Umbral calculus and invariant sequences

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Abstract

In this paper, we study invariant sequences by umbral method, and give some identities which are similar with the identities of Bernoulli numbers.

Keywords: Umbral calculus; Invariant sequences; Bernoulli numbers

1. Introduction

The umbral calculus originated with Blissard in the nineteenth century in informal calculations involving the "lowering" and "raising" of indices. Although widely used, the umbral calculus was nothing more than a "magic rule". Rota [2] first used operators methods to free umbral calculus from its mystery. After more than twenty years, Rota and Taylor [3] gave a rigorous, simple presentation of umbral calculus by a linear functional.

We work in a formal power series ring $K[\alpha][[x]]$, where $K[\alpha]$ is a polynomial ring of one variable umbrae α. And we have a linear functional $eval : K[\alpha][[x]] \to K[[x]]$. We use the $\simeq$ symbol to stress equivalent under linear functional *eval*. For example, if $eval(\alpha^n) = a_n$, then we have $\alpha^n \simeq a_n$. If we have to deal with several umbrae together, we use the symbol *eval* to present many different such functionals.

Sun [4] and Wang [5] studied the invariant sequences which are the sequences $\{a_n\}$ satisfying

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k = a_n \tag{1}$$

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and the inverse invariant sequences which are the sequences $\{a_n\}$ satisfying

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k = -a_n. \tag{2}$$

For example, $\{\frac{1}{2^n}\}$, $\{(-1)^n B_n\}$, $\{L_n\}$ are invariant sequences, where B_n denote Bernoulli numbers and L_n denote Lucas numbers; the Fibonacci sequence $\{F_n\}$ is an inverse invariant sequence.

If we use the umbrae α to represent the sequences $\{a_n\}$, i.e., $\alpha^n \simeq a_n$, then $\{a_n\}$ is an invariant sequence if and only if $(1-\alpha)^n \simeq \alpha^n$, and $\{a_n\}$ is an inverse invariant sequence if and only if $(1-\alpha)^n \simeq -\alpha^n$. Using the classical umbral method, Wang [5] proved the following identity:

$$\sum_{k=0}^{n}\binom{n}{k}(f_k - (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f_s)a_{n-k} = 0, \tag{3}$$

where $\{f_n\}$ is a sequence and $\{a_n\}$ is an invariant sequence. In this paper, using umbral calculus, we recreate some other results presented by Sun [4] and Wang [5], and give some interesting identities which are similar with the identities of Bernoulli numbers.

2. The generating functions and transformation formulas

We say that the formal power series $\sum_{n=0}^{\infty} a_n x^n$ is the ordinary generating function of sequence $\{a_n\}$, and denote $A_{ord}(x) = \sum_{n=0}^{\infty} a_n x^n$. Similarly, the formal power series $\sum_{n=0}^{\infty} a_n x^n/n!$ is called the exponential generating function of sequence $\{a_n\}$, and we write $A_{exp}(x) = \sum_{n=0}^{\infty} a_n x^n/n!$.

The next two theorems are the relationship of the generating function of invariant sequences.

Theorem 1 ([4, Theorem 3.1]). *$\{a_n\}$ is an invariant sequence (resp., an inverse invariant sequence) if and only if $A_{ord}(\frac{x}{x-1}) = (1-x)A_{ord}(x)$ (resp., $A_{ord}(\frac{x}{x-1}) = -(1-x)A_{ord}(x)$).*

Proof. Let $\alpha^n \simeq a_n$. Then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = \pm a_n \iff (1-\alpha)^n \simeq \pm\alpha^n$$

$$\iff \quad A_{ord}(x) \simeq \sum_{n=0}^{\infty}\alpha^n x^n \simeq \sum_{n=0}^{\infty}\pm(1-\alpha)^n x^n = \pm\frac{1}{1-(1-\alpha)x}.$$

