The sum numbers and the integral sum numbers of the graph $K_n \backslash E(C_{n-1})$

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Abstract The concept of the sum graph and integral sum graph were introduced by F.Harary. Let N denote the set of all positive integers. The sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A simple graph G is said to be a sum graph if it is isomorphic to a sum graph of some $S \subset N$. The sum number $\sigma(G)$ of G is the smallest number of isolated vertices which when added to G result in a sum graph. Let Z denote the set of all integers. The integral sum graph $G^+(S)$ of a finite subset $S \subset Z$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A simple graph G is said to be an integral sum graph if it is isomorphic to an integral sum graph of some $S \subset Z$. The integral sum number $\zeta(G)$ of G is the smallest number of isolated vertices which when added to G result in an integral sum graph. In this paper, we investigate and determine the sum number and the integral sum number of the graph $K_n \backslash E(C_{n-1})$. The results are presented as follows: $\zeta(K_n \backslash E(C_{n-1})) = \begin{cases} 0, & n = 4,5,6,7 \\ 2n-7, & n \geq 8 \end{cases}$ and $\sigma(K_n \backslash E(C_{n-1})) = \begin{cases} 1, & n = 4 \\ 2, & n = 5 \\ 5, & n = 6 \\ 7, & n = 7 \\ 2n-7, & n \geq 8 \end{cases}$

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1. Introduction

For a simple graph $G = (V, E)$, let V denote its vertex set and E its edge set. All other notation and terminology are referred to [1].

The concept of the sum graph and the integral sum graph were introduced by F.Harary ([2][3]). Let N denote the set of all positive integers. The sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A simple graph G is said to be a sum graph if it is isomorphic to the sum graph of some $S \subset N$. We say that S gives a sum labeling for G ([7]). The sum number $\sigma(G)$ of G is the smallest number of the isolated vertices, which result in a sum graph when added to G. The integral sum graph and an integral sum number $\zeta(G)$ of a graph G are also defined when S is extended from the positive integers set N to the integer set Z. It is obvious that $\zeta(G) \leq \sigma(G)$ for a graph G.

To better understand the notions of the sum graph and the integral sum graph, we give them the equivalent definitions ([7]). For a simple graph G and a positive integer m, a labeling of $G \cup mK_1$ is a mapping L from $V(G \cup mK_1)$ to N. A graph $G \cup mK_1$ is called a sum graph if there exists a labeling L such that $uv \in E$ if and only if there exists $w \in V(G \cup mK_1)$ with $L(w) = L(u) + L(v)$ for every pair of distinct vertices u and v of $G \cup mK_1$. Then L is said a sum labeling of $G \cup mK_1$, and L is an optimal sum labeling if $m = \sigma(G)$. Similarly, we also get the definitions of the integral sum labeling and the optimal integral sum labeling.

To simplify notations, we may assume that the vertices of a sum graph and integral sum graph are identified with their labeling throughout this paper.

As we know, it is very difficult to determine $\zeta(G)$ and $\sigma(G)$ in general. But for special classes of graphs, their sum numbers and integral sum numbers have still been derived, such as complete graphs, complete bipartite graph, cocktail party graph and $K_n \backslash E(K_r)$ with $r < n$ and so on ([2-12]), and it is sure that all observations will be useful to the research on a (integral) sum graph.

Let $K_n \backslash E(C_{n-1})$ denote the graph in which all edges of a cycle of $n-1$ vertices are deleted from a complete graph with n vertices. In this paper, the sum number and the integral sum number of the graph $K_n \backslash E(C_{n-1})$ is investigated and determined.

2. Main results

Let $K_n \backslash E(C_{n-1}) = (V, E)$, $V = A \cup B$ and $S = V \cup C$, where the edge-deleted cycle $C_{n-1} = a_1 a_2 a_3 \cdots a_{n-1} a_1$, $A = \{a_1, a_2, \cdots, a_{n-1}\}$, $B = \{b_1\}$ and C is the isolated vertex set (see Figure 1). Then $V = \{a_1, a_2, \cdots, a_{n-1}, b_1\}$ with $d_G(b_1) = n - 1$ and $d_G(a_i) = n - 3$ for any $i = 1, 2, \cdots, n-1$.

In this paper, some special signs are used. For some vertex $u_i \in V$, let u_i' and u_i'' denote the vertices such that $u_i u_i' \notin E$ and $u_i u_i'' \notin E$. Of course, according to the structure of the graph $K_n \backslash E(C_{n-1})$, if $d_G(u_i) = n - 1$ then $u_i = b_1$, which implies u_i' and u_i'' do not exist at all; if $d_G(u_i) = n - 3$ then both vertices u_i' and u_i'' exist.

In this section, we investigate and determine the sum numbers and the integral sum numbers of the graphs $K_n \backslash E(C_{n-1})$ for all positive integral $n \geq 4$.

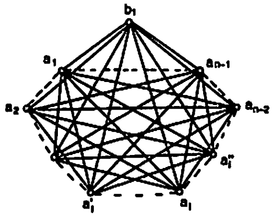

Figure 1

Lemma 2.1 For $n = 4, 5, 6, 7$, $K_n \backslash E(C_{n-1})$ is an integral sum graph.

Proof: The integral sum labels are given respectively (see Figure 2,3,4,5). Thus, Lemma 2.1 holds. □

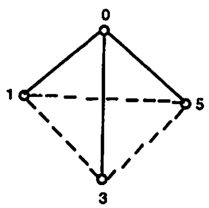

Figure 2

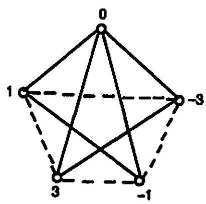

Figure 3

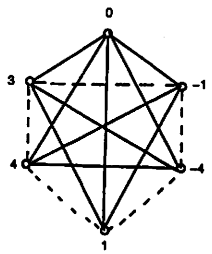

Figure 4

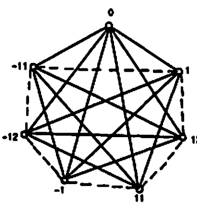

Figure 5

Lemma 2.2 Let $n \geq 8$. Then $K_n \backslash E(C_{n-1})$ is not an integral sum graph.

Proof: Let $n \geq 8$. We argue by contradiction. Assume that $K_n \backslash E(C_{n-1}) = (V, E)$ is an integral sum graph. Then $a + a' \in V$ for any edge $aa' \in E$.

