Locating and Liar Domination of Circulant Networks

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Abstract

A *monitor* is a computer in the network which is able to detect a fault computer among its neighbors. There are two stages of monitoring fault computer: (1) Sensing a fault among its neighbors and (2) Locating the fault computer. A sensitive computer network requires double layer monitoring system where monitors are monitored. This problem is modeled using the graph theory concept of dominating set. In graph theory, there are two variations of domination concepts which represent double layer monitoring system. One concept is locating-domination and the other is liar domination. It has been recently demonstrated that circulant network is a suitable topology for the design of On-Chip Multiprocessors and has several advantages over torus and hypercube from the perspectives of VLSI design. In this paper, we study both locating-domination and liar domination in circulant networks. In addition to characterization of locating-dominating set and liar dominating set of circulant networks, sharp lower and upper bounds of locating-dominating set and liar dominating set of circulant networks are presented.

Keywords: Dominating set in graph theory, locating-dominating set, liar dominating set, circulant networks.

1. Introduction

A computer network is represented by a graph $G = (V, E)$ where each vertex $v \in V(G)$ is a computer and edge $e \in E(G)$ is a link between two adjacent computers. A *fault* computer is the one which is affected by malicious virus or is under the influence of hacker. An example of a fault computer is a liar which lies to its neighbors. Another example is a spy computer which behaves like a normal computer but transfers sensitive data illegally from the network to some outsider. A *monitor* is a computer in the network which is able to detect a fault computer among its neighbors. There are two stages of monitoring a fault:

1. Sensing a fault among its neighbors: $N[v] \cap L \neq \emptyset$ for every vertex v.
2. Locating the fault: $N[u] \cap L \neq N[v] \cap L$

where $N(v) = \{x \in V(G) / (x, v) \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. In graph theory, this set of monitors is called *locating-dominating* set [17]. In other words, a set L of vertices is a locating-dominating set if

1. $N[v] \cap L \neq \emptyset$ for every vertex v.
2. $N[u] \cap L \neq N[v] \cap L$ for every pair of vertices $u \neq v$

A sensitive computer network requires *double layer monitoring system* where monitors are monitored. Peter Slater [19, 20] models double layer monitoring system graph theoretically and calls it *liar domination*. A set L of vertices is liar dominating set if

1. For every vertex v, $|N[v] \cap L| \geq 2$.
2. For every pair u, v of distinct vertices, $|(N[u] \cup N[v]) \cap L| \geq 3$.

Circulant network is a natural generalization of double loop network [25]. A *circulant network* with N vertices and jumps $\{j_1, j_2 \ldots j_m\}$ is an undirected graph in which each vertex k, $0 \leq k \leq N-1$, is adjacent to all the vertices $k \pm j_i \bmod N$, with $1 \leq i \leq m$. This graph is denoted as $C_N(j_1, j_2 \ldots j_m)$ [15, 25]. The adjacency matrix of $C_N(j_1, j_2 \ldots j_m)$ is such that its each row is the periodic rotation of the previous row moved one place to the right. Figure 1 is circulant graph $C_8(1, 2)$. Martınez et al [16] demonstrate that circulant network is the most suitable topology for the design of On-Chip Multiprocessors. It is also shown that circulant networks have several advantages over torus and hypercube from the perspectives of VLSI design [16, 23]. Circulant networks have been efficiently used in ILLIAC IV, FDDI-token, SILK and SONET rings, Intel Paragon, Cray T3D, MPP, and MICROS [23].

Both problems are NP-complete for general graphs [2, 3, 20]. In this paper, we study both problems in circulant networks. The interesting part of the paper is the characterization of locating-dominating set and liar dominating set of circulant networks. Using those characterizations, we derive sharp lower and upper bounds of locating-dominating set and liar dominating set of circulant networks. We also illustrate two kinds of locating-dominating set and liar dominating set of circulant networks whose cardinalities are very close to lower bounds.

