A Note on Eigenvalues of the Derangement Graph *

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Abstract

In this note, we determine the exact value for the second largest eigenvalue of the derangement graph, by deriving a formula for all the eigenvalues corresponding to the 2-part partitions. This result is then used to obtain lower bounds for the vertex connectivity and Cheeger constant, and an upper bound for the bipartite density. Also, the exact value of the Shannon capacity of the derangement graph is obtained.

Key words: Derangement graph; eigenvalue; symmetric group; Shannon capacity.
AMS Classifications: 05C25, 05C50

1 Introduction

Let G be a finite group and $S \subseteq G \setminus \{1\}$ be a subset of generators closed under inverses. The *Cayley graph* $\Gamma(G, S)$ on G with respect to S is defined

*National Natural Science Foundation of China (No.10971137), National Basic Research Program of China 973 Program (No.2006CB805900), National Research Program of China 863 Program (No.2006AA11Z209) and a grant of Science and Technology Commission of Shanghai Municipality (STCSM No: 09XD1402500).

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by

$$V(\Gamma(G,S)){=}G,\ E(\Gamma(G,S)){=}\{(v,sv)\mid v{\in}G,\ s{\in}S\}.$$

Thus $\Gamma(G,S)$ is regular of vertex degree $|S|$. Let S_n be the *symmetric group of permutations* of $X = \{1,2,\cdots,n\}$, and let $\mathcal{D}_n := \{\alpha \in S_n : \alpha(x) \neq x, \forall x \in X\}$ denote the *derangements* on X, namely the set of fixed point free permutations of S_n. Note that $\mathcal{D}_n$ is closed under inverses, as the inverse of a derangement is a derangement. $\Gamma_n := \Gamma(S_n, \mathcal{D}_n)$ is called the *derangement graph* on X. It is known that $\Gamma_n\,(n > 3)$ is connected, as S_n can be generated by $\mathcal{D}_n$. We denote $|\mathcal{D}_n|$ by D_n.

The derangement graph has been studied by many researchers. For example, Eggleton and Wallis [4] showed that $\Gamma_n\,(n > 3)$ is Hamiltonian; Deza and Frankl [3] proved that the independence number $\alpha(G) = (n-1)!$; Renteln [9] observed that the chromatic number $\chi(G) = n$ and the clique number $\omega(G) = n$. Renteln also derived several formulae for the eigenvalues of Γ_n and used these to confirm a conjecture of Ku regarding the least eigenvalue. Recently, Ku and Wales [6] investigated various bounds on the eigenvalues of Γ_n and obtained a simple expression for their signs.

Motivated by these studies, we obtain a simple formula for all eigenvalues of the derangement graph corresponding to 2-part partitions. The main result of this note can be stated as follows:

Theorem 1.1 *The second largest eigenvalue of the adjacency matrix of the derangement graph* $\Gamma_n(n \geq 4)$ *is given by*

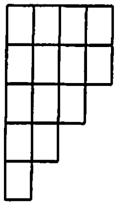
$$\eta_{(n-2,2)} = \frac{n-1}{n-3}D_{n-2}.$$

The rest of this note is organized as follows: In Section 2, general notation and various lemmas are presented. In Section 3, we derive an explicit formula for the eigenvalues of the derangement graph corresponding to all the 2-part partitions. In Section 4, we present a proof of Theorem 1.1. In Section 5, we present lower bounds on the vertex connectivity and Cheeger constant and an upper bound on the bipartite density of Γ_n. Lastly, the exact value of the Shannon capacity of the derangement graph is obtained. (In essence, this result was implied in [9].)

2 Preliminaries

A *partition* of n of length l is a sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ such that $\lambda_i \geq \lambda_{i+1}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Each λ_i is a *part* of the partition. Partitions are represented by Ferrers diagrams

$$(4,4,3,2,1) \quad \longleftrightarrow$$



or by multiplicity notation

$$(4,4,3,2,1) \quad \longleftrightarrow \quad 4^2 3^1 2^1 1^1.$$

Following Renteln [9], we introduce some terminology. For any Ferrers diagram corresponding to partition λ, we may assign xy-coordinates to each of the boxes by defining the upper-left-most box to be $(1,1)$, with the x axis increasing to the right and the y axis increasing downwards. The *hook* through the box (x,y) is the union of the boxes (x',y) and (x,y'), where $x' \geq x$ and $y' \geq y$. The hook through the box $(1,1)$ is called the *principal hook* (of λ). By abuse of notation, let h denote either the principal hook itself or its cardinality (its meaning will be clear from context). Denote by $\lambda - h$ the partition obtained from λ by removing the principal hook. The first column of the Ferrers diagram corresponding to partition λ is called the *principal ladder* (of λ), and $\lambda - 1$ is the partition obtained from λ by removing its principal ladder. Next we need some lemmas from [9].

