Commutative weakly distance-regular digraphs of circle with fixed length

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Abstract

We obtain some new examples of weakly distance-regular digraphs. Moreover, a class of commutative weakly distance-regular digraphs of valency 4 and girth 2 is characterized.

1 Introduction

A digraph Γ is a pair (X, A), where X is a finite set of vertices and $A \subseteq X^2$ is a set of arcs. Throughout this paper we use the term digraph to mean a finite directed graph with no loops. We often write $V\Gamma$ for X and $A\Gamma$ for A, respectively. An arc(u, v) of Γ is said to be an *edge* if $(v, u) \in A\Gamma$. A *path* of length r from u to v is a finite sequence of vertices $(u = w_0, w_1, \cdots, w_r = v)$ such that $(w_{t-1}, w_t) \in A\Gamma$, for $t = 1, 2, \cdots, r$. A path $(w_0, w_1, \cdots, w_r)$ is said to be a *circuit* of length $r+1$, if $(w_r, w_0) \in A\Gamma$. A shortest circuit is called a *minimal circuit*. The *girth* of Γ is the length of a minimal circuit. If a digraph contains an edge, its girth is 2. The number of arcs traversed in a shortest path from u to v is called the *distance* from u to v in Γ, denoted by $\partial(u, v)$. The maximum value of the distance function in Γ is called the *diameter* of Γ. A digraph is said to be *strongly connected* if, for any two distinct vertices x and y, there is a path from x to y. For vertices x and y of Γ, define $\tilde{\partial}(x, y) = (\partial(x, y), \partial(y, x))$. For a digraph Γ, we assume that $\tilde{\partial}(\Gamma)$ denote the set $\{\tilde{\partial}(x, y) | x, y \in V\Gamma\}$.

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Definition 1.1 *A strongly connected digraph Γ is said to be weakly distance-regular, if for all $\tilde{h}, \tilde{i}, \tilde{j} \in \tilde{\Gamma}$ and $\tilde{\partial}(x,y) = \tilde{h}$,*

$$p^{\tilde{h}}_{\tilde{i},\tilde{j}} = |\{z \in V\Gamma | \tilde{\partial}(x,z) = \tilde{i} \text{ and } \tilde{\partial}(z,y) = \tilde{j}\}|$$

depends only on $\tilde{h}, \tilde{i}, \tilde{j}$. The nonnegative $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$ are called the intersection numbers We say that Γ is commutative, if $p^{\tilde{h}}_{\tilde{i},\tilde{j}} = p^{\tilde{h}}_{\tilde{j},\tilde{i}}$ for all $\tilde{h}, \tilde{i}, \tilde{j} \in \tilde{\Gamma}$.

For a strongly connected digraph Γ, let $A_{i,j}$ denote a square matrix of degree $|V\Gamma|$, whose rows and columns are indexed by vertices of Γ such that

$$(A_{i,j})_{x,y} = \begin{cases} 1, & \text{if } \tilde{\partial}(x,y) = (i,j), \\ 0, & \text{otherwise.} \end{cases}$$

We say that $A_{i,j}$ is the (i,j)th *adjacency matrix* of Γ. It is easy to see that Γ is a weakly distance-regular graph iff the span of the $\{A_{i,j} | (i,j) \in \tilde{\partial}(\Gamma)\}$ is closed under multiplication. Let

$$\Gamma_{i,j} = \{(x,y) \in V\Gamma \times V\Gamma | \tilde{\partial}(x,y) = (i,j)\},$$

and

$$\Gamma_{i,j}(x) = \{y \in V\Gamma | \tilde{\partial}(x,y) = (i,j)\}.$$

Let Γ be a weakly distance-regular digraph. For vertices x and y of Γ, let

$$P_{\tilde{i},\tilde{j}}(x,y) = \{x \in V\Gamma | \tilde{\partial}(x,z) = \tilde{i} \text{ and } \tilde{\partial}(z,y) = \tilde{j}\}.$$

If $\tilde{\partial}(x,y) = \tilde{h}$, the $|P_{\tilde{i},\tilde{j}}(x,y)| = p^{\tilde{h}}_{\tilde{i},\tilde{j}}$. Note that $|\Gamma_{i,j}(x)|$ does not depend on the choice of x and will be denoted by $k_{i,j}$.