Moreover, we have

$$A_{ord}(\frac{x}{x-1}) \simeq \sum_{n=0}^{\infty}\alpha^n\frac{x^n}{(x-1)^n} = \frac{1-x}{1-(1-\alpha)x}. \quad \square$$

Theorem 2 ([4, Theorem 3.2]). *$\{a_n\}$ is an invariant sequence (resp., an inverse invariant sequence) if and only if $A_{exp}(x)e^{-\frac{x}{2}}$ is an even function (resp., an odd function).*

Proof. Let $\alpha^n \simeq a_n$. Then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = \pm a_n \Longleftrightarrow (1-\alpha)^n \simeq \pm\alpha^n$$

$$\Longleftrightarrow \quad A_{exp}(x) \simeq \sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!} \simeq \sum_{n=0}^{\infty}\pm(1-\alpha)^n\frac{x^n}{n!} \Longleftrightarrow e^{\alpha x} \simeq \pm e^{(1-\alpha)x}$$

$$\Longleftrightarrow \quad A_{exp}(x)e^{-\frac{x}{2}} \simeq e^{(\alpha-\frac{1}{2})x} \simeq \pm e^{(\frac{1}{2}-\alpha)x} \simeq \pm A_{exp}(-x)e^{\frac{x}{2}}. \quad \square$$

The following theorem is an equivalent form of invariant sequences.

Theorem 3 ([5, Theorem 2.4]). *The sequences $\{a_n\}$ satisfy $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = \pm a_n$ if and only if there exist a sequence $\{\lambda_n\}$ such that*

$$a_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\lambda_k \pm \lambda_n. \tag{4}$$

Proof. Let $\alpha^n \simeq a_n$. Suppose that $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = \pm a_n$. Then

$$(1-\alpha)^n \simeq \pm\alpha^n.$$

We denote $\lambda_n \simeq \beta^n \simeq \frac{1}{2}\alpha^n$. So we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{2}\alpha^k \pm \frac{1}{2}\alpha^n = \frac{1}{2}(1-\alpha)^n \pm \frac{1}{2}\alpha^n \simeq \pm\frac{1}{2}\alpha^n \pm \frac{1}{2}\alpha^n = \pm\alpha^n.$$

Conversely, we have

$$\begin{aligned}
(1-\alpha)^n &\simeq \sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^i \\
&\simeq \sum_{i=0}^{n}\binom{n}{i}(-1)^i\Big(\sum_{k=0}^{i}\binom{i}{k}(-1)^k\beta^k \pm \beta^i\Big) \\
&= \sum_{i=0}^{n}\binom{n}{i}(-1)^i((1-\beta)^i \pm \beta^i) = (1-(1-\beta))^n \pm (1-\beta)^n \\
&= \beta^n \pm (1-\beta)^n \simeq \pm\alpha^n. \quad \square
\end{aligned}$$

Sun [4] gave some transformation formulas using generating functions. We recreate two of them by umbral calculus directly.

Theorem 4 ([4, Theorem 5.1]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonzero sequences satisfying $c_n = \frac{1}{n+1}\sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0, 1, \cdots)$. If $\{a_n\}$ is an invariant sequence, we have $\{b_n\}$ is an invariant sequence if and only if $\{c_n\}$ is an invariant sequence.*

Proof. Let $\alpha^n \simeq a_n$, $\beta^n \simeq b_n$, and $\gamma^n \simeq c_n$. Suppose $\{b_n\}$ is an invariant sequence. Then we have $(1-\alpha)^n \simeq \alpha^n$ and $(1-\beta)^n \simeq \beta^n$. So

$$\gamma^n \simeq \frac{1}{n+1}\sum_{k=0}^{n} \alpha^k \beta^{n-k} = \frac{1}{n+1}\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}.$$