Lemma 2.1 [9] *For any partition λ, the eigenvalues of the derangement graph satisfy the following recurrence:*

$$\eta_\lambda = (-1)^h(\eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1}) \tag{1}$$

with initial condition $\eta_\phi = 1$.

Lemma 2.2 [9] *Let $\lambda = j1^{n-j}$ denote the hook shape with first part j and remaining parts 1. Then for $1 \leq j \leq n$,*

$$\eta_{j1^{n-j}} = (-1)^n(1 + (-1)^j n D_{j-1}). \tag{2}$$

Lemma 2.3 [9] *The derangement numbers satisfy the following properties:*

(i). The first six derangement numbers are $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$.

(ii). $D_n = [n!/e]$, *where* $[x]$ *is the nearest integer to* x. *In particular, the derangement numbers are monotonic increasing for* $n \geq 1$.

(iii). For $n \geq 1$ *the derangement numbers satisfy the following recursions:*

$$D_n = nD_{n-1} + (-1)^n, \tag{3}$$

$$D_n = (n-1)(D_{n-1} + D_{n-2}). \tag{4}$$

3 The eigenvalues corresponding to 2-part partitions

In this section, we derive an explicit formula for the eigenvalues of the derangement graph Γ_n corresponding to all the 2-part partitions of n. Lemma 2.2 gives an expression for the eigenvalue corresponding to partitions of the form $(n-1,1)$. Hence, we only need to derive a formula for the eigenvalues corresponding to partitions $\lambda = (n-k,k)(2 \leq k \leq \lfloor n/2 \rfloor)$, where $\lfloor x \rfloor$ is the largest integer less than x. The main result of this section is as follows:

Theorem 3.1

$$\eta_{(n-k,k)} = \frac{(-1)^k(n-k+1)D_{n-k} + (-1)^{n+k}kD_{k-1}}{n-2k+1} \tag{5}$$

for $2 \leq k \leq \lfloor n/2 \rfloor$. *In particular,*

$$\eta_{(n-2,2)} = \frac{n-1}{n-3}D_{n-2}. \tag{6}$$

Proof. By Theorem 4.2 of [9], the eigenvalues corresponding to $(n-k,k)(2 \leq k \leq \lfloor n/2 \rfloor)$ can be written

$$\eta_{(n-k,k)} = \frac{(-1)^n}{\mu_1 - \mu_2} \begin{vmatrix} (-1)^{\mu_1}b(\mu_1;1) & 1 \\ (-1)^{\mu_2}b(\mu_2;1) & 1 \end{vmatrix}$$

where

$$\mu_1 = (n-k) + 2 - 1 = n-k+1,$$

$$\mu_2 = k + 2 - 2 = k,$$

$$b(\mu_1; 1) = -\mu_1 D_{\mu_1 - 1} = -(n-k+1)D_{n-k},$$

$$b(\mu_2; 1) = -\mu_2 D_{\mu_2 - 1} = -kD_{k-1}.$$

Hence

$$\eta_{(n-k,k)} = \frac{(-1)^n}{n-2k+1}\begin{vmatrix} (-1)^{n-k}(n-k+1)D_{n-k} & 1 \\ (-1)^{k+1}kD_{k-1} & 1 \end{vmatrix}$$

$$= \frac{(-1)^n}{n-2k+1}[(-1)^{n-k}(n-k+1)D_{n-k} - (-1)^{k+1}kD_{k-1}]$$

$$= \frac{(-1)^k(n-k+1)D_{n-k} + (-1)^{n+k}kD_{k-1}}{n-2k+1}.$$

In particular,

$$\eta_{(n-2,2)} = \frac{(-1)^2(n-2+1)D_{n-2} + (-1)^{n+2}2D_{2-1}}{n-2\cdot 2+1}$$

$$= \frac{(n-1)D_{n-2} + 0}{n-3}$$

$$= \frac{n-1}{n-3}D_{n-2}.$$

The assertion holds. ■

4 Proof of Theorem 1.1

We first introduce two more results about the eigenvalues of the derangement graph Γ_n.