Assume that $V = \{u_1, u_2, \cdots, u_n\}$, where $u_1 < u_2 < \cdots < u_n$. Then we obtain at least $(2n-3)$ numbers $u_1 + u_2 < u_1 + u_3 < \cdots < u_1 + u_n < u_2 + u_n < u_3 + u_n < \cdots < u_{n-1} + u_n$. Let $U = \{u_1 + u_2, u_1 + u_3, \cdots, u_1 + u_n, u_2 + u_n, u_3 + u_n, \cdots, u_{n-1} + u_n\}$. According to the structure of the graph $K_n \backslash E(C_{n-1})$, each vertex has at most 2 non-adjacent vertices. Then in the set U there are at least $(2n-3) - 2 - 2$ numbers which must be edge sums of the graph $K_n \backslash E(C_{n-1})$. Since $K_n \backslash E(C_{n-1})$ is an integral sum graph, $(2n-3) - 2 - 2 \leq n$, that is, $n \leq 7$, but $n \geq 8$, a contradiction. Thus, Lemma 2.2 holds. □

Lemma 2.3 Let $n \geq 8$. Then $0 \notin S$.

Proof: By Lemma 2.2, $\zeta(K_n \backslash E(C_{n-1})) \geq 1$, which implies the isolated set $C \neq \emptyset$. Let $c_1 \in C$. We argue by contradiction. If $0 \in V$, then $0 + c_1 = c_1 \in S$, which implies there is an edge between the isolated vertex and a vertex in the set V, a contradiction. Similarly, it appears a contradiction if $0 \in C$. Thus, Lemma 2.3 holds. □

Lemma 2.4. Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Assume $a_{\max}$ be a vertex which absolute value is maximum in the set V. Then there exists one edge adjacent to $a_{\max}$ such that their label sum is belong to the isolated set.

Proof: Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Assume $a_{\max}$ be a vertex which absolute value is maximum in the set V. Without loss of generality, we may assume $a_{\max} \in V$ and $a_{\max} > 0$ (Otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

We argue by contradiction. Suppose that $a + a_{\max} \in V$ for all edges $aa_{\max} \in E$. According to the choice of $a_{\max}$, we have $a < 0$ and $a + a_{\max} > 0$. Notice that $d_G(a_{\max}) \in \{n-3, n-2, n-1\}$. Then assume $v_1, v_2, \cdots, v_{n-3}$ are its adjacent vertices, where $v_1 < v_2 < \cdots < v_{n-3} < 0$. Then $0 < v_1 + a_{\max} < v_2 + a_{\max} < \cdots < v_{n-3} + a_{\max}$ and they belong to the vertices set V.

Up to now, there are in all three vertices v_{n-2}, v_{n-1}, v_n, which signs may be positive. So $n - 3 \leq 3$, that is $n \leq 6$, but $n \geq 8$, which is a contradiction. Thus, Lemma 2.4 holds. □

Lemma 2.5. Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Assume $a_{\max}$ be a vertex which absolute value is maximum in the set V. Then $a + a_{\max} \in C$ for all edges $aa_{\max} \in E$.

Proof: Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Assume that $a_{\max}$ is a vertex which absolute value is maximum in the set V. Without loss of generality, we may assume that and $a_{\max} \in V$ and $a_{\max} > 0$ (Otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

According to the structure of the graph $K_n \backslash E(C_{n-1})$, $d_G(a_{\max}) \in \{n-3, n-1\}$. Since $n \geq 8$, there are at least 5 edges adjacent to the vertex $a_{\max}$. By Lemma 2.4, there exists an edge $a_{i_j} a_{\max} \in E$ such that $a_{i_j} + a_{\max} \in C$. Then for any edge $aa_{\max} \in E \backslash \{a_{i_j} a_{\max}\}$, $a_{i_j} + (a + a_{\max}) = (a_{i_j} + a_{\max}) + a \notin S$, which implies there is no edge between $(a + a_{\max})$ and a_{i_j}, that is, $a + a_{\max} \in (\{a_{i_j}\} \cup \{a'_{i_j}\} \cup \{a''_{i_j}\}) \cup C$, denoted (*). Thus, there are at most 3 such edges such that their sums belong to the vertices subset $(\{a_{i_j}\} \cup \{a'_{i_j}\} \cup \{a''_{i_j}\})$. At the same time, there must exist at least 2 edges adjacent to $a_{\max}$, denoted $a_{i_1} a_{\max}, a_{i_2} a_{\max} \in E$, such that their sums belong to the isolated set C, that is, $a_{i_1} + a_{\max}, a_{i_2} + a_{\max} \in C$. Similarly (*), the sums of other edges adjacent to $a_{\max}$ must belong to $[(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\})] \cup C$. Since $|(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\})| \in \{0, 1, 2\}$, there must at least 3 edges adjacent to $a_{\max}$, denoted $a_{i_1} a_{\max}, a_{i_2} a_{\max}, a_{i_3} a_{\max} \in E$, such that their sums belong to the isolated set C, that is, $a_{i_1} + a_{\max}, a_{i_2} + a_{\max}, a_{i_3} + a_{\max} \in C$. Similarly (*), the sums of other edges adjacent to $a_{\max}$ must belong to $[(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\}) \cap (\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\})] \cup C$. Since $|(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\}) \cap (\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\})| \in \{0, 1\}$, there must at least 4 edges adjacent to $a_{\max}$, denoted $a_{i_1} a_{\max}, a_{i_2} a_{\max}, a_{i_3} a_{\max}, a_{i_4} a_{\max} \in E$, such that their sums belong to the isolated set C, that is, $a_{i_1} + a_{\max}, a_{i_2} + a_{\max}, a_{i_3} + a_{\max}, a_{i_4} + a_{\max} \in C$. Similarly (*), the sums of other edges adjacent to $a_{\max}$ must belong to $[(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\}) \cap (\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\}) \cap (\{a_{i_4}\} \cup \{a'_{i_4}\} \cup \{a''_{i_4}\})] \cup C$. Since $(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\}) \cap (\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\}) \cap (\{a_{i_4}\} \cup \{a'_{i_4}\} \cup \{a''_{i_4}\}) = \emptyset$, all edges adjacent to $a_{\max}$ such that their sums belong to the isolated set C.

Thus, Lemma 2.5 holds. □

Lemma 2.6. Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Then $a_i + a_j \in C$ for any edge $a_i a_j \in E$.

