating set of $G_1[G_2]$. Hence $\gamma_t(G_1[G_2]) \leq |D| = |D_1| = \gamma_t(G_1)$. From above we have $\gamma_t(G_1[G_2]) = \gamma_t(G_1)$. □

Next, we study the restrained domination number of $G_1[G_2]$.

Theorem 2.4. *Let $G_1 = (V_1, E_1)$ be a graph with no isolated vertex, and let $G_2 = (V_2, E_2)$ be a graph with at least two vertices. Then $\gamma_r(G_1[G_2]) = \gamma(G_1[G_2])$.*

Proof. We consider two cases.

Case 1. $|V_2| = 2$. Let $V_2 = \{y_1, y_2\}$. If G_2 is a path with two vertices, then $\gamma(G_1[G_2]) = \gamma(G_1)$ by Theorem 2.1. Let D_1 is a minimum dominating set of G_1, then $D_1 \times \{y_i\}$ is a dominating set of $G_1[G_2]$ for any $y_i \in V_2$. It is easy to see that $D_1 \times \{y_i\}$ ia also a restrained dominating set.

If G_2 is an empty graph with two vertices, then we claim that there exists a minimum dominating set D of $G_1[G_2]$ such that $|D \cap V(G_2^x)| \leq 1$ for each vertex of $x \in V_1$. In fact, If $|D \cap V(G_2^x)| = 2$ for some vertex of $x \in V_1$, that is, $(x, y_1), (x, y_2) \in D$, then there exists a vertex x_i such that

$xx_i \in E_1$ and $D' = (D - \{(x, y_1)\}) \cup \{(x_i, y_2)\}$ is a minimum dominating set of $G_1[G_2]$. Therefore, we can find a minimum dominating set D' of $G_1[G_2]$ such that $|D' \cap V(G_2^x)| \leq 1$ for each vertex of $x \in V_1$. Let (x, y) be any vertex of $G_1[G_2] - D'$, then one of two vertices (x_i, y_1) and (x_i, y_2) belongs to $G_1[G_2] - D'$ for a vertex $x_i \in V_1$ with $xx_i \in E_1$. Therefore D' is a restrained dominating set of $G_1[G_2] - D'$. Hence $\gamma_r(G_1[G_2]) = \gamma(G_1[G_2])$.

Case 2. $|V_2| \geq 3$. We claim that there exists a dominating set D such that $|D \cap V(G_2^x)| \leq 2$ for each vertex of $x \in V_1$. In fact, by Theorem 2.2, we can find a minimum dominating set of $D = (D_1 \times \{y_1\}) \cup (U' \times \{y_2\})$, where D_1 is a $\gamma(G_1)$-set and U' is a subset of U_1(the monitor set of D_1). It is easy to see that $|D \cap V(G_2^x)| \leq 2$ for each vertex of $x \in V_1$. Let (x, y) be any vertex of $G_1[G_2] - D$, then there exists a vertex $(x_i, y_j) \in G_1[G_2] - D$ such that $(x, y)(x_i, y_j) \in E(G_1[G_2])$. Therefore D is a restrained dominating set of $G_1[G_2] - D$. Hence $\gamma_r(G_1[G_2]) = \gamma(G_1[G_2])$.

□

It is clear that the dominating sets of $P_4[P_4]$ and $P_6[P_4]$ are restrained dominating sets in Fig.1 and Fig.2.

We will discuss the total restrained domination number of $G_1[G_2]$ in the following theorem 2.5.

Theorem 2.5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with at least two vertices. Then $\gamma_{tr}(G_1[G_2]) \leq \gamma_{tr}(G_1)$.*

Proof. Let D_1 be a $\gamma_{tr}(G_1)$-set of G_1. Set $D = D_1 \times \{y_1\} \subseteq V(G_1[G_2])$ for some vertex $y_1 \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1$. There exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$. If $y = y_1$, then $(x, y) \in D$. We have $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. If $y \neq y_1$, then $(x, y) \notin D$. We have $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$, $(x, y)(x_i, y) \in E(G_1[G_2])$ and $(x_i, y) \notin D$.

Case 2. $x \notin D_1$. Clearly, $(x, y) \notin D$. Therefore there exist two vertices $x_i \in D_1$ and $x_j \notin D_1$ such that $xx_i, xx_j \in E_1$. We have $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$, $(x, y)(x_j, y) \in E(G_1[G_2])$ and $(x_j, y) \notin D$. Thus, every vertex in $V(G_1[G_2]) - D$ is adjacent to some vertex in D and to some vertex in $V(G_1[G_2]) - D$, and every vertex in D is adjacent to some vertex in D, D is a total restricted dominating set of $G_1[G_2]$. Hence $\gamma_{tr}(G_1[G_2]) \leq |D| = |D_1| = \gamma_{tr}(G_1)$. From above we have $\gamma_{tr}(G_1[G_2]) \leq \gamma_{tr}(G_1)$.

□

Finally, we consider the connected domination number of $G_1[G_2]$.

Theorem 2.6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with at least two vertices. If $\gamma_c(G_1) = 1$ and $\gamma(G_2) \geq 2$, then $\gamma_c(G_1[G_2]) = 2$. Otherwise, $\gamma_c(G_1[G_2]) = \gamma_c(G_1)$.*

Proof. Clearly, $\gamma_c(G_1[G_2]) \geq \gamma_c(G_1)$. We consider three cases.

Case 1. $\gamma(G_2) = 1$. Let D_1 be a $\gamma_c(G_1)$-set of G_1 and $D_2 = \{y_1\}$ be a $\gamma(G_2)$-set of G_2. Set $D = D_1 \times \{y_1\} \subseteq V(G_1[G_2])$. We know that D is a dominating set of $G_1[G_2]$ from the proof of Theorem 2.1. Since $\langle D_1 \rangle$ is connected, $\langle D \rangle$ is also connected. Thus, D is a connected dominating set of $G_1[G_2]$. Hence $\gamma_c(G_1[G_2]) \leq |D| = |D_1| = \gamma_c(G_1)$.

Case 2. $\gamma(G_2) \geq 2$ and $\gamma_c(G_1) = 1$. It is easy to see that $\gamma_c(G_1[G_2]) = 2$.

Case 3. $\gamma(G_2) \geq 2$ and $\gamma_c(G_1) \geq 2$. Let D_1 be a $\gamma_c(G_1)$-set of G_1. Set $D = D_1 \times \{y_1\} \subseteq V(G_1[G_2])$ for some vertex $y_1 \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$. Since D_1 is a connected dominating set of G_1, there exists a vertex $x_i \in D_1$ such that $xx_i \in E_1$. Thus $(x, y)(x_i, y_1) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. Hence D is a dominating set of $G_1[G_2]$. Since $\langle D_1 \rangle$ is connected, $\langle D \rangle$ is also connected. Thus, D is a connected dominating set of $G_1[G_2]$. Hence $\gamma_c(G_1[G_2]) \leq |D| = |D_1| = \gamma_c(G_1)$. From above we have $\gamma_c(G_1[G_2]) = \gamma_c(G_1)$.

□

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