Minimal weak separators of chordal graphs

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Abstract

A minimal separator of a graph is an inclusion-minimal set of vertices whose removal disconnects some pair of vertices. We introduce a new notion of minimal weak separator of a graph, whose removal merely increases the distance between some pair of vertices. The minimal separators of a chordal graph G have been identified with the edges of the clique graph of G that are in some clique tree, while we show that the minimal weak separators can be identified with the edges that are in no clique tree. We also show that the minimal weak separators of a chordal graph G can be identified with pairs of minimal separators that have nonempty intersection without either containing the other—in other words, the minimal weak separators can be identified with the edges of the overlap graph of the minimal separators of G.

1 Introduction

A subset S of vertices of a connected graph G is a *minimal u,v-separator* of vertices u and v of G if S is inclusion-minimal such that u and v are in different components of the subgraph induced by $V(G) - S$; such an S is also called a *minimal separator*. (The empty set of vertices is frequently taken to be a minimal separator of a graph that is not connected, and that convention could have been carried through in the present paper—but, for convenience and simplicity, separators are only being defined here for connected graphs.) Not unsurprisingly, there are related concepts in the literature of graph vulnerability and connectivity; [6] is an early paper and [15] is a recent paper with updated references.

We introduce and study a new notion of 'minimal weak separator' (previously occurring only in Exercise 2.9 of [14]). Define a subset S of vertices of a connected graph G to be a *minimal weak u,v-separator* of vertices u and v of G if S is inclusion-minimal such that u and v are still in a common component of the subgraph induced by $V(G) - S$ but the distance between

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them in that subgraph is greater than the distance between them in G; also call such an S a *minimal weak separator* of G. As examples, each vertex v of C_5 forms a minimal weak separator $\{v\}$, while C_4 has no minimal weak separators. Pairs of nonadjacent vertices that are in common 4-cycles form the twelve minimal weak separators present in the 3-cube $K_2 \times K_2 \times K_2$. A graph G has no minimal weak separators if and only if G is *distance-hereditary*, where this means that the distance between every two vertices in every connected induced subgraph of G is the same as it is in G; see [4]. (By contrast, G has no minimal separators if and only if it is complete.)

Our results on minimal weak separators will all be in the restricted context of *chordal graphs*, meaning graphs with no induced cycles of length four or more (see [4, 14] for details). This mirrors the frequent restriction to chordal graphs in the extensive work that has been done on minimal separators. One motivation for that restriction is that the number of minimal separators grows exponentially in the number of vertices in general, but at most linearly for chordal graphs. Similarly, the number of minimal weak separators grows exponentially in general (indeed, for hypercubes), but at worst quadratically for chordal graphs (which will follow from Theorem 5 below and the well-known fact that the number of maximal complete subgraphs of a chordal graph is bounded by the number of vertices).

Another (historical) motivation for restricting the study of minimal separators to chordal graphs is Dirac's well-known characterization of chordal graphs as those in which minimal separators induce complete subgraphs. There cannot be a similar chordal characterization property that every minimal weak separator satisfies, since C_4 has no minimal weak separators. But Proposition 7 will be a weak separator analog of another, more recent characterization of chordal graphs in terms of minimal separators.

References [5, 12] locate all 'hinge vertices'—which, in our language, are the singleton vertex subsets that are either minimal separators or minimal weak separators—for classes that are more restrictive than chordal graphs (namely, strongly chordal graphs and interval graphs, respectively). Theorems 5 and 12 below will show ways to locate minimal weak separators of all cardinalities for all chordal graphs. (Lemma 3 will review one well-known way to locate all minimal separators of chordal graphs.)

Our results will include the following, for connected chordal graphs G:

- The minimal weak separators of G can be identified with the edges of the clique graph of G that are not in any clique tree of G (clique graphs and clique trees will be defined in section 2).
- A vertex belongs to a minimal weak separator of G if and only if it is the degree-4 vertex of an induced 'gem' subgraph of G (a *gem* results from a length-5 cycle by inserting additional edges from one vertex to the two opposite vertices).

- The minimal weak separators of G can be identified with the edges of the separator overlap graph of G (separator overlap graphs will be defined in section 4).

Section 2 will establish our notation and review the necessary background concerning chordal graphs and clique trees. Section 3 will show how minimal weak separators of chordal graphs can be located using 'clique thickets' (unions of all clique trees); section 4 will do the same using 'separator overlap graphs.' (Both methods correspond to quadratic algorithms.)