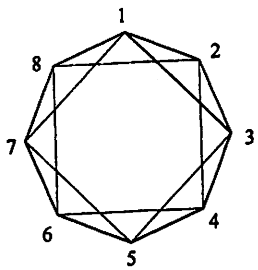


Fig 1: Undirected Circulant graph $C_8(1, 2)$

2. Background

A vertex set $D \subseteq V(G)$ is a *k-dominating set* if $|N[u] \cap D| \geq k$ for every $x \in V(G)$ [13]. Fei Dai and Jie Wu [6] provide rich references how virtual backbone nodes form a connected dominating set of the wireless network. They also demonstrate that k-connected k-dominating set forms a backbone of wireless network system to balance efficiency and fault tolerance. Fei Dai and Jie Wu [6] listed three applications of a connected dominating set in wireless networks:

> *Reducing routing overhead*: By removing all links between non-backbone nodes, the size and maintenance cost of routing tables can be reduced. By using only backbone nodes to forward broadcast packets, the excessive broadcast redundancy can be avoided.
>
> *Energy-efficient routing*: By putting non-backbone nodes into periodical sleep mode, the energy consumption is greatly reduced while network connectivity is still maintained by backbone nodes.
>
> *Area coverage*: In densely deployed sensor networks, the node coverage of a connected dominating set is a good approximation of area coverage. That is, the deployment area is within the sensing range of backbone nodes with high probability.

Another interesting application of domination theory is that there is a relationship between dominating set and average distance of a network [7, 22].

A 2-dominating set is also called a double-dominating set [8]. A liar dominating set is a 2-dominating set and every 3-dominating set is a liar dominating set. Thus a liar dominating set is also called as 2½-dominating set [19, 20]. The first part is "sensing task". In [5], it is described how dominating sets help the sensing task in sensor networks. The second part is "locating task" which is also widely studied [9, 17, 21].

Theoretical properties of circulant graphs have been studied extensively and are surveyed in [14]. Circulant graphs have deserved significant attention during the last decades, the traditional ring and the complete graph topologies belong to this class of graphs. The circulant graphs are studied extensively as reliable interconnection networks for the multiprocessor systems. When selecting an appropriate topology for a computer network, circulant graphs represent an intermediate choice between simple but unreliable ring topology and reliable but expensive (and sometimes technologically unfeasible) fully connected topology. Due to their favorable properties among which are symmetry, scalability, reliability, small diameter, and small average node distance, circulant graphs are widely studied as suitable topologies for local area networks and parallel computer architectures [25]. The diameter and average distance of circulant networks are smaller than those of torus [16, 23]. Variations of domination problem such as distance paired-domination [24], efficient domination problem [18], and total and connected domination [4] are solved for circulant graphs. Martinez et al [15] study perfect t-dominating set problem in Martinez graphs.

Peter Slater et al [19, 20] have shown that the liar domination problem is NP-complete for general graphs and they have provided polynomial solution for trees. The locating-domination problem is NP-complete for general graphs [2, 3]. The problem has been studied for paths [11], chains and cycles [1], trees [9], infinite grids [12, 21], infinite triangular grid [10].

3. Locating-domination of Circulant Networks $C_N(1, 2 \ldots j)$

In this section, we characterize locating-dominating set of circulant graphs $C_N(1, 2 \ldots j)$.

Definition 1: A set $\ell, \ell+1, \ell+2 \ldots \ell+k$ of consecutive vertices is called an *arc* and is denoted by $A(\ell, \ell+k)$. Throughout the paper, we reserve notation A to denote an arc. The two vertices ℓ and $\ell+k$ are called *end vertices* of the arc. See Figure 6.

A locating-dominating set of $C_N(1, 2 \ldots j)$ is expressed in terms of end points of arc $A(\alpha, \beta)$ of $2j+2$ consecutive vertices.

Theorem 1: A set L of vertices is a locating-dominating set of $C_N(1, 2 \dots j)$ if and only if each arc $A(\alpha, \beta)$ of $2j+2$ consecutive vertices satisfies the following two conditions:

1. One of the end vertices α and β is in L.
2. $A(\alpha, \beta)$ contains at least 2 vertices of L.

Proof: Suppose L is a locating dominating set $C_N(1, 2 \dots j)$. Let us prove that each arc $A(\alpha, \beta)$ of $2j+2$ consecutive vertices satisfies both conditions. Without loss of generality, let $A(\alpha, \beta) = \{\ell-j \dots \ell-2, \ell-1, \ell, \ell+1, \ell+2 \dots \ell+j, \ell+j+1\}$ where the numbers are taken modulo N. Here $\alpha = \ell-j$ and $\beta = \ell+j+1$. Let us first prove condition 1.