Proof: Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Without loss of generality, we may assume that $A = \{a_1, a_2, \cdots, a_{n-1}\} = \{a'_1, a'_2, a'_3, ..., a'_{n-1}\}$ with $a'_1 < a'_2 < ... < a'_{n-1}$. Then $b_1 + a'_1 < b_1 + a'_2 < ... < b_1 + a'_{n-1}$.

Firstly, there is at least one edge $b_1 a_{i_1}$ such that $b_1 + a_{i_1} \in C$. Otherwise, $\{b_1 + a'_1, b_1 + a'_2, ..., b_1 + a'_{n-1}\} \subseteq \{a'_1, a'_2, a'_3, ..., a'_{n-1}, b_1\}$. However, if there was an edge sum $b_1 + a'_i = b_1$, then $a'_i = 0$, which contradicts with Lemma 2.3. Then only $\{b_1 + a'_1, b_1 + a'_2, ..., b_1 + a'_{n-1}\} = \{a'_1, a'_2, a'_3, ..., a'_{n-1}\}$. Since $a'_1 < a'_2 < ... < a'_{n-1}$ and $b_1 + a'_1 < b_1 + a'_2 < ... < b_1 + a'_{n-1}$, we have $b_1 = 0$, which is a contradiction with Lemma 2.3.

Secondly, to prove that $b_1 + a_j \in C$ for any edge $b_1 a_j \in E$. In fact, for any other edge $b_1 a_j \in E \backslash b_1 a_{i_1}$, $(b_1 + a_{i_1}) + a_j = (b_1 + a_j) + a_{i_1} \notin S$, which implies that $b_1 + a_j \in (\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cup C$, denoted (*). Then there are at most 3 edges adjacent to b_1 such that their sums belong to the vertices subset $(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\})$ and others belong to the isolated set C. Since $(n-1) - 1 - 3 = n - 5 \geq 3$, there are at least 3 edges adjacent to b_1 besides $b_1 a_{i_1}$, denoted $b_1 + a_{i_2}$, $b_1 + a_{i_3}$, $b_1 + a_{i_4}$, such that their sums belong to C. Similarly (*), if there exists an edge sum adjacent to b_1 which does not belong to C, then it not only belong to the vertices subset $(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\})$, but also belongs to $(\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\})$, $(\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\})$ and $(\{a_{i_4}\} \cup \{a'_{i_4}\} \cup \{a''_{i_4}\})$. Since $(\{a_{i_1}\} \cup \{a'_{i_1}\} \cup \{a''_{i_1}\}) \cap (\{a_{i_2}\} \cup \{a'_{i_2}\} \cup \{a''_{i_2}\}) \cap (\{a_{i_3}\} \cup \{a'_{i_3}\} \cup \{a''_{i_3}\}) \cap (\{a_{i_4}\} \cup \{a'_{i_4}\} \cup \{a''_{i_4}\}) = \emptyset$. Then there is no edge adjacent to b_1 such that its sum belongs to the vertices set V and all belong to the isolated set C. This claim holds.

Finally, to prove $a_i + a_j \in C$ for any edge $a_i a_j \in E$ for $n \geq 8$. We consider it from two cases as follows.

(I) Consider any edge $a_i a_j$ with $\{a_i, a_j\} \neq \{a'_{\max}\} \cup \{a''_{\max}\}$. In this case, it shows that there is at least one element in $\{a_i, a_j\}$ such that there is an edge with the vertex $a_{\max}$. Assume a_i. Since $(a_i + a_j) + a_{\max} = (a_{\max} + a_i) + a_j \notin S$, $a_i + a_j$ has no edge with the vertex $a_{\max}$. So $a_i + a_j \in (\{a_{\max}\} \cup \{a'_{\max}\} \cup \{a''_{\max}\}) \cup C$, denoted (@).

(I.1) Case 1.1: $b_1 = a_{\max}$ (see Figure 6). Then $(\{a'_{\max}\} \cup \{a''_{\max}\}) = \emptyset$ and $a_i + a_j \in \{a_{\max}\} \cup C$ for any edge $a_i a_j \in E$.

If there is one edge, denoted $a_{i_1} a_{j_1} \in E$ such that $a_{i_1} + a_{j_1} = a_{\max}$, then others with endpoint a_{i_1} belong to the isolated set C. Since $d_G(a_{i_1}) \in \{n - 1, n - 3\}$ and $n \geq 8$, there exists at least one vertex belong to the vertices subset $V \backslash (\{a_{j_1}\} \cup (\{a'_{i_1}\} \cup \{a''_{i_1}\}))$, denoted a_k, such that $a_{i_1} a_k \in E$ and $a_{\max} a_k \in E$. Then $a_{i_1} + a_k \in C$ and $a_{\max} + a_{i_1} \in C$. So $(a_{i_1} + a_k) + a_{j_1} = (a_{i_1} + a_{j_1}) + a_k = a_{\max} + a_k \in S$, which is a contradiction with $a_{i_1} + a_k \in C$. Thus, $a_i + a_j \neq a_{\max}$ for any edge $a_i a_j \in E$.

(I.2) Case 1.2: $b_1 \neq a_{\max}$ (see Figure 7 and Figure 8). In this case $b_1 a_{\max} \in E$ and $b_1 \notin (\{a_{\max}\} \cup \{a'_{\max}\} \cup \{a''_{\max}\})$. Since $b_1 + a_i \in C$ for any edge $b_1 a_i \in E$. Then $(a_i + a_j) + b_1 = (b_1 + a_i) + a_j \notin S$, which implies $a_i + a_j \in \{b_1\} \cup C$. Combined (@), we have $a_i + a_j \in C$.