2 Notation and background

We use standard graph-theoretic notation. In particular, for any $v \in V(G)$, $N(v)$ denotes the set of neighbors of v in G. For any subgraph H of a graph G—possibly a path P or a spanning tree T of the graph—$V(H)$ and $E(H)$ denote, respectively, the vertex and edge sets of H. For any $S \subset V(G)$, $G - S$ denotes the subgraph induced by $V(G) - S$. For any $uv \in E(G)$, $G - uv$ denotes the subgraph of G obtained by deleting the edge uv; if u and v are in a common component of G but not of $G - uv$, then uv is a *cut-edge* of G. For any $u, v \in V(H)$, $d_H(u, v)$ denotes the *distance* between u and v—the length of a shortest path connecting u and v—in the subgraph H. The *maxcliques* of G are the inclusion-maximal subsets of $V(G)$ that induce complete subgraphs.

The remainder of this section reviews needed background from chordal graph theory [14]. We begin with two simple observations that will be used tacitly throughout the remainder of this paper.

Lemma 1 *Suppose G is chordal with nonadjacent vertices u and v where $N(u) \cap N(v) \neq \emptyset$. Then $N(u) \cap N(v)$ is either a minimal separator or a minimal weak separator.*

Proof. Suppose G, u, and v are as stated and $S = N(u) \cap N(v) \neq \emptyset$. Since S is contained in every minimal u,v-separator, either S will be a minimal u,v-separator or u and v will still be in a common component of $G - S$, connected there only by paths of length three or more. In the latter case, $d_G(u, v) = 2$ will have increased to $d_{G-S}(u, v) \geq 3$, making S a minimal weak u,v-separator. □

Lemma 2 *Suppose G is chordal with nonadjacent vertices u and v where $N(u) \cap N(v) \neq \emptyset$. Then there exist maxcliques Q_u and Q_v of G such that $u \in Q_u$, $v \in Q_v$, and $Q_u \cap Q_v = N(u) \cap N(v)$.*

Proof. Suppose G, u, and v are as stated and $S = N(u) \cap N(v) \neq \emptyset$. Since S is contained in every minimal u,v-separator and every minimal u,v-separator induces a complete subgraph of G (by Dirac's characterization of chordal), both $S \cup \{u\}$ and $S \cup \{v\}$ will induce complete subgraphs and so will be contained in maxcliques Q_u and Q_v, respectively. So $S \subseteq Q_u \cap Q_v$. Conversely, any $x \in Q_u \cap Q_v$ will be adjacent to both u and v, and so $x \in S$. Therefore, $Q_u \cap Q_v = S$. □

The *clique graph* $\mathcal{K}(G)$ of a connected graph G has all the maxcliques of G as nodes—calling the vertices of the clique graph *nodes* will lessen their confusion with the vertices of G—with two nodes Q and Q' adjacent whenever $Q \cap Q' \neq \emptyset$. The *weighted clique graph* $\mathcal{K}^w(G)$ assigns each edge QQ' of the clique graph the weight $|Q \cap Q'| > 0$. (Weight zero edges could have been used if disconnected graphs G had been allowed.) Suppose T is a maximum-weight spanning tree of $\mathcal{K}^w(G)$. For any two nodes Q and Q' of $\mathcal{K}(G)$, let $T[Q, Q']$ denote the subgraph of T that consists of the path in T between Q and Q'.

Figure 1 shows a graph G_1, its weighted clique graph $\mathcal{K}^w(G_1)$—where for instance the node '*abe*' abbreviates the maxclique $\{a, b, e\}$ and the weight of an edge is indicated using multiple edges—and one maximum spanning tree T_1 of $\mathcal{K}^w(G_1)$; any one of the three edges from the node *ef* could have been chosen. (The 'clique thicket' Θ_1 will be defined at the beginning of section 3.)

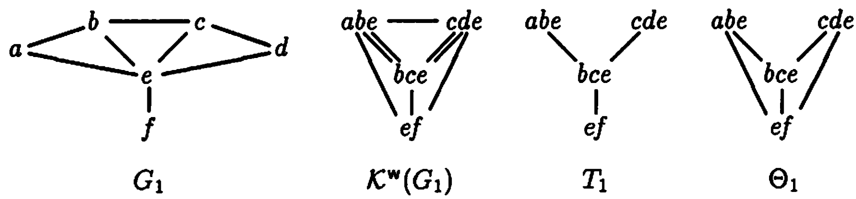


Figure 1: A chordal graph G_1 and its weighted clique graph $\mathcal{K}^w(G_1)$, with a clique tree T_1 and the clique thicket Θ_1.