Condition 1: Suppose neither α nor β is in L. Since neither α nor β is in L, $N[\ell] \cap L$ is a subset of $\{\ell-j+1 \dots \ell-2, \ell-1, \ell, \ell+1, \ell+2 \dots \ell+j\}$ and $N[\ell+1] \cap L$ is also a subset of $\{\ell-j+1 \dots \ell-2, \ell-1, \ell, \ell+1, \ell+2 \dots \ell+j\}$. Moreover, $N[\ell] \cap N[\ell+1] = \{\ell-j+1 \dots \ell-2, \ell-1, \ell, \ell+1, \ell+2 \dots \ell+j\}$. Thus $N[\ell] \cap L = N[\ell] \cap N[\ell+1] \cap L$ and $N[\ell+1] \cap L = N[\ell] \cap N[\ell+1] \cap L$. This leads to a contradiction to our assumption that $N[\ell] \cap L \neq N[\ell+1] \cap L$.

Next let us prove condition 2.
Condition 2: Suppose $A(\alpha, \beta)$ and L have at most one vertex in common. By condition 1, one of the end vertices α and β of $A(\alpha, \beta)$ is in L. Thus L has exactly only one vertex of $A(\alpha, \beta)$ which is either α or β. If L contains α, then $N[\ell+1] \cap L = \emptyset$. If L contains β, then $N[\ell] \cap L = \emptyset$. Either way it contradicts to the fact that L is locating-dominating set.

Now let us move on to prove the converse part. Suppose $N[\ell] \cap L = \emptyset$ for some ℓ. By definition, $N[\ell] \cup \{\ell+j+1\} = A(\alpha, \beta)$ where $\alpha = \ell-j$ and $\beta = \ell+j+1$. Since $N[\ell] \cap L = \emptyset$, $A(\alpha, \beta)$ has at most one vertex of L which is a contradiction to our assumption.

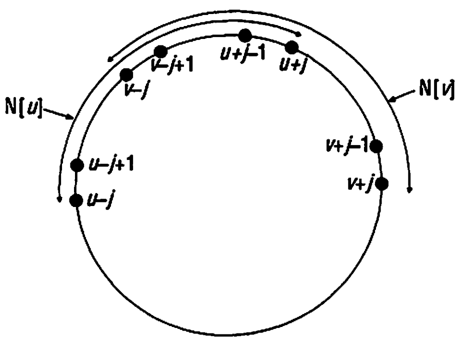


Fig 2: The vertices of N[u] and N[v].

Suppose $N[u] \cap L = N[v] \cap L$ for some pair of vertices $u \neq v$. Then $N[u] \cap N[v] \neq \emptyset$. This implies that $N[u] \cup N[v]$ is an arc of more than $2j+2$ consecutive vertices. See Figure 2. The end vertices of arc $N[u] \cup N[v]$ are $u-j$ and $v+j$. Since $N[u] \cap L = N[v] \cap L$, L does not intersect the set $(N[u] \cup N[v]) \setminus (N[u] \cap N[v])$. Thus the end vertices of arc $A(u-j, u+j+1)$ of $2j+2$ consecutive vertices are not in L. This is a contradiction to our assumption.

Thus L is a locating-dominating set. □

Now we estimate a tight lower bound of locating-dominating set of $C_N(1, 2 \dots j)$.

Theorem 2: A locating-dominating set of $C_N(1, 2 \dots j)$ has at least $(2j+1)\lfloor N/(4j+2) \rfloor$ vertices.

Proof: Let L be a locating-dominating set of $C_N(1, 2 \dots j)$. The vertex set $\{0, 1 \dots N-1\}$ is partitioned into $V_1, V_2 \dots V_d$. Each V_i is a set of $(2j+2)+(2j) = 4j+2$ consecutive vertices. See figure 3. Thus $d = \lfloor N/(4j+2) \rfloor$. Each set V_i is further partitioned into $(2j+1)$ pair of vertices. Each pair is k and $k+(2j+1)$.