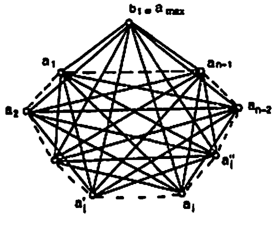

Figure 6

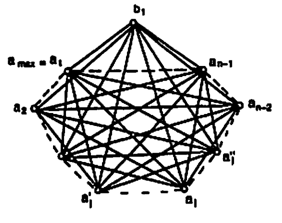

Figure 7

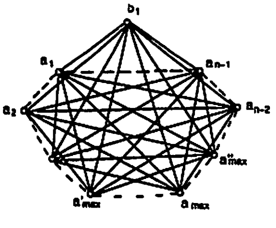

Figure 8

(II) $\{a_i, a_j\} = \{a'_{\max}\} \cup \{a''_{\max}\}$. Since $n \geq 8$, there is at least one vertex, denoted a_i, such that the edge $d_G(a_i) = n-3$ and $a''_{\max}a_i \in E$. By the results of (I), we have $a''_{\max} + a_i \in C$. Since $a''_{\max} + a_i \in C$ and $a''_{\max} + b_1 \in C$, $(a'_{\max} + a''_{\max}) + a_i = (a''_{\max} + a_i) + a'_{\max} \notin S$ and $(a'_{\max} + a''_{\max}) + b_1 = a'_{\max} + (a''_{\max} + b_1) \notin S$, then $(a'_{\max} + a''_{\max}) \in \{a_i, a'_i, a''_i\} \cup C$ and $(a'_{\max} + a''_{\max}) \in \{b_1\} \cup C$, which implies $a'_{\max} + a''_{\max} \in C$.

Thus, $a_i + a_j \in C$ for any edge $a_ia_j \in E$. Thus, Lemma 2.6 holds. □

Lemma 2.7. Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Then $\zeta(K_n \backslash E(C_{n-1})) \geq 2n - 7$.

Proof: Let $n \geq 8$ and $K_n \backslash E(C_{n-1}) = (V, E)$. Without loss of generality, we may assume that $V = \{x_1, x_2, x_3, ..., x_n\}$, where $x_1 < x_2 < ... < x_n$ and $x_n > 0$ (otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S). According to Lemma 2.6, $x_i + x_j \in C$ for all edges $x_ix_j \in E$. On the other hand, since $x_1 + x_2 < x_1 + x_3 < ... < x_1 + x_n < x_2 + x_n < ... < x_{n-1} + x_n$, these $2n - 3$ numbers are distinct and there are at most 4 sums are not the sums of the edges of the graph $K_n \backslash E(C_{n-1})$. So $\zeta(K_n \backslash E(C_{n-1})) \geq (2n-3) - 4$, that is, $\zeta(K_n \backslash E(C_{n-1})) \geq 2n - 7$. □

Lemma 2.8. For $n \geq 8$, $\sigma(K_n \backslash E(C_{n-1})) \leq 2n - 7$.

Proof: Let $K_n \backslash E(C_{n-1}) = (V, E)$ and $V = \{x_1, x_2, x_3, \cdots, x_n\}$ and $S = V \cup C$, where C is the isolated set. Firstly, let $x_i = (i-1) \times 10 + 1$ and $c_j = j \times 10 + 2$. Then $V = \{(i-1) \times 10 + 1 : i = 1, 2, ..., n\}$ and the isolated set $C = \{c_j : j = 1, 2, ..., 2n-3\} - \{c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}\}$. This key is to find $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$ and the cycles C_{n-1}. Secondly, let us look for them and verify that this is an optimal sum labeling in detail.

Case 1. n is odd. Let $\{c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}\} = \{c_{n-4}, c_{n-6}, c_{2n-6}, c_{2n-3}\}$. Then the isolated set $C = \{c_j : j = 1, 2, ..., 2n-3\} - \{c_{n-4}, c_{n-6}, c_{2n-6}, c_{2n-3}\}$ and $E(C_{n-1}) = \{x_1x_{n-3}, x_2x_{n-4}, x_3x_{n-5}, \cdots, x_{\frac{n-3}{2}}x_{\frac{n-1}{2}}; x_1x_{n-5}, x_2x_{n-6}, x_3x_{n-7}, \cdots, x_{\frac{n-5}{2}}x_{\frac{n-3}{2}}; x_{n-3}x_{n-1}, x_{n-4}x_n; x_{n-1}x_n\}$. In fact, we have

(1) The vertices in S are distinct.

(2) For any vertices $x_i \in \{x_1, x_2, x_3, \cdots, x_n\}$ and $c_k \in C$, since $x_i + c_k \equiv 3(\text{mod}10)$, $x_i + c_k \notin S$.

(3) For any distinct vertices $c_i, c_k \in C$, since $c_i + c_k \equiv 4(\text{mod}10)$, $c_i + c_k \notin S$.

(4) Let $1 \leq i \neq j \leq n$. For any distinct vertices $x_i, x_j \in \{x_1, x_2, x_3, \cdots, x_n\}$, $x_i + x_j = (i + j - 2) \times 10 + 2 = c_{i+j-2}$.

Since $x_i + x_j = c_{n-4} \iff i + j - 2 = n - 4 \iff i + j = n - 2 \iff (i, j) \in \{(1, n-3), (2, n-4), (3, n-5), \cdots, (\frac{n-3}{2}, \frac{n-1}{2})\}$.

Since $x_i + x_j = c_{n-6} \iff i + j - 2 = n - 6 \iff i + j = n - 4 \iff (i,j) \in \{(1, n-5), (2, n-6), (3, n-7), \cdots, (\frac{n-5}{2}, \frac{n-3}{2})\}$.

Since $x_i + x_j = c_{2n-6} \iff i + j - 2 = 2n - 6 \iff i + j = 2n - 4 \iff (i,j) \in \{(n-4, n), (n-3, n-1)\}$.

Since $x_i + x_j = c_{2n-3} \iff i+j-2 = 2n-3 \iff i+j = 2n-4 \iff (i,j) \in \{(n-1, n)\}$. So it is an optimal sum labeling of $(K_n \backslash E(C_{n-1})) \cup (2n-7)K_1$, where n is odd.

Case 2. n is even. Let $\{c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}\} = \{c_{n-7}, c_{n-2}, c_{2n-6}, c_{2n-4}\}$.

Then the isolated set $C=\{c_j : j = 1, 2, ..., 2n-3\} - \{c_{n-7}, c_{n-2}, c_{2n-6}, c_{2n-4}\}$ and $E(C_{n-1}) = \{x_1x_{n-6}, x_2x_{n-7}, \cdots, x_{\frac{n-8}{2}}x_{\frac{n-2}{2}}, x_{\frac{n-6}{2}}x_{\frac{n-4}{2}}; x_1x_{n-1}, x_2x_{n-2}, \cdots, x_{\frac{n-4}{2}}x_{\frac{n+4}{2}}, x_{\frac{n-2}{2}}x_{\frac{n+2}{2}}; x_{n-3}x_{n-1}, x_{n-4}x_n; x_{n-2}x_n\}$. In fact, we have

(1) The vertices in S are distinct.