Lemma 4.1 [9] *(i).The maximum eigenvalue of the derangement graph* Γ_n *is* $\eta_{(n)} = D_n$.

(ii). The least eigenvalue of the derangement graph Γ_n *is* $\eta_{(n-1,1)} = \frac{-D_n}{n-1}$*, and it is unique for* $n \geq 5$.

We divide the proof of Theorem 1.1 into four parts. First we show Theorem 1.1 holds for all the 2-part partitions $(n-k,k)$. Next we show Theorem 1.1 holds for hooks and all *near hooks,* that is, partitions of the form $j21^{n-j-2}$, where $2 \leq j \leq n-2$. Finally, we show that it holds for all other partitions. First note that, by direct computation using Lemma 2.1

it is easily seen that Theorem 1.1 holds for $n = 4$ and $n = 5$. So we assume $n \geq 6$ in the following proof. We will complete the proof by a series of lemmas.

Lemma 4.2 *Theorem 1.1 holds for all the 2-part partitions* $(n-k, k)$.

Proof. By Lemma 4.1 (ii),

$$\eta_{(n-1,1)} < \eta_{(n-2,2)}.$$

By Theorem 3.1 and Lemma 2.3 (iii), we have

$$\begin{aligned}
\alpha &:= \left|\eta_{(n-k,k)}\right| - \eta_{(n-2,2)} \\
&= \left|\frac{(-1)^k(n-k+1)D_{n-k} + (-1)^{n+k}kD_{k-1}}{n-2k+1}\right| - \frac{n-1}{n-3}D_{n-2} \\
&\leq \left(\frac{n-k+1}{n-2k+1}D_{n-k} + \frac{k}{n-2k+1}D_{k-1}\right) - \frac{n-1}{n-3}D_{n-2} \\
&= \left(\frac{n-k+1}{n-2k+1}D_{n-k} + \frac{k}{n-2k+1}D_{k-1}\right) - \frac{n-1}{n-3}(n-3)(D_{n-3} + D_{n-4}) \\
&= \left(\frac{n-k+1}{n-2k+1}D_{n-k} + \frac{k}{n-2k+1}D_{k-1}\right) - (n-1)(D_{n-3} + D_{n-4}).
\end{aligned}$$

If $k = 2$, then $\alpha = 0$.
If $3 \leq k \leq \lfloor n/2 \rfloor$, then

$$k - 1 \leq \lfloor n/2 \rfloor - 1 \leq n - \lfloor n/2 \rfloor - 1 \leq n - 3 - 1 = n - 4.$$

By lemma 2.3 (ii), we have $D_{n-k} \leq D_{n-3}$, $D_{k-1} \leq D_{n-4}$.
Hence

$$\begin{aligned}
\alpha &\leq \left(\frac{n-k+1}{n-2k+1}D_{n-3} + \frac{k}{n-2k+1}D_{n-4}\right) - (n-1)(D_{n-3} + D_{n-4}) \\
&= \left[\frac{n-k+1}{n-2k+1} - (n-1)\right]D_{n-3} + \left[\frac{k}{n-2k+1} - (n-1)\right]D_{n-4} \\
&< 0.
\end{aligned}$$