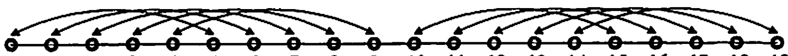
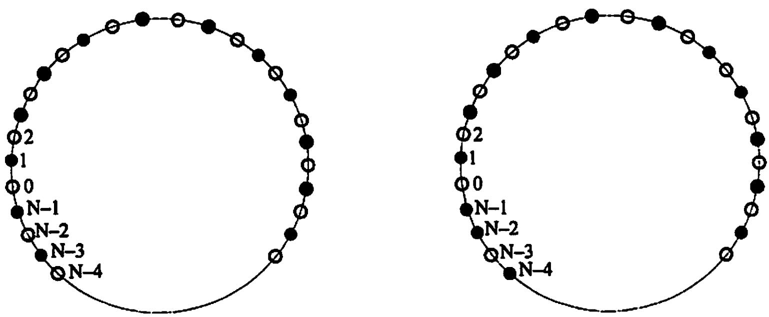


Fig 3: Here vertex set of $C_N(1, 2)$ is partitioned into pair of vertices. In $C_N(1, 2)$, $4j+2 = 10$. Each set of 10 vertices is partitioned into 5 pair of vertices. Each pair is k and $k+5$.

Each pair $\{k, k+(2j+1)\}$ induces an arc $A(k, k+(2j+1))$ of $2j+2$ consecutive vertices. By Theorem 1, each arc $A(k, k+(2j+1))$ has one of its end vertices k and $k+(2j+1)$ in L. Thus

$$|V_i \cap L| \geq 2j+1 \text{ for } i=1, 2 \dots d.$$

Since $d = \lfloor N/(4j+2) \rfloor$, the cardinality of L is at least $(2j+1)\lfloor N/(4j+2) \rfloor$. □



Bold bullets are members of locating-dominating set LOC of $C_N(1, 2, 3)$. The vertices N–2 and N–2 are members of LOC if N is odd.

Fig 4a: N is even. **Fig 4b: N is odd.**

Locating-dominating set LOC: Consider a locating-dominating set LOC with alternative vertices of $C_N(1, 2 \ldots j)$. See Figures 4a and 4b. The cardinality of LOC is $\lceil N/2 \rceil$.

Theorem 3: The cardinality of minimum locating-dominating set of $C_N(1, 2 \ldots j)$ is between $(2j+1)\lfloor N/(4j+2) \rfloor$ and $\lceil N/2 \rceil$.

Proof: The lower bound of a minimum locating-dominating set of $C_N(1, 2 \ldots j)$ is by Theorem 2. The upper bound is obtained by locating-dominating set LOC whose cardinality is $\lceil N/2 \rceil$. □

4. Liar domination of Circulant Networks $C_N(1, 2 \ldots j)$

First we characterize liar dominating sets of $C_N(1, 2 \ldots j)$.

Theorem 4: A set L of vertices of circulant graph $C_N(1, 2 \ldots j)$ is a liar dominating set if and only if $|A \cap L| \geq 3$ for every arc A on $2j+2$ consecutive vertices of $C_N(1, 2 \ldots j)$.

Proof: Let L be a liar dominating set of $C_N(1, 2 \ldots j)$. Let A be an arc of $2j+2$ consecutive vertices. Without loss of generality, we assume that A is $\{\ell-j \ldots \ell-2, \ell-1, \ell, \ell+1, \ell+2 \ldots \ell+j, \ell+j+1\}$ where the numbers are taken modulo N. We claim that $|A \cap L| \geq 3$. It is easy to see that

$$N[\ell] \cup N[\ell+1] = A \qquad \text{(E1)}$$

Since L is liar dominating set of $C_N(1, 2 \ldots j)$, we have

$$|(N[\ell] \cup N[\ell+1]) \cap L| \geq 3. \qquad \text{(E2)}$$

Combining E1 and E2, we get $|A \cap L| \geq 3$.

Now we prove the converse. Assuming that $|A \cap L| \geq 3$ for every arc A on $2j+2$ consecutive vertices, we shall prove that L is liar dominating set. In order to prove that L is liar dominating set, we need to prove the following two conditions:

1. For every vertex v, $|N[v] \cap L| \geq 2$.
2. For every pair u, v of distinct vertices, $|(N[u] \cup N[v]) \cap L| \geq 3$.

Condition (1): Suppose there exists a vertex u such that $|N[u] \cap L| < 2$ where $N[u] = \{u-j \ldots u-2, u-1, u, u+1, u+2 \ldots u+j\}$. Consider the arc $A = \{u-j \ldots u-2, u-1, u, u+1, u+2 \ldots u+j, u+j+1\}$. Since $|N[u] \cap L| < 2$, $|A \cap L| < 3$. This is a contradiction to hypothesis. Hence $|N[u] \cap L| \geq 2$ which is condition (1).