(2) For any vertices $x_i \in \{x_1, x_2, x_3, \cdots, x_n\}$ and $c_k \in C$, since $x_i + c_k \equiv 3(\text{mod}10)$, $x_i + c_k \notin S$.

(3) For any distinct vertices $c_i, c_k \in C$, since $c_i + c_k \equiv 4(\text{mod}10)$, $c_i + c_k \notin S$.

(4) Let $1 \le i \neq j \le n$. For any distinct vertices $x_i, x_j \in \{x_1, x_2, x_3, \cdots, x_n\}$, $x_i + x_j = (i + j - 2) \times 10 + 2 = c_{i+j-2}$.

Since $x_i + x_j = c_{n-7} \iff i + j - 2 = n - 7 \iff i + j = n - 5 \iff (i,j) \in \{(1, n-6), (2, n-7), \cdots, (\frac{n-8}{2}, \frac{n-2}{2}), (\frac{n-6}{2}, \frac{n-4}{2})\}$.

Since $x_i + x_j = c_{n-2} \iff i + j - 2 = n - 2 \iff i + j = n \iff (i,j) \in \{(1, n-1), (2, n-2), \cdots, (\frac{n-4}{2}, \frac{n+4}{2}), (\frac{n-2}{2}, \frac{n+2}{2})\}$.

Since $x_i + x_j = c_{2n-6} \iff i + j - 2 = 2n - 6 \iff i + j = 2n - 4 \iff (i,j) \in \{(n-4, n), (n-3, n-1)\}$.

Since $x_i + x_j = c_{2n-4} \iff i+j-2 = 2n-4 \iff i+j = 2n-2 \iff (i,j) \in \{(n-2, n)\}$. Then it is an optimal sum labeling of $(K_n \backslash E(C_{n-1})) \cup (2n-7)K_1$, where n is even. Thus, $\sigma(K_n \backslash E(C_{n-1})) \le 2n - 7$. □

Lemma 2.9. For $n = 4$, $\sigma(K_n \backslash E(C_{n-1})) = 1$.

Proof: According to the definition of the sum graph, $\sigma(G) \ge \delta(G)$ for any graph G. When $n = 4$, $\delta(K_n \backslash E(C_{n-1})) = 1$. Then $\sigma(K_n \backslash E(C_{n-1})) \ge 1$. On the other hand, Figure 9 gives one sum labeling of the graph $(K_4 \backslash E(C_3)) \cup 1 \cdot K_1$. Then $\sigma(K_n \backslash E(C_{n-1})) \le 1$. Thus, $\sigma(K_4 \backslash E(C_3)) = 1$ holds.

Lemma 2.10. For $n = 5$, $\sigma(K_n \backslash E(C_{n-1})) = 2$.

Proof: According to the definition of the sum graph, $\sigma(G) \ge \delta(G)$ for any graph G. When $n = 5$, $\delta(K_n \backslash E(C_{n-1})) = 2$. Then $\sigma(K_n \backslash E(C_{n-1})) \ge 2$. On the other hand, Figure 10 gives one sum labeling of the graph $(K_5 \backslash E(C_4)) \cup 2 \cdot K_1$. Then $\sigma(K_n \backslash E(C_{n-1})) \le 2$. Thus, $\sigma(K_5 \backslash E(C_4)) = 2$ holds. □

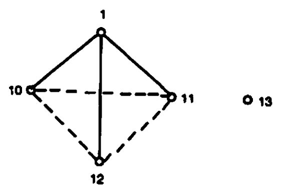

Figure 9

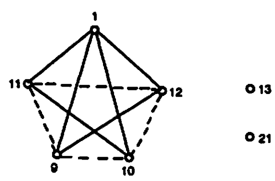

Figure 10

Lemma 2.11. For $n = 6$, $\sigma(K_n \backslash E(C_{n-1})) = 5$.

Proof: First, Figure 11 gives one sum labeling of the graph $(K_6 \backslash E(C_5)) \cup 5 \cdot K_1$. Then $\sigma(K_n \backslash E(C_{n-1})) \leq 5$.

Second, we just show $\sigma(K_6 \backslash E(C_5)) \geq 5$ in the following. Let $V(K_6 \backslash E(C_5)) = \{a_1, a_2, a_3, a_4, a_5, b_1\}$ and $E(C_5) = a_1a_2a_3a_4a_5a_1$. Then $d(b_1) = 5$ and $d(a_i) = 3$ for any $i = 1, 2, 3, 4, 5$ (see Figure 12).

Assume $a_{\max}$ be a vertex which value is maximum in the set V. According to the definition of sum labeling, $a + a_{\max} \in C$ for all edges $aa_{\max} \in E$.

Case 1. $a_{\max} \in \{b_1\}$ (see Figure13). According to the definition of sum labeling, $a + a_{\max} \in C$ for all edges $aa_{\max} \in E$ and $d(a_{\max}) = 5$, we have $\sigma(K_6 \backslash E(C_5)) \geq 5$.

Case 2. $a_{\max} \in V(C_5)$, that is, $a_{\max} \in \{a_1, a_2, a_3, a_4, a_5\}$. In this case, according to the symmetry of the vertices a_1, a_2, a_3, a_4, a_5, without loss of generality we may assume $a_{\max} = a_5$ (see Figure14). In order to improve it, Claim 1 below is very important.

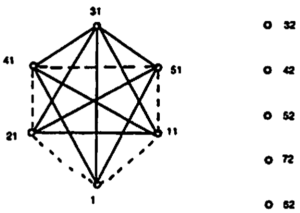

Figure 11

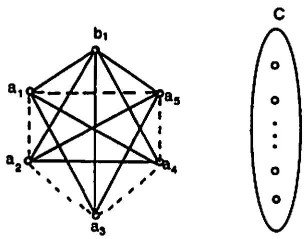

Figure 12

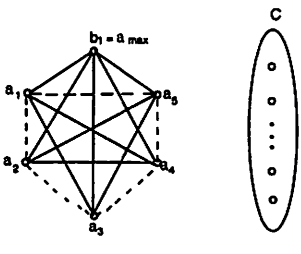

Figure 13

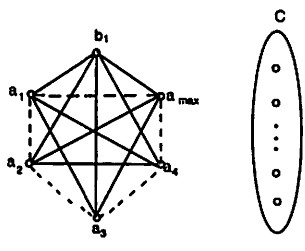

Figure 14

Claim 1. For any edge $ab \in E(K_6 \backslash E(C_5))$, $a + b \in C$. In fact, as $a_{\max}$ is a vertex which sum labeling value is maximum in the set V, $a_{\max} + b_1 \in C$ and $a_{\max} + a_i \in C$ for $i = 2, 3$.