Moreover, we have

$$\begin{aligned}
(1-\gamma)^n &\simeq \sum_{i=0}^{n}\binom{n}{i}(-1)^i \frac{1}{i+1}\frac{\beta^{i+1}-\alpha^{i+1}}{\beta-\alpha} \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i+1}(-1)^i \frac{\beta^{i+1}-\alpha^{i+1}}{\beta-\alpha} \\
&= \frac{1}{(n+1)(\alpha-\beta)}\sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^j(\beta^j-\alpha^j) \\
&= \frac{1}{(n+1)(\alpha-\beta)}\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j(\beta^j-\alpha^j) \\
&= \frac{1}{n+1}\frac{(1-\beta)^{n+1}-(1-\alpha)^{n+1}}{(1-\beta)-(1-\alpha)} \simeq \frac{1}{n+1}\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} \simeq \gamma^n .
\end{aligned}$$

Conversely, if $\{c_n\}$ is an invariant sequence, then $(1-\gamma)^n \simeq \gamma^n$. And we have

$$\begin{aligned}
\frac{1}{n+1}\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} &\simeq \frac{1}{n+1}\frac{(1-\beta)^{n+1}-(1-\alpha)^{n+1}}{(1-\beta)-(1-\alpha)} \\
&\simeq \frac{1}{n+1}\frac{(1-\beta)^{n+1}-\alpha^{n+1}}{(1-\beta)-\alpha}.
\end{aligned}$$

By induction on n, we derive that $(1-\beta)^{n+1} \simeq \beta^{n+1}$. □

Theorem 5 ([4, Theorem 5.2]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonzero sequences satisfying $c_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}$ $(n = 0, 1, \cdots)$. If $\{a_n\}$ is an invariant sequence, we have $\{b_n\}$ is an invariant sequence if and only if $\{c_n\}$ is an invariant sequence.*

Proof. Let $\alpha^n \simeq a_n$, $\beta^n \simeq b_n$, and $\gamma^n \simeq c_n$. Then

$$\gamma^n \simeq \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k} = \frac{1}{2^n}(\alpha+\beta)^n .$$

Suppose $\{b_n\}$ is an invariant sequence. Then we have

$$\begin{aligned}(1-\gamma)^n &\simeq \sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{1}{2^i}(\alpha+\beta)^i = (1-\frac{\alpha+\beta}{2})^n \\ &\simeq (1-\frac{1-\alpha+1-\beta}{2})^n = \frac{1}{2^n}(\alpha+\beta)^n \simeq \gamma^n.\end{aligned}$$

Conversely, if $\{c_n\}$ is an invariant sequence, then $(1-\gamma)^n \simeq \gamma^n$. So we have

$$\begin{aligned}(1-\gamma)^n &\simeq (1-\frac{\alpha+\beta}{2})^n \simeq (1-\frac{1-\alpha+\beta}{2})^n \\ &= \frac{1}{2^n}(\alpha+1-\beta)^n \simeq \gamma^n \simeq \frac{1}{2^n}(\alpha+\beta)^n.\end{aligned}$$

By induction on n, we derive that $(1-\beta)^n \simeq \beta^n$. □

3. Some identities of invariant sequences

In this section, using umbral calculus, we derive some identities of invariant sequences which cannot be found in Sun[4] and Wang[5]. When the invariant sequences are Bernoulli numbers, we get the remarkable identities of Bernoulli numbers.

Let $\alpha^n \simeq (1-\alpha)^n$, and $f(x)$ be a one variable polynomial or a formal power series. Obviously, we have $f(\alpha) \simeq f(1-\alpha)$. By selecting different $f(x)$, we get the following formulas.