Condition (2): We prove that $|(N[u] \cup N[v]) \cap L| \geq 3$ for each pair of vertices u and v. There are two cases:

Case 1 ($N[u] \cap N[v] \neq \emptyset$): We use a structural property of $C_N(1, 2 \ldots j)$. If $N[u] \cap N[v] \neq \emptyset$, then $(N[u] \cup N[v])$ induces an arc of consecutive vertices and $|N[u] \cup N[v]| \geq 2j+2$. This implies that $|(N[u] \cup N[v]) \cap L| \geq 3$ by the hypothesis of Lemma.

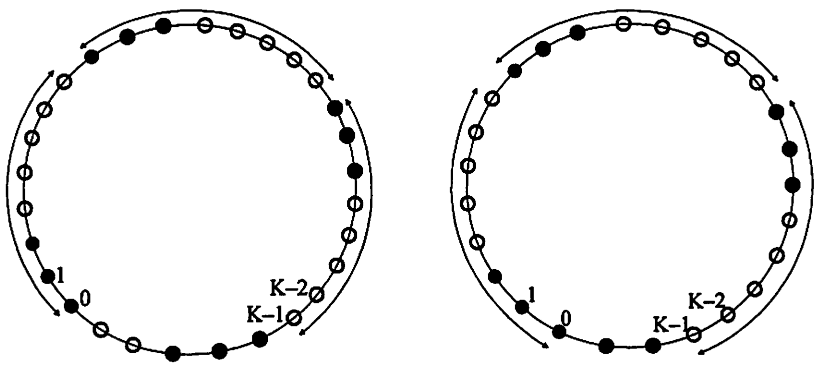
Case 2 ($N[u] \cap N[v] = \emptyset$): We have just proved condition (1) that $|N[u] \cap L| \geq 2$ for every vertex u of V. Therefore, $N[u] \cap N[v] = \emptyset$, $|N[u] \cap L| \geq 2$ and $|N[v] \cap L| \geq 2$. This implies that $|(N[u] \cup N[v]) \cap L| \geq 4$ which proves condition (2). □

Corollary 1: An arc of $2j+1$ ($2j$) consecutive vertices contains at least two (one) vertices of a liar dominating set of $C_N(1, 2 \ldots j)$. □

Now we estimate a tight lower bound of liar dominating set of $C_N(1, 2 \ldots j)$.

Liar dominating set of $C_N(1, 2 \ldots j)$: Consider a liar dominating set *LIA* has 3 consecutive vertices in an arc of $2j+2$ consecutive vertices. Let $K = (2j+2)\lfloor N/(2j+2) \rfloor$. The cardinality of *LIA* is $3(\lfloor N/(2j+2) \rfloor + 1)$ if $(N-K) \geq 3$. See Figure 5a. The cardinality of *LIA* is $3\lfloor N/(2j+2) \rfloor + (N-K)$ if $(N-K) < 3$. See Figure 5b. In other words, the cardinality of *LIA* is between $3\lfloor N/(2j+2) \rfloor$ and $3(\lceil N/(2j+2) \rceil)$.

Now we provide a tight bound of minimum liar dominating set of $C_N(1, 2 \ldots j)$.



Liar dominating set *LIA* of $C_N(1, 2, 3)$. Bold bullets are members of L. $|L| \leq 3(\lceil N/(2j+2) \rceil)$.

Fig 5a: $(N-K) \geq 3$. Here N = 29 **Fig 5b: $(N-K) < 3$. Here N = 26**

Theorem 5: The cardinality of a minimum liar dominating set of $C_N(1, 2 \ldots j)$ is between $3\lfloor N/(2j+2) \rfloor$ and $3(\lceil N/(2j+2) \rceil)$.

Proof: Let L be a liar dominating set of $C_N(1, 2 \ldots j)$. By Theorem 4, an arc of $(2j+2)$ consecutive vertices has 3 vertices of L. Hence a liar dominating set of $C_N(1, 2 \ldots j)$ has at least $3\lfloor N/(2j+2) \rfloor$ vertices. Since the cardinality of liar dominating set

LIA is $3(\lceil N/(2j+2) \rceil)$, a minimum liar dominating set has at most $3(\lceil N/(2j+2) \rceil)$ vertices.. □