(i) Consider the edges sums $b_1 + a_1, b_1 + a_2, b_1 + a_3, b_1 + a_4$. For $i = 1, 2, 3, 4$, since $a_{\max} + b_1 \in C$, $(a_{\max} + b_1) + a_i = a_{\max} + (b_1 + a_i) \notin S$, which implies $b_1 + a_i \in \{a_{\max}, a_1, a_4\} \cup C$, denoted (1). This shows that there is at least one edge, which is adjacent to b_1, such that its sum must belong to the isolated set C. Let $\{b_1 + a_1, b_1 + a_2, b_1 + a_3, b_1 + a_4\} = \{b_1 + a_{i_1}, b_1 + a_{i_2}, b_1 + a_{i_3}, b_1 + a_{i_4}\}$. Assume that $b_1 + a_{i_1} \in C$. Then for any $j = 2, 3, 4$, $(b_1 + a_{i_j}) + a_{i_1} = (b_1 + a_{i_1}) + a_{i_j} \notin S$, which implies that $b_1 + a_{i_j} \in \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cup C$, denoted (2).

Since $a_{\max}$ and a_{i_1} are distinct, $|\{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cap \{a_{\max}, a_1, a_4\}| \leq 2$. Combined (1) and (2), there is at least one edge in the subset $\{b_1 + a_{i_2}, b_1 + a_{i_3}, b_1 + a_{i_4}\}$, denoted $b_1 + a_{i_2}$, such that its sum must belong to the isolated set C. Then for any $j = 3, 4$, $(b_1 + a_{i_j}) + a_{i_2} = (b_1 + a_{i_2}) + a_{i_j} \notin S$, which implies that $b_1 + a_{i_j} \in \{a_{i_2}, a'_{i_2}, a''_{i_2}\} \cup C$, denoted (3).

Since $a_{\max}$, a_{i_1} and a_{i_2} are distinct, $|\{a_{i_2}, a'_{i_2}, a''_{i_2}\} \cap \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cap \{a_{\max}, a_1, a_4\}| \leq 1$. Combined (1) (2) and (3), there is at least one edge in the subset $\{b_1 + a_{i_3}, b_1 + a_{i_4}\}$, denoted $b_1 + a_{i_3}$, such that its sum must belong to the isolated set C. Then for any $j = 4$, $(b_1 + a_{i_j}) + a_{i_3} = (b_1 + a_{i_3}) + a_{i_j} \notin S$, which implies that $b_1 + a_{i_j} \in \{a_{i_3}, a'_{i_3}, a''_{i_3}\} \cup C$, denoted (4).

Since $a_{\max}$, a_{i_1}, a_{i_2} and a_{i_3} are distinct, $|\{a_{i_3}, a'_{i_3}, a''_{i_3}\} \cap \{a_{i_2}, a'_{i_2}, a''_{i_2}\} \cap \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cap \{a_{\max}, a_1, a_4\}| = 0$. Combined (1)(2)(3)(4), $b_1 + a_{i_4} \in C$. Thus, $\{b_1 + a_1, b_1 + a_2, b_1 + a_3, b_1 + a_4\} \subseteq C$.

(ii) Consider the edges sums $a_1 + a_3$ and $a_2 + a_4$. Since $b_1 + a_1 \in C$, $(b_1 + a_1) + a_3 = b_1 + (a_1 + a_3) \notin S$, which implies $a_1 + a_3 \in \{b_1\} \cup C$, denoted (5). Since $a_3 + a_{\max} \in C$, $(a_3 + a_{\max}) + a_1 = (a_1 + a_3) + a_{\max} \notin S$, which implies $a_1 + a_3 \in \{a_1, a_4, a_{\max}\} \cup C$, denoted (6). Combined (5)(6), we have $a_1 + a_3 \in C$. Similarly, $a_2 + a_4 \in C$.

(iii) Consider the edge sum $a_1 + a_4$. Since $b_1 + a_1 \in C$, $(b_1 + a_1) + a_4 = b_1 + (a_1 + a_4) \notin S$, which implies $a_1 + a_4 \in \{b_1\} \cup C$. Since $a_1 + a_3 \in C$, $(a_1 + a_3) + a_4 = (a_1 + a_4) + a_3 \notin S$, which implies $a_1 + a_4 \in \{a_2, a_3, a_4\} \cup C$. Thus, only $a_1 + a_4 \in C$.

Up to now, we have proved that $a + b \in C$ for any $ab \in E(K_6 \backslash E(C_5))$, that is, Claim 1 holds.

Let $K_6 \backslash E(C_5) = (V, E)$. Without loss of generality, we may assume that $V = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, where $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$.

According to Claim 1, $x_i + x_j \in C$ for all edges' sum $x_i x_j \in E(K_6 \backslash E(C_5))$. On the other hand, since $x_1 + x_2 < x_1 + x_3 < x_1 + x_4 < x_1 + x_5 < x_1 + x_6 < x_2 + x_6 < x_3 + x_6 < x_4 + x_6 < x_5 + x_6$, these $2n - 3 = 9$ numbers are distinct and there are at most 4 sums are not the sums of the edges of the graph $K_6 \backslash E(C_5)$. So $\sigma(K_6 \backslash E(C_5)) \geq (2n - 3) - 4 = 5$, that is, $\sigma(K_6 \backslash E(C_5)) \geq 2n - 7 = 5$.

Thus, Lemma 2.11 holds. □

Lemma 2.12. For $n = 7$, $\sigma(K_n \backslash E(C_{n-1})) = 7$.

Proof: Firstly, Figure 15 gives one sum labeling of the graph $(K_7 \backslash E(C_6)) \cup 7 \cdot K_1$. Then $\sigma(K_7 \backslash E(C_6)) \leq 7$.

Secondly, we just show $\sigma(K_7 \backslash E(C_6)) \geq 7$ in the following. Let $V(K_7 \backslash E(C_6)) = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1\}$ and $E(C_5) = a_1 a_2 a_3 a_4 a_5 a_6 a_1$. Then $d(b_1) = 6$ and $d(a_i) = 4$ for any $i = 1, 2, 3, 4, 5, 6$ (see Figure 16).