Definition 1.2 *Let Γ_1 and Γ_2 be digraphs. The lexicographic product $\Gamma_1[\Gamma_2]$ from Γ_1 to Γ_2 is a digraph with the vertex set $\{(u_1,u_2) | u_1 \in V\Gamma_1$ and $u_2 \in V\Gamma_2\}$ and the arc set $\{((u_1,u_2),(u_1',u_2')) | (u_1,u_1') \in A\Gamma_1$ or $u_1 = u_2$ and $(u_2,u_2') \in A\Gamma_2\}$. The directed product $\Gamma_1 \times \Gamma_2$ from Γ_1 to Γ_2 is a digraph with the vertex set $\{(u_1,u_2) | u_1 \in V\Gamma_1$ and $u_2 \in V\Gamma_2\}$ and the arc set $\{((u_1,u_2),(u_1',u_2')) | (u_1, u_1') \in A\Gamma_1$ and $u_2 = u_2'$ or $u_1 = u_1'$ and $(u_2,u_2') \in A\Gamma_2\}$.*

Let $\chi = (X, \{R_i\}_{0 \le i \le d})$ denote an association scheme of class d. As regards association schemes, we refer readers to [4]. For any two nonempty subsets $E, F \subseteq R := \{R_i | 0 \le i \le d\}$, define

$$EF := \{R_h | \sum_{R_i \in E} \sum_{R_j \in F} p^h_{i,j} \neq 0\},$$

and write R_iR_j instead of $\{R_i\}\{R_j\}$. For each nonempty subset F of R, define $< F >$ to be the minimal equivalence relation containing F. For any nonempty subset $F \subseteq R$ and $x \in R$, let

$$F(x) := \{y \in X | (x,y) \in \bigcup_{f \in F} f\}, \quad X/F := \{F(x) | x \in X\},$$

and

$$R_i^F := \{(F(x), F(y)) | y \in FR_iF(x)\}.$$

Let Γ be a strongly connected digraph of diameter d. In the rest of this paper, we always assume that $R = \{\Gamma_{i,j} | (i,j) \in \tilde{\partial}(\Gamma)\}$. For any equivalency relation $F \subseteq R$, the digraph $(V\Gamma/F, \bigcup_{(1,j)\in\tilde{\partial}(\Gamma)} \Gamma_{1,j}^F)$ is said to be the *quotient digraph* of Γ over F, denoted by Γ/F, where $\Gamma_{i,j} = \{(x,y) \in V\Gamma \times V\Gamma | \tilde{\partial}(x,y) = (i,j)\}$.

Proposition 1.1 ([3]) *Let Γ be a commutative weakly distance-regular digraph of valency k, diameter d, and girth 2. If $k - k_{1,1} = 1$, then Γ is isomorphic to one of the following digraphs:*
(1) $Cay(Z_4, \{1,2\})$.
(2) $K_n[C_4]$.
(3) $\Delta[C_3]$.
(4) $\Delta \times C_r$, *where r is odd or $d < r/2$.*
Here K_n is a complete graph with n vertices, $C_r = Cay(Z_r, \{1\})$, and Δ is a distance-regular graph.

In [4] Kaishun Wang and Hiroshi Suzuki defined weakly distance-regular digraphs and determined all commutative 2-valent weakly distance-regular digraphs. In [2] Suzuki proved the nonexistence of noncommutative 2-valent weakly distance-regular digraphs. In [3] and [5] Kaishun Wang gave a classification of weakly distance-regular digrphs with girth 2 and $k-k_{11} = 1$. In this paper, a family of commutative weakly distance-regular digraphs of valency 4 and girth 2 is characterized.

2 Main Results

Proposition 2.1 *Let $C_r = Cay(Z_r, 1)$, and let $\bar{K}_t$ be a coclique. Then $C_r[\bar{K}_t]$ is a weakly distance-regular digraph with valency t and diameter r.*

Proof. Let $\Gamma = C_r[\bar{K}_t]$. Let $A_{i,j}$ be the (i,j)th adjacency matrix of C_r. By choosing the suitable ordering of rows and columns , the (i,j)th adjacency matrix of Γ is as follows:

$$\tilde{A}_{i,j} = \begin{cases} I_r \bigotimes I_t, & \text{if } i = j = 0, \\ A_{i,r-i} \bigotimes J_t, & \text{if } j = r - i, 1 \le i \le r-1, \\ I_r \bigotimes (J_t - I_t), & \text{if } i = j = r, \\ 0, & \text{otherwise,} \end{cases}$$

where $A_{i,r-i} = \begin{pmatrix} 0 & I_{r-i} \\ I_i & 0 \end{pmatrix}$. So for any $i, j \in \{0, 1, \cdots, r\}$, we have

$$\tilde{A}_{0,0}\tilde{A}_{i,j} = \tilde{A}_{i,j}\tilde{A}_{0,0} = \tilde{A}_{i,j}.$$