Reference [14] contains the history and proof of the statements made in the remaining part of this section.

One standard characterization of chordal graphs [14, Theorem 2.1] is that a connected graph G is chordal if and only if, for some (or, equivalently, for every) maximum spanning tree T of $\mathcal{K}^w(G)$ and for every $v \in V(G)$, the subgraph T_v that is induced by those nodes that contain v is connected.

Such a tree T is called a *clique tree* of G. One characterization of a graph G being chordal is that G is the intersection graph of a family of subtrees of some tree—this is used as the definition of chordal graph in [14]—and, indeed, a chordal graph G will always be the intersection graph of the subtrees T_v of any clique tree T of G.

The clique trees of a connected chordal graph G can be constructed from $\mathcal{K}^w(G)$ as follows: At each step, select any one of the next-largest-weight edges that would not form a cycle with previously-chosen edges [14, Theorem 2.3]. (This is, of course, just Kruskal's standard greedy maximum spanning tree algorithm.)

The following is a standard result of chordal graph theory, originating in [2] and [11]; also see [14].

Lemma 3 *Every edge in a clique tree T of a connected chordal graph G corresponds to a minimal separator of G, and every minimal separator of G corresponds to an edge in every clique tree of G.* Specifically, $\{Q \cap Q' : QQ' \in E(T)\}$ is the set of minimal separators of G. □

In Figure 1, for instance, the three sets $\{b, e\}$, $\{c, e\}$, and $\{e\}$ correspond to the three edges of T and are the three minimal separators of G. Notice that Lemma 3 also shows that the set $\{Q \cap Q' : QQ' \in E(T)\}$ is uniquely determined by G even though, as in Figure 1, G can have more than one clique tree T.

3 Clique thickets and their chords

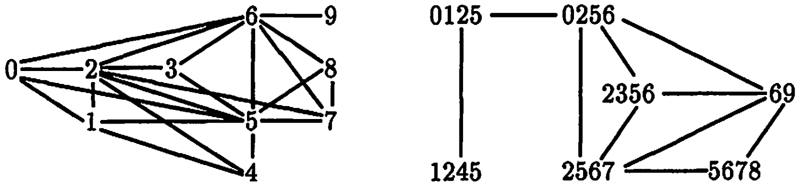
Define the *clique thicket* Θ of a chordal graph G to be the subgraph of $\mathcal{K}(G)$ that is the union of all the clique trees of G. Clique thickets were introduced and studied in [7] (where they were, against standard practice, called 'clique graphs'; our name 'thicket'—an entanglement of trees—should help avoid confusion). Thus, Lemma 3 can be restated as: *Every edge of the clique thicket Θ of a chordal graph G corresponds to a minimal separator of G, and every minimal separator of G corresponds to an edge of Θ.*

Lemma 4 *The clique thicket Θ of a connected chordal graph G can be constructed from $\mathcal{K}^w(G)$, starting from any clique tree T of G as follows: Include an edge QQ' of $\mathcal{K}^w(G)$ in Θ if and only if $|Q \cap Q'| = |R \cap R'|$ for some edge RR' in $T[Q, Q']$.*

Proof. This is a special case of a general construction from [16] for the union of all maximum (or all minimum)spanning trees of any graph. □

Define the *chords* of a clique tree T or of a clique thicket Θ to be the edges of $\mathcal{K}(G)$ that are not edges of, respectively, T or Θ. In other words, the chords of the clique thicket are the edges of $\mathcal{K}(G)$ that are chords of every clique tree. Notice that if Q, R, R' and Q' are nodes in that order along $T[Q, Q']$, then $Q \cap Q' \subseteq R \cap R'$ (since T_v is connected for every $v \in Q \cap Q'$).

Figure 2 shows another example of a chordal graph G_2 and its clique thicket Θ_2. There are nine chords QQ' of Θ_2: five with $Q \cap Q' = \{2,5\}$, two with $Q \cap Q' = \{5,6\}$, and two with $Q \cap Q' = \{5\}$.



G_2 Θ_2

Figure 2: A chordal graph G_2 and its clique thicket Θ_2.