Theorem 6. *If $\{a_n\}$ is an invariant sequence, then*

$$\sum_{i=0}^{m}\binom{m}{i}(-1)^i a_{n+i} = \sum_{i=0}^{n}\binom{n}{i}(-1)^i a_{m+i}. \tag{5}$$

Proof. Let $f(x) = x^m(x-1)^n$ and $\alpha^n \simeq a_n$. Then $(1-\alpha)^n \simeq \alpha^n$ and $f(1-\alpha) \simeq f(\alpha)$. We have

$$f(1-\alpha) = (1-\alpha)^m\alpha^n = \sum_{i=0}^{m}\binom{m}{i}(-\alpha)^{n+i},$$

and

$$f(\alpha) = \alpha^m(\alpha-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\alpha^{m+i}.$$

Then

$$\sum_{i=0}^{m}\binom{m}{i}(-1)^{n+i}\alpha^{n+i} \simeq \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\alpha^{m+i}.$$

Moreover, we get (5). □

Because $\{(-1)^n B_n\}$ is an invariant sequence, we have:

Corollary 1.

$$\sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^{m+n} \sum_{j=0}^{n} \binom{n}{i} B_{m+j}. \tag{6}$$

Similarly, we can get the following theorem.

Theorem 7. *If $\{a_n\}$ is an inverse invariant sequence, then*

$$\sum_{i=0}^{m} \binom{m}{i} (-1)^i a_{n+i} = -\sum_{i=0}^{n} \binom{n}{i} (-1)^i a_{m+i}. \tag{7}$$

Theorem 8. *If $\{a_n\}$ is an invariant sequence, then*

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (m+1+i)(-1)^i a_{m+i} = -\sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i (n+1+i) a_{n+i}. \tag{8}$$

Proof. Let $g(x) = (x^{n+1}(x-1)^{m+1})' = (n+1)x^n(x-1)^{m+1} + (m+1)x^{n+1}(x-1)^m$ and $\alpha^n \simeq a_n$. Then we have $(1-\alpha)^n \simeq \alpha^n$ and $g(1-\alpha) \simeq g(\alpha)$. Moreover, we get

$$\begin{aligned} g(1-\alpha) &= (n+1)(1-\alpha)^n(-\alpha)^{m+1} + (m+1)(1-\alpha)^{n+1}(-\alpha)^m \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (m+1+j)(-1)^{m+j} \alpha^{m+j}, \end{aligned}$$

and

$$\begin{aligned} g(\alpha) &= (n+1)\alpha^n(\alpha-1)^{m+1} + (m+1)\alpha^{n+1}(\alpha-1)^m \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} (n+1+j)(-1)^{m+1-j} \alpha^{n+j}. \end{aligned}$$

So we have

$$\begin{aligned} &\sum_{j=0}^{n+1} \binom{n+1}{j} (m+1+j)(-1)^{m+j} \alpha^{m+j} \\ \simeq &\sum_{j=0}^{m+1} \binom{m+1}{j} (n+1+j)(-1)^{m+1-j} \alpha^{n+j}. \end{aligned}$$

Consequently, we derive (8). □

Corollary 2. (see [1])

$$(-1)^m \sum_{k=0}^{m} \binom{m+1}{k} (n+k+1) B_{n+k} = -(-1)^n \sum_{k=0}^{n} \binom{n+1}{k} (m+k+1) B_{m+k}, \tag{9}$$

where $m + n > 0$.

Proof. We take $a_n = (-1)^n B_n$ in Theorem 8, and note that $B_{2k+1} = 0$ for $k \geq 1$. □

If we take $m = n$ in Theorem 8, then we get:

Corollary 3.

$$\sum_{l=0}^{n+1} \binom{n+1}{l} (n+l+1)(-1)^l a_{n+l} = 0. \tag{10}$$

For Bernoulli numbers, we have:

Corollary 4.

$$\sum_{k=0}^{n} \binom{n+1}{k} (n+k+1) B_{n+k} = 0. \tag{11}$$

Similarly, we have following theorem for inverse invariant sequences.

Theorem 9. *If* $\{a_n\}$ *is an inverse invariant sequence, then*

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (m+1+i)(-1)^i a_{m+i} = \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i (n+1+i) a_{n+i}. \tag{12}$$

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