Assume $a_{\max}$ be a vertex which value is maximum in the set V. According to the definition of the sum labeling, $a + a_{\max} \in C$ for all edges $a a_{\max} \in E$. In order to prove $\sigma(K_7 \backslash E(C_6)) \geq 7$, Claim 2 below plays a key role and we will give it the proof.

Claim 2. For any edge $ab \in E(K_7 \backslash E(C_6))$, $a + b \in C$. In fact, we will discuss it in the following two cases.

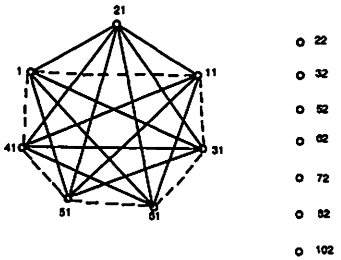

Figure 15

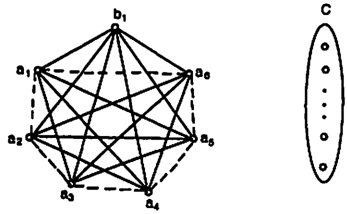

Figure 16

Case 1. $a_{\max} \in \{b_1\}$ (see Figure17). Since $a_{\max}$ is a vertex which sum labeling value is maximum in the set V, $b_1 + a_i \in C$ for any $i = 1,2,3,4,5,6$.

(i) Consider the edge sums $a_1 + a_3, a_1 + a_4, a_1 + a_5$. Since $(a_1 + a_i) + b_1 = (a_1+b_1)+a_i \notin S$, $a_1+a_i \in \{b_1\}\cup C$ for any $i = 3,4,5$. Then there exists at least two edges adjacent to a_1, denoted $a_1 + a_{i_2}$ and $a_1 + a_{i_3}$, such that $a_1 + a_{i_2} \in C$ and $a_1 + a_{i_3} \in C$.

Let $\{a_1 + a_{i_1}, a_1 + a_{i_2}, a_1 + a_{i_3}\} = \{a_1 + a_3, a_1 + a_4, a_1 + a_5\}$. At the same time, there exists at most one edge adjacent to a_1 such that its edge sum equals b_1. If yes, then it must be $a_1 + a_{i_1}$ such that $a_1 + a_{i_1} = b_1$. Since $a_1 + a_{i_1} = b_1$ and $a_1 + a_{i_2} \in C$, $b_1 + a_{i_2} = (a_1 + a_{i_1}) + a_{i_2} = (a_1 + a_{i_2}) + a_{i_1} \notin S$, but $b_1 + a_{i_2} \in S$, a contradiction. Thus, $a_1 + a_3, a_1 + a_4, a_1 + a_5 \in C$.

(ii) Consider the edge sums $a_2 + a_4, a_2 + a_5, a_2 + a_6$. The proof of (i) is also suitable to prove $a_2+a_4, a_2+a_5, a_2+a_6 \in C$. Thus, $a_2+a_4, a_2+a_5, a_2+a_6 \in C$.

(iii) Consider the edge sums $a_3 + a_5, a_3 + a_6$. Since $b_1 + a_6 \in C$, $(b_1 + a_6) + a_3 = b_1 + (a_3 + a_6) \notin S$, which implies $a_3 + a_6 \in \{b_1\} \cup C$, denoted (2). Similarly, $a_3 + a_5 \in \{b_1\} \cup C$, denoted (3). Since $a_2 + a_6 \in C$, $(a_2 + a_6) + a_3 = a_2 + (a_3 + a_6) \notin S$, which implies $a_3 + a_6 \in \{a_1, a_2, a_3\} \cup C$. Combined (2), $a_3+a_6 \in C$. Since $a_3+a_6 \in C$, $(a_3+a_6)+a_5 = (a_3+a_5)+a_6 \notin S$, which implies $a_3+a_5 \in \{a_1, a_5, a_6\}\cup C$. Combined (3), $a_3+a_5 \in C$. Thus, $a_3+a_5, a_3+a_6 \in C$.

(iv) Consider the edge sums a_4+a_6. Since $b_1+a_4 \in C$, $(b_1+a_4)+a_6 = b_1+(a_4+a_6) \notin S$, which implies $a_4+a_6 \in \{b_1\}\cup C$, denoted (4). Since $a_3+a_6 \in C$, $(a_3 + a_6) + a_4 = a_3 + (a_4 + a_6) \notin S$, which implies $a_4 + a_6 \in \{a_2, a_3, a_4\} \cup C$. Combined (4), $a_4 + a_6 \in C$.

Case 2. $a_{\max} \in V(C_6)$, that is, $a_{\max} \in \{a_1, a_2, a_3, a_4, a_5, a_6\}$. In this case, without loss of generality we may assume $a_{\max} = a_6$ (see Figure18). Since $a_{\max}$ is a vertex which sum labeling value is maximum in the vertices set V, we have $a_i + a_{\max}, b_1 + a_{\max} \in C$ for any $i = 2,3,4$ with $b_1 + a_{\max} = b_1 + a_6$.

(i) Consider the edges $a_1 + b_1, a_2 + b_1, a_3 + b_1, a_4 + b_1, a_5 + b_1$. For any $i = 1,2,3,4,5$, since $b_1 + a_{\max} \in C$, $(a_i + b_1) + a_{\max} = a_i + (b_1 + a_{\max}) \notin S$, which implies $a_i + b_1 \in \{a_1, a_5, a_{\max}\} \cup C$, denoted (5). Then there are at least two edge sums in the subset $\{a_1 + b_1, a_2 + b_1, a_3 + b_1, a_4 + b_1, a_5 + b_1\}$, denoted $a_{i_1} + b_1$ and $a_{i_2} + b_1$, such that $a_{i_1} + b_1 \in C$ and $a_{i_2} + b_1 \in C$.

Let $\{a_{i_1} + b_1, a_{i_2} + b_1, a_{i_3} + b_1, a_{i_4} + b_1, a_{i_5} + b_1\} = \{a_1 + b_1, a_2 + b_1, a_3 + b_1, a_4 + b_1, a_5 + b_1\}$, where $a_{i_1} + b_1, a_{i_2} + b_1 \in C$.