Theorem 5 *Every chord of the clique thicket Θ of a connected chordal graph G corresponds to a minimal weak separator of G, and every minimal weak separator of G corresponds to a chord of Θ.* Specifically, $\{Q \cap Q' : QQ' \in E(\mathcal{K}(G)) - E(\Theta)\}$ is the set of minimal weak separators of G.

Proof. Suppose QQ' is a chord of the clique thicket Θ of a chordal graph G and $Q = Q_1, Q_2, \ldots, Q_k = Q'$, $k \geq 3$, is the path $T[Q,Q']$ in some clique tree T inside Θ. Because each Q_i is a maxclique of G, there exist nonadjacent vertices $v \in Q - Q_2$ and $v' \in Q' - Q_{k-1}$ with $d_G(v,v') = 2$ (since v and v' are common neighbors of the vertices in $Q \cap Q' \neq \emptyset$).

Suppose x is any common neighbor of v and v'. Then $xv \in E(G)$ implies that x and u are in some common maxclique of G, and so that x is in some node of T_v. Similarly, x is in some node of $T_{v'}$. Therefore T_x will contain every node in $T[Q,Q']$, and so $x \in Q \cap Q'$. Therefore, every common neighbor of v and v' will be in $Q \cap Q'$, and so $d_{G-(Q\cap Q')}(v,v') \geq 3$. This makes $Q \cap Q'$ a minimal weak v,v'-separator.

Conversely, suppose S is a minimal weak v,v'-separator in a chordal graph G. Let N and N' consist, respectively, of all the neighbors of v and of v' in $G - S$. Note that $N \cap N' = \emptyset$ (since $x \in N \cap N'$ would imply that $w \in S$). Let R and R' be minimal v,v'-separators that properly contain S and that are contained in, respectively, $S \cup N$ and $S \cup N'$. Also note that $R \cap R' = S$. Suppose T is any clique tree of G. By Lemma 3, T contains an edge Q_1Q_2 with $Q_1 \cap Q_2 = R$ and an edge $Q'_1Q'_2$ with $Q'_1 \cap Q'_2 = R'$. Pick $Q \in \{Q_1, Q_2\}$ and $Q' \in \{Q'_1, Q'_2\}$ such that Q_1, Q_2, Q'_1, and Q'_2 are all nodes in the path $T[Q,Q']$. The minimal separators R and R' will correspond to the edges of $T[Q,Q']$ that are incident to, respectively, Q and Q'. So $S \subset R \subset Q$ and $S \subset R' \subset Q'$, and so $S \subseteq Q \cap Q'$. Since $Q \cap Q'$ is contained in every node of $T[Q,Q']$, we have $Q \cap Q' \subseteq R \cap R' = S$. Thus Q and Q' are nonadjacent nodes of T—and so QQ' is a chord of T—with

$Q \cap Q' = S$. Because S is properly contained in both R and R' (and so is properly contained in every edge of $T[Q, Q']$), edge QQ' will not be in θ (by Lemma 4). Hence QQ' is a chord of the clique thicket of G. □

In the example shown in Figure 1, notice that the set $\{e\} \subseteq V(G_1)$ corresponds to edges in the clique thicket Θ_1 and also to a chord of Θ_1. This reflects that $\{e\}$ is both a minimal a, f-separator and a minimal weak a, d-separator in G_1. Notice as well that each edge [and chord] QQ' of Θ_1 determines nonadjacent vertices $u \in Q - Q'$ and $v \in Q' - Q$ for which $Q \cap Q'$ is a minimal [weak] u,v-separator.

Corollary 6 *Every edge of the clique graph of a connected chordal graph G corresponds to either a minimal separator or a minimal weak separator of G.* □

Corollary 6 can also be rephrased to say that: *Every edge not in a clique tree (or the clique thicket) of a connected chordal graph G corresponds to a minimal weak separator of G, and every minimal weak separator of G corresponds to an edge that is not in every clique tree (or the clique thicket) of G.* This resembles Lemma 3 with 'minimal weak separators' replacing 'minimal separators' and with the insertion of negations. There is a similar transformation of a recent characterization of chordal graphs in terms of minimal separators into a characterization of chordal in terms of minimal weak separators. Reference [3] contains the following: *A graph G is chordal if and only if, for every $uv \in E(G)$, either uv is a cut-edge of G or $N(u) \cap N(v) \neq \emptyset$ is a minimal u,v-separator in $G - uv$.* Proposition 7 is the minimal weak separator analog.