For any $i, j \in \{1, 2, \cdots, r-1\}$, if $i + j \le r$, we have

$$\tilde{A}_{i,r-i}\tilde{A}_{j,r-j} = A_{i,r-i}A_{j,r-j} \otimes J_tJ_t = A_{i+j,r-(i+j)} \otimes tJ_t = t\tilde{A}_{i+j,r-(i+j)},$$

and if $i + j > r$, we have

$$\begin{aligned} & \tilde{A}_{i,r-i}\tilde{A}_{j,r-j} \\ = \; & A_{i,r-i}A_{j,r-j} \otimes J_tJ_t \\ = \; & A_{2r-(i+j),(i+j)-r} \otimes tJ_t \\ = \; & t\tilde{A}_{2r-(i+j),(i+j)-r}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{A}_{i,r-i}\tilde{A}_{r,r} \\ = \; & \tilde{A}_{r,r}\tilde{A}_{i,r-i} \\ = \; & ((I_r \otimes (J_t - I_t))(\tilde{A}_{i,r-i} \otimes J_t) \\ = \; & (t-1)\tilde{A}_{i,r-i} \otimes J_t = (t-1)\tilde{A}_{i,r-i}. \end{aligned}$$

Note that

$$\begin{aligned} & \tilde{A}^2_{r,r} \\ = \; & (I_r \otimes ((J_t - I_t))^2 \\ = \; & I_r \otimes (t-2)J_t + I_t \\ = \; & I_r \otimes (t-2)(J_t - I_t) + (t-1)I_t = (t-2)\tilde{A}_{r,r} + (t-1)\tilde{A}_{0,0}. \end{aligned}$$

Hence the desired result follows. □

Proposition 2.2 *Let $C_r = Cay(Z_r, 1)$, and let K_t be a clique. Then $C_r[K_t]$ is a weakly distance-regular digraph.*

Proof. Let $\Gamma = C_r[K_t]$. If $r \neq 2$, then suppose that $A_{i,j}$ is (i,j)th adjacency matrix of C_r. By choosing the suitable ordering of rows and columns, the (i,j)th adjacency matrix of Γ is as follows:

$$\tilde{A}_{i,j} = \begin{cases} I_r \bigotimes I_t, & \text{if } i = j = 0, \\ I_r \bigotimes (J_t - I_t), & \text{if } i = j = 1, \\ A_{i,r-i} \bigotimes J_t, & \text{if } j = r - i, 1 \le i \le r-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $A_{i,r-i} = \begin{pmatrix} 0 & I_{r-i} \\ I_i & 0 \end{pmatrix}$.

Since

$$\begin{aligned} & \tilde{A}_{1,1}^2 \\ = \ & I_r \otimes (tJ_t - 2J_t - I_t) \\ = \ & (t-2)I_r \otimes (J_t - I_t) + (t-3)I_r \otimes I_t \\ = \ & (t-2)\tilde{A}_{1,1} + (t-3)\tilde{A}_{0,0}, \end{aligned}$$

for any $i,j \in \{0,1,\cdots,r\}$, we have $\tilde{A}_{0,0}\tilde{A}_{i,j} = \tilde{A}_{i,j}\tilde{A}_{0,0} = \tilde{A}_{i,j}$.

For any $i,j \in \{1,2,\cdots,r-1\}$, if $i+j \le r$, then

$$\begin{aligned} & \tilde{A}_{i,r-i}\tilde{A}_{j,r-j} \\ = \ & A_{i,r-i}A_{j,r-j} \otimes J_tJ_t \\ = \ & A_{i+j,r-(i+j)} \otimes tJ_t \\ = \ & t\tilde{A}_{i+j,r-(i+j)} \end{aligned}.$$

If $i+j > r$, then

$$\begin{aligned} & \tilde{A}_{i,r-i}\tilde{A}_{j,r-j} \\ = \ & A_{i,r-i}A_{j,r-j} \otimes J_tJ_t \\ = \ & A_{2r-(i+j),(i+j)-r} \otimes tJ_t \\ = \ & t\tilde{A}_{2r-(i+j),(i+j)-r}, \end{aligned}$$

$$\begin{aligned} & \tilde{A}_{1,1}\tilde{A}_{i,r-i} \\ = \ & A_{i,r-i} \otimes (J_t - I_t)J_t \\ = \ & (t-1)\tilde{A}_{i,r-i}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{A}_{i,r-i}\tilde{A}_{1,1} \\ = \ & A_{i,r-i} \otimes J_t(J_t - I_t) \\ = \ & (t-1)\tilde{A}_{i,r-i}. \end{aligned}$$

Hence if $r \neq 2$, $C_r[K_t]$ is a weakly distance-regular digraph.