Since $a_{i_1}+b_1 \in C$, for any $j=3,4,5$, $(a_{i_1}+b_1)+a_{i_j}=(a_{i_j}+b_1)+a_{i_1} \notin S$, which implies that $a_{i_j}+b_1 \in \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cup C$, denoted (6).

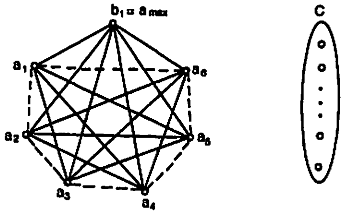

Figure 17

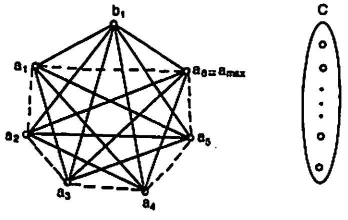

Figure 18

Since $a_{i_1}+b_1 \in C$, for any $j=3,4,5$, $(a_{i_1}+b_1)+a_{i_j}=(a_{i_j}+b_1)+a_{i_1} \notin S$, which implies that $a_{i_j}+b_1 \in \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cup C$, denoted (6).

Since $a_{i_2}+b_1 \in C$, for any $j=3,4,5$, $(a_{i_2}+b_1)+a_{i_j}=(a_{i_j}+b_1)+a_{i_2} \notin S$, which implies that $a_{i_j}+b_1 \in \{a_{i_2}, a'_{i_2}, a''_{i_2}\} \cup C$, denoted (7).

Since $a_{\max}$, a_{i_1} and a_{i_2} are distinct, $|\{a_1, a_5, a_{\max}\} \cap \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cap \{a_{i_2}, a'_{i_2}, a''_{i_2}\}| \leq 1$. Combined (5)(6)(7), there are at least two edges, which belong to the subset $\{a_{i_3}+b_1, a_{i_4}+b_1, a_{i_5}+b_1\}$ such that their sums belong to the isolated set C. Assume that $a_{i_3}+b_1 \in C$ and $a_{i_4}+b_1 \in C$.

Since $a_{i_3}+b_1 \in C$, $(a_{i_3}+b_1)+a_{i_5}=(a_{i_5}+b_1)+a_{i_3} \notin S$, which implies that $a_{i_5}+b_1 \in \{a_{i_3}, a'_{i_3}, a''_{i_3}\} \cup C$, denoted (8).

Since $a_{i_4}+b_1 \in C$, $(a_{i_4}+b_1)+a_{i_5}=(a_{i_5}+b_1)+a_{i_4} \notin S$, which implies that $a_{i_5}+b_1 \in \{a_{i_4}, a'_{i_4}, a''_{i_4}\} \cup C$, denoted (9).

As the above, $a_{i_5}+b_1$ satisfies (5)(6)(7)(8)(9). Since $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ are distinct, $|\{a_1, a_5, a_{\max}\} \cap \{a_{i_1}, a'_{i_1}, a''_{i_1}\} \cap \{a_{i_2}, a'_{i_2}, a''_{i_2}\} \cap \{a_{i_3}, a'_{i_3}, a''_{i_3}\} \cap \{a_{i_4}, a'_{i_4}, a''_{i_4}\}| = 0$. Then $a_{i_5}+b_1 \in C$. Thus, $a_1+b_1, a_2+b_1, a_3+b_1, a_4+b_1, a_5+b_1, a_{\max}+b_1 \in C$.

(ii) Consider the edge sums $a_1+a_3, a_1+a_4, a_1+a_5$. Since $a_1+b_1 \in C$, $(a_1+b_1)+a_3=b_1+(a_1+a_3) \notin S$, which implies $a_1+a_3 \in \{b_1\} \cup C$, denoted (10). Since $a_3+a_{\max} \in C$, $(a_3+a_{\max})+a_1=a_{\max}+(a_1+a_3) \notin S$, which implies $a_1+a_3 \in \{a_1, a_5, a_{\max}\} \cup C$, denoted (11). Combined (10)(11), $a_1+a_3 \in C$. Similarly, we have $a_1+a_4, a_1+a_5 \in C$.

(iii) Consider the edge sums a_2+a_4, a_2+a_5. Since $a_2+a_{\max} \in C$, $(a_2+a_{\max})+a_4=a_{\max}+(a_2+a_4) \notin S$, which implies $a_2+a_4 \in \{a_1, a_5, a_{\max}\} \cup C$, denoted (12). Since $a_2+b_1 \in C$, $(a_2+b_1)+a_4=b_1+(a_2+a_4) \notin S$, which implies $a_2+a_4 \in \{b_1\} \cup C$, denoted (13). Combined (12)(13), $a_2+a_4 \in C$. Similarly, $a_2+a_5 \in C$.

Up to now, we have proved that $a+b \in C$ for any $ab \in E(K_7 \backslash E(C_6))$ in Case 1 and Case 2 respectively, that is, this claim holds.

Let $K_7 \backslash E(C_6)=(V,E)$. Without loss of generality, we may assume that $V=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, where $0<x_1<x_2<x_3<x_4<x_5<x_6<x_7$.

According to Claim, $x_i+x_j \in C$ for all edges' sum $x_ix_j \in E$. On the other hand, since $x_1+x_2<x_1+x_3<x_1+x_4<x_1+x_5<x_1+x_6<x_1+x_7<x_2+x_7<x_3+x_7<x_4+x_7<x_5+x_7<x_6+x_7$, these $2n-3=11$ numbers are distinct and

there are at most 4 sums are not the sums of the edges of the graph $K_7\backslash E(C_6)$. So $\sigma(K_7\backslash E(C_6)) \geq (2n-3)-4 = 7$, that is, $\sigma(K_7\backslash E(C_6)) \geq 2n-7 = 7$.

Thus, Lemma 2.12 holds. □

Theorem 2.1. For $n \geq 4$, $\sigma(K_n\backslash E(C_{n-1})) = \begin{cases} 1, & n=4 \\ 2, & n=5 \\ 5, & n=6 \\ 7, & n=7 \\ 2n-7, & n \geq 8. \end{cases}$ and

$\zeta(K_n\backslash E(C_{n-1})) = \begin{cases} 0, & n=4,5,6,7 \\ 2n-7, & n \geq 8. \end{cases}$

Proof: For $n \geq 4$, by Lemma 2.1, 2.7, 2.8, 2.9, 2.10, 2.11 and 2.12, Theorem 2.1 holds. □

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