Proposition 7 *A graph G is chordal if and only if, for every $uv \in E(G)$, either uv is a cut-edge of G or $N(u) \cap N(v) \neq \emptyset$ is* not *a minimal weak u,v-separator in $G - uv$.*

Proof. First suppose G is chordal and $uv \in E(G)$ is not a cut-edge of G. Then $N(u) \cap N(v) \neq \emptyset$ is a minimal u,v-separator in $G - uv$ (by the above-cited result from [3]), and so $N(u) \cap N(v)$ cannot be a minimal weak u,v-separator in $G - uv$ (by the definition of minimal weak separator).

Conversely, suppose G is not chordal; say C is an induced cycle of G of length four or more with $uv \in E(C)$. Let P be the induced u,v-path $C - uv$. Then $N(u) \cap N(v) \cap V(P) = \emptyset$, and so $N(u) \cap N(v)$ is either empty or is a minimal weak u,v-separator. □

Recall that a *gem* is a graph that is isomorphic to the subgraph $G - \{f\}$ shown in Figure 1.

Theorem 8 *A vertex of a chordal graph is in a minimal weak separator if and only if it is the degree-4 vertex of an induced gem.*

Proof. First suppose G is a chordal graph that has a vertex v that is in a minimal weak separator S. By Theorem 5, $S = Q \cap Q'$ where QQ' is a chord of the clique thicket of G. Then, for any clique tree T of G, v will be adjacent to every vertex in every node Q_i along the path $T[Q, Q'] = Q_1, Q_2, \dots, Q_k$ ($k \geq 3$), each $Q_i \cap Q_{i+1}$ with $1 \leq i < k$ will properly contain S, and every vertex of G that is in every node of $T[Q, Q']$ will be in $Q_1 \cap Q_k = S$. Therefore, there will exist j with $1 < j < k$ and $x \in (Q_{j-1} \cap Q_j) - Q_{j+1}$ and $y \in (Q_{j+1} \cap Q_j) - Q_{j-1}$ with $xy \in E(Q)$. Since the nodes of T are the maxcliques of G, there exist vertices $w \in Q_{j-1} - Q_j$ and $z \in Q_{j+1} - Q_j$; so w is adjacent to x but not to y, y is adjacent to z but not to x, and w is not adjacent to z. Thus, v is the degree-4 vertex of the gem induced by $\{v, w, x, y, z\}$.

Conversely, suppose $\{v, w, x, y, z\}$ induces a gem in G with v adjacent to each vertex of the induced path w, x, y, z. Let $S = N(w) \cap N(z)$. Then $v \in S$ and S is a minimal weak w, z-separator. □

Corollary 9 *If S is a minimal weak separator in a chordal graph G, then S is a minimal weak u,v-separator where $d_G(u, v) = 2$ and $d_{G-S}(u, v) = 3$.*

Proof. This is a direct consequence of Theorem 8, where u and v are the simplicial vertices of a gem. □

A graph is *ptolemaic* [4, 10] if it is chordal and distance-hereditary. A graph G is *trivially perfect*—other names used include *HT-*, *nested interval*, and *quasi-threshold* [4, 14]—if, for every induced subgraph G' of G, the cardinality of the largest independent set in G' equals the number of maxcliques in G'.

Corollary 10 *A connected graph G is ptolemaic if and only if G is chordal with clique thicket $\Theta \cong \mathcal{K}(G)$.*

Proof. This follows from Theorem 5 together with the result in [10] that *a graph is ptolemaic if and only if $Q \cap Q'$ is a minimal separator for every two* nondisjoint *maxcliques Q and Q'*. (Theorem 5 also proves the result from [10] that a graph is ptolemaic if and only if it is chordal and gem-free.) □

Corollary 11 *A connected graph G is trivially perfect if and only if G is chordal with clique thicket Θ complete.*

Proof. This follows from Theorem 5 together with the fact that *a graph is trivially perfect if and only if $Q \cap Q'$ is a minimal separator for every two* *maxcliques Q and Q'*, which itself is a straightforward consequence of the result from [8] that a graph is trivially perfect if and only if it is chordal and P_4-free. □

4 Separator overlap graphs and their edges

Define the *separator overlap graph* $\Sigma = \Sigma(G)$ of a connected graph G as follows: the nodes of Σ are the minimal separators of G, with two nodes Q and Q' adjacent if and only if both $Q \cap Q' \neq \emptyset$ and also Q and Q' are *incomparable* (meaning that neither $Q \subseteq Q'$ nor $Q' \subseteq Q$). (This graph can of course also be viewed as the intersection graph of the minimal separators minus their comparability graph.) Nonempty intersection ($\emptyset \neq Q \cap Q'$) is thus strengthened to overlap ($\emptyset \neq Q \cap Q' \neq Q, Q'$). Overlap graphs date back at least to [9], but separator overlap graphs have not been specifically considered before now.