The assertion holds. ■

Lemma 4.3 *Theorem 1.1 holds for hooks.*

Proof. Let $\lambda = j1^{n-j}$ be a hook. By Lemma 4.1 we may assume that $j \le n-2$ (if $j = n-1$, then $\lambda = (n-1,1)$). By Lemmas 2.2, 2.3 and Theorem 3.1, we have

$$\begin{aligned}
& \eta_{j1^{n-j}} - \eta_{(n-2,2)} \\
= & \ (-1)^n(1+(-1)^j nD_{j-1}) - \frac{n-1}{n-3}D_{n-2} \\
\le & \ (-1)^n(1+(-1)^n nD_{(n-2)-1}) - \frac{n-1}{n-3}D_{n-2} \qquad ((-1)^{n+j} \le (-1)^{2n}) \\
= & \ (-1)^n(1+(-1)^n nD_{n-3}) - \frac{n-1}{n-3}((n-2)D_{n-3} + (-1)^{n-2}) \\
= & \ \left(n - \frac{(n-1)(n-2)}{n-3}\right) D_{n-3} + (-1)^n\left(1 - \frac{n-1}{n-3}\right) \\
= & \ \frac{-2}{n-3}D_{n-3} + (-1)^n\frac{-2}{n-3} \\
= & \ \frac{-2}{n-3}(D_{n-3} + (-1)^n) \\
\le & \ 0 \qquad\qquad (n \ge 5 \text{ and Lemma 2.3}).
\end{aligned}$$

The assertion holds. ■

Lemma 4.4 *Theorem 1.1 holds for near hooks.*

Proof. Let $\lambda = j21^{n-j-2}$ be a near hook. By Lemma 4.2, we may assume $j \le n-3$ (if $j = n-2$, then $\lambda = (n-2,2)$).

Applying Lemmas 2.1, 2.3, 4.1 and Theorem 3.1 gives

$$\begin{aligned}
& \eta_{j21^{n-j}} - \eta_{(n-2,2)} \\
= & \ (-1)^{n-1}\left(\eta_1 + (-1)^j(n-1)\eta_{(j-1,1)}\right) - \frac{n-1}{n-3}D_{n-2} \\
= & \ (-1)^{n+j-1}(n-1)\eta_{(j-1,1)} - \frac{n-1}{n-3}D_{n-2} \\
= & \ (-1)^{n+j-1}(n-1)\frac{-D_j}{j-1} - \frac{n-1}{n-3}D_{n-2} \\
= & \ (n-1)\left((-1)^{n+j}\frac{D_j}{j-1} - \frac{D_{n-2}}{n-3}\right) \\
= & \ (n-1)\left((-1)^{n+j}\frac{(j-1)(D_{j-1}+D_{j-2})}{j-1} - \frac{(n-3)(D_{n-3}+D_{n-4})}{n-3}\right) \\
= & \ (n-1)\left((-1)^{n+j}(D_{j-1}+D_{j-2}) - (D_{n-3}+D_{n-4})\right) \\
< & \ 0.
\end{aligned}$$

The proof is complete. ■

Now we present a proof of Theorem 1.1.

Proof. (of Theorem 1.1). Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a partition of n. By Lemmas 4.1 - 4.4, we may assume $l \geq 3$ and that λ is neither a hook (in which case we would have $n = h$), nor a near hook (in which case we would have $n = h + 1$). So we may assume $n \geq h + 2$ and $h > l \geq 3$. Then we get the following chain of equalities and inequalities:

$$
\begin{aligned}
|\eta_\lambda| &= \left|\eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1}\right| && \text{(Lemma 2.1)} \\
&\leq |\eta_{\lambda-h}| + h\,|\eta_{\lambda-1}| \\
&\leq D_{n-h} + hD_{n-l} && \text{(Lemma 4.1)} \\
&< (1+h)D_{n-l} && \text{(Lemma 2.3 and } h > l) \\
&\leq (n-1)D_{n-l} \\
&\leq (n-1)D_{n-3} \\
&= (n-1)\left(\frac{D_{n-2}}{n-3} - D_{n-4}\right) && \text{(Lemma 2.3 } (iii)) \\
&= \frac{n-1}{n-3}D_{n-2} - (n-1)D_{n-4} \\
&= \eta_{(n-2,2)} - (n-1)D_{n-4} && \text{(Theorem 3.1)} \\
&< \eta_{(n-2,2)}.
\end{aligned}
$$

The proof is complete. ■

5 Some applications

In this section, we present several applications from Theorem 1.1.