If $r = 2$, we have $C_r[K_t]$ is a clique. Thus $C_r[K_t]$ is a weakly distance-regular digraph. □

Theorem 2.3 *Let Γ be a weakly distance-regular digraph of valency t. If every arc of Γ is contained in a minimal circuit with length r, and $k_{2,r-2} = p^{(1,r-1)}_{(2,r-2),(r-1,1)} \ge 1$. Then $\Gamma \simeq C_r[\bar{K}_t]$.*

Proof. If $r = 2$, we have $k_{2,0} \ge 1$; but $k_{2,0} = 0$, this is a contradiction, hence $r \ge 3$. Suppose that $(x_0, x_1, \cdots, x_{r-1})$ is a minimal circuit, for all i, where all subscripts of x are taken modulo r. Since $k = t$ and every arc of Γ is contained in a minimal circuit with length r, $k_{1,r-1} = k = t$. Note that

$$k_{2,r-2} = p^{(1,r-1)}_{(2,r-2),(r-1,1)},$$

and

$$k_{2,r-2}p^{(2,r-2)}_{(1,r-1),(1,r-1)} = k_{1,r-1}p^{(1,r-1)}_{(2,r-2),(r-1,1)}.$$

Hence $p^{(2,r-2)}_{(1,r-1),(1,r-1)} = t$.

Since $x_i \in P^{(2,r-2)}_{(1,r-1),(1,r-1)}(x_{i-1}, x_{i+1}), i = 0, 1, \cdots r-1$, for every $i \in \{0, 1, \cdots r-1\}$, there are only $t-1$ vertices $y_i^{(j)}$ other than vertices x_i such that

$$y_i^{(j)} \in P^{(2,r-2)}_{(1,r-1),(1,r-1)}(x_{i-1}, x_{i+1}),\ j = 0, 1, \cdots t-2,$$

Since $k = t$ and every arc of Γ is contained in a minimal circuit with length r, $k_{1,r-1} = k = t$. It follows that $\Gamma_{1,r-1}(x_{i-1}) = \{x_i, y_i^0, y_i^1, \cdots y_i^{(t-2)}\}$, $i = 0, 1, \cdots r-1$. By proposition 2.1, we have $\Gamma \simeq C_r[\bar{K}_t]$. □

Theorem 2.4 *Let Γ be a commutative weakly distance-regular digraph of valency 4, diameter d, and girth 2. If every arc of Γ is contained in a minimal circuit with length r not containing any edge, and $k_{2,r-2} = p^{(1,r-1)}_{(2,r-2),(r-1,1)} \geq 1$. Then Γ is isomorphic to one of the following digraphs:*

(1) Γ is a distance-regular graph of valency 4.

(2) $K_2[C_3]$.

(3) $\Delta \times C_r$, where r is odd or $d < r/2$ and Δ is a distance-regular graph of valency 3.

(4) $\Delta \times [C_r[\bar{K}_2]]$, where Δ is a circuit.

(5) $K_2 \times [C_r[\bar{K}_3]]$.

Proof. If $k_{1,1} = 4$, then (1) is obvious.

If $k_{1,1} = 3$, since $k = 4$, then (2), (3) holds, by proposition 1.1(3)(4).

If $k_{1,1} = 2$, we have $k_{1,r-1} = 2$. Let $F = <\Gamma_{1,r-1}>$. Since every arc of Γ is contained in a minimal circuit with length r not containing any edge, Γ/F is a circuit. It follows that the valency of $\Gamma(x)$ is $k = k_{1,r-1} = 2$ and $k_{2,r-2} = p^{(1,r-1)}_{(2,r-2),(r-1,1)} \geq 1$. By Theorem 2.3, we have $F(x) \simeq C_r[\bar{K}_2]$. Therefore $\Gamma \simeq \Delta \times [C_r[\bar{K}_2]]$, where Δ is a circuit.

If $k_{1,1} = 1$, then $k_{1,r-1} = 3$. Let $F = <\Gamma_{1,r-1}>$. Similar to the proof of above, we have Γ/F is a edge. It follows that the valency of $\Gamma(x)$ is $k = k_{1,r-1} = 3$, and $k_{2,r-2} = p^{(1,r-1)}_{(2,r-2),(r-1,1)} \geq 1$. By Theorem 2.3, we have $F(x) \simeq C_r[\bar{K}_3]$. Hence $\Gamma \simeq K_2 \times [C_r[\bar{K}_3]]$. □

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