5. Liar domination of Circulant Networks $C_N(1, 2)$

In this section, we study circulant graphs $C_N(1, 2)$ of restricted class. In the case of $C_N(1, 2)$, the gap between the lower bound and upper bound of minimum liar dominating set is reduced further. Let us recall some number theory properties of the "floor" function of an integer:

$$\lfloor N/2 \rfloor = 3\ell + 0 \quad \text{if } N = 6\ell \text{ or } N = 6\ell+1.$$

$$\lfloor N/2 \rfloor = 3\ell + 1 \quad \text{if } N = 6\ell+2 \text{ or } N = 6\ell+3.$$

$$\lfloor N/2 \rfloor = 3\ell + 2 \quad \text{if } N = 6\ell+4 \text{ or } N = 6\ell+5.$$

In what follows, we improve the lower bound for circulant networks $C_N(1, 2)$ and prove that the bound is sharp.

Lemma 1: The cardinality of a liar dominating set L of circulant graph $C_N(1, 2)$ is at least $\lfloor N/2 \rfloor$. □

Proof: Let $N = 6\ell+k$. We first prove the result for $k = 2$. Since $N = 6\ell+2$, $\lfloor N/2 \rfloor = 3\ell+1$. Suppose L has only 3ℓ vertices. Since $N = 6\ell+2$ and L has only 3ℓ vertices, there is a pair of consecutive vertices x and $x+1$ of $C_N(1, 2)$ which are not in L. Without loss of generality, we assume that these two vertices are 0 and 1. See Figure 6. Then the arc of consecutive vertices 2, 3, 4, 5 has al least 3 vertices of L by Theorem 4. Subsequently each arc $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, $6k+5$ has al least 3 vertices of L by Theorem 4 for every $k = 1, 2 \ldots \ell$. The set L has only 3ℓ vertices and all the vertices of L are lying between 2 and $6\ell-1$. Since the cardinality of $C_N(1, 2)$ is $6\ell+2$, there are still remaining 2 vertices, which are 6ℓ and $6\ell+1$. Notice that the vertices 6ℓ and $6\ell+1$ are nothing but $N-2$ and $N-1$ respectively. These vertices $N-2$ and $N-1$ are not in L because all the vertices of L are lying between 2 and $6\ell-1$. Thus the arc of four consecutive vertices $N-2$, $N-1$, 0, and 1 has no vertex of L which is a contradiction to Corollary 1.

The proof is similar for other cases $k = 3, 4, 5$ and hence we prove the lemma. □

Theorem 6: The cardinality of a minimum liar dominating set of circulant graph $C_N(1, 2)$ is between $\lfloor N/2 \rfloor$ and $\lceil N/2 \rceil$.

Proof: By Lemma 1, a minimum liar dominating set of $C_N(1, 2)$ has at least $\lfloor N/2 \rfloor$ vertices. Also we know that the cardinality of liar dominating set of type I is $\lceil N/2 \rceil$. See figure 7. Thus, a minimum liar dominating set of $C_N(1, 2)$ has at most $\lceil N/2 \rceil$ vertices. Hence the theorem is proved. □

Remark: Readers will note that Theorem 6 does not follow from Theorem 5. □

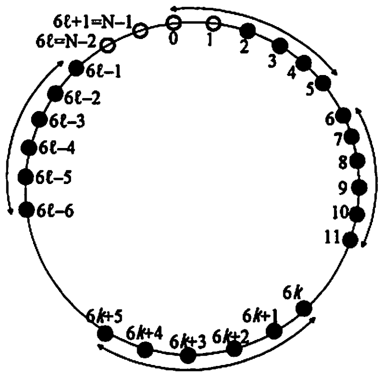


Fig 10: Liar dominating set of circulant graph $C_N(1, 2)$

Conclusion: We have derived sharp lower and upper bounds of minimum locating-dominating set and minimum liar dominating set of $C_N(1, 2 \ldots j)$. Moreover, we have also identified locating-dominating set and liar dominating set of $C_N(1, 2 \ldots j)$ which are lying between this lower and upper bound. In the case of $C_N(1, 2)$, we go very close to the answer that the difference between the lower bound and the upper bound of a minimum liar dominating set of $C_N(1, 2)$ is just 1. However, we have not solved the problem of finding minimum locating-dominating set and minimum liar dominating set of $C_N(1, 2 \ldots j)$. We leave it to the reader to show that the problem of finding minimum locating-dominating set and minimum liar dominating set of $C_N(1, 2 \ldots j)$ is polynomially solvable. □

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