Theorem 5.1 *Let Γ_n be the derangement graph. Then the vertex connectivity $\kappa(\Gamma_n)$ is at least*

$$D_n - \frac{n-1}{n-3}D_{n-2}.$$

Proof. The Laplacian matrix of Γ_n is $L(\Gamma_n) = D_n I - A(\Gamma_n)$. Then, by Theorem 1.1, the algebraic connectivity of Γ_n, i.e, the second smallest eigenvalue of $L(\Gamma_n)$, is $D_n - \frac{n-1}{n-3}D_{n-2}$. Now it follows from [5] that the assertion holds. ■

The *Cheeger constant* of a simple graph G is defined as follows:

$$h(G) = \min_{S \subseteq V(G)} \frac{|\partial S|}{\min\{vol(S),\ vol(G) - vol(S)\}}, \tag{7}$$

where $\partial S = \{(u,v) \in E(G),\ u \in S, v \notin S\}$ and $vol(S) = \sum_{v \in S} d(v)$, where $d(v)$ is the degree of vertex v.

Theorem 5.2 *Let Γ_n be the derangement graph. Then*

$$\frac{1}{2}\left(1 - \frac{(n-1)D_{n-2}}{(n-3)D_n}\right) \leq\ h(G) \leq\ \sqrt{2\left(1 - \frac{(n-1)D_{n-2}}{(n-3)D_n}\right)}.$$

Proof. The normal Laplacian matrix of Γ_n is $\mathcal{L}(\Gamma_n) = I - \frac{1}{D_n}A(\Gamma_n)$. Then the second smallest eigenvalue μ of $\mathcal{L}(\Gamma_n)$ is $1 - \frac{(n-1)D_{n-2}}{(n-3)D_n}$. On the other hand, by Lemma 2.1 and Theorem 2.2 of page. 25-26 in [2], we have $\frac{\mu}{2} \leq h(G) \leq \sqrt{2\mu}$. Hence the assertion holds. ■

Let G be a graph and H be any bipartite subgraph of G with the maximum number of edges. Then

$$b(G) = \frac{|E(H)|}{|E(G)|}$$

is called the *bipartite density* of G (for example, see [1]).

Theorem 5.3 *Let Γ_n be the derangement graph. Then*

$$b(G) \leq \frac{n}{2(n-1)}.$$

Proof. The Laplacian matrix of Γ_n is $D_nI - A(\Gamma_n)$. Since the smallest eigenvalue of Γ_n is $\frac{-D_n}{n-1}$ by [9], the largest eigenvalue of the Laplacian matrix is $\frac{nD_n}{n-1}$. By [8], for any bipartite subgraph H of G, we have

$$|E(H)| \leq \frac{|S_n| n D_n}{4(n-1)}.$$

But $|E(G)| = \frac{1}{2}D_n|S_n|$. Hence the assertion holds. ■

Before stating the next result we recall the definition of the Shannon capacity. Let G be a graph. Denote by G^l the product of l copies of G, i.e., the graph with vertex set $\{1, \cdots, n\}^l$ and edge set consisting of pairs

$\{(x_1, \cdots, x_l), (y_1, \cdots, y_l)\}$ for which either $x_i = y_i$ or x_i is adjacent to y_i in G. The number

$$\Theta(G) = \sup_l \sqrt[l]{\alpha(G^l)}$$

is called the *Shannon capacity* of G. To date there are just a few graphs whose Shannon capacity has been determined. For example, Lovász in [7] proved that the Shannon capacity of the cycle of order 5 is $\sqrt{5}$. Here, we give the exact value of the Shannon capacity of the derangement graph. (This result was observed but left implicit in [9].)

Theorem 5.4 *Let Γ_n be the derangement graph. Then*

$$\Theta(\Gamma_n) = (n-1)!.$$

Proof. By Theorem 9 in [7], we have

$$\alpha(\Gamma_n) \leq \Theta(\Gamma_n) \leq \frac{|S_n|\eta_{min}}{D_n - \eta_{min}},$$

where η_{min} is the smallest eigenvalue of Γ_n. By Theorem 1.1 in [9], we have $\eta_{min} = \frac{-D_n}{n-1}$. On the other hand, by [9] $\alpha(G) = (n-1)!$. Hence

$$(n-1)! = \alpha(\Gamma_n) \leq \Theta(\Gamma_n) \leq \frac{|S_n|\eta_{min}}{D_n - \eta_{min}} = (n-1)!.$$

Therefore, $\Theta(\Gamma_n) = (n-1)!$. ■

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