While much work has been done with intersection graphs (and their maximum spanning trees) of various families of subgraphs of a graph, see [13], similar questions concerning overlap graphs of families of subgraphs of a graph have not been studied. This partially reflects that, in the most traditional case—when the family consists of the maxcliques of a graph—the overlap graph is precisely the clique graph (since maxcliques are always incomparable). But Theorem 12 will show that the overlap graph Σ of the family of weak separators of a chordal graph G is of interest in that the edges of Σ correspond precisely to the minimal weak separators of G. This can be viewed as an independent approach to minimal weak separators that uses overlap graphs (of minimal separators) instead of intersection graphs (of maxcliques).

Figure 3 shows two examples of (weighted) separator overlap graphs. For instance, the graph G shown in Figure 2 has minimal separators {0,2,5}, {1, 2, 5}, {2, 5, 6}, {2, 6, 7}, and {6}. The weighted separator overlap graph Σ_2^w shown in Figure 3 has edges corresponding to the three minimal weak separators of G_2: {2, 5} (three times), {5, 6} (once), and {5} (twice).

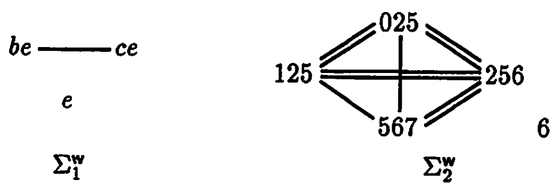


Figure 3: The weighted separator overlap graphs Σ_i^w of the graphs G_i in Figures 1 and 2 (with multiple overlap indicated by multiple edges).

Theorem 12 *Every edge of the separator overlap graph Σ of a connected chordal graph G corresponds to a minimal weak separator of G and every minimal weak separator of G corresponds to an edge of Σ.* Specifically, $\{Q \cap Q' : QQ' \in E(\Sigma)\}$ is the set of minimal weak separators of G.

Proof. First suppose that R and R' are two incomparable minimal separators of a chordal graph G with separator graph Σ and that $RR' \in E(\Sigma)$. Using Lemma 3, say $R = Q_1 \cap Q_2$ and $R' = Q_1' \cap Q_2'$ where Q_1Q_2 and $Q_1'Q_2'$ are edges of an arbitrary clique tree T of G and where, without loss of generality, the nodes Q_1, Q_1', Q_2, and Q_2' come in that order along $T[Q_1, Q_2']$ (possibly with $Q_1' = Q_2$). Because T is a clique tree, $Q_1 \cap Q_2' = R \cap R'$, and so $R \cap R'$ corresponds to the chord Q_1Q_2' of T. Since T was arbitrary, $R \cap R'$ corresponds to a chord of every clique tree of G, and so, by Theorem 5, $R \cap R'$ is a minimal weak separator of G.

Conversely, suppose S is a minimal weak separator of a chordal graph G with clique thicket Θ and a clique tree T. Theorem 5 implies that $S = Q_1 \cap Q_2$ where Q_1Q_2 is a chord of Θ, and so that S is properly contained in $Q \cap Q'$ for every edge QQ' along $T[Q_1, Q_2]$. Let Q_1^+ and Q_2^- be the neighbors of, respectively, Q_1 and Q_2 along $T[Q_1, Q_2]$ (possibly with $Q_1^+ = Q_2^-$). Because Q_1Q_2 is a chord of the clique tree T, sets $Q_1 \cap Q_1^+$ and $Q_2^- \cap Q_2$ contain $Q_1 \cap Q_2$, and Lemma 4 implies that those containments are proper containments. This makes $Q_1 \cap Q_1^+$ and $Q_2^- \cap Q_2$ incomparable ($Q_1 \cap Q_1^+ \subseteq Q_2^- \cap Q_2$ would imply $Q_1 \cap Q_1^+ = S$). Thus, $(Q_1 \cap Q_1^+) \cap (Q_2^- \cap Q_2) = S$ corresponds to an edge of Σ. □

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List-colouring the square of an outerplanar graph

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Abstract

It is proved that if G is a $K_{2,3}$-minor-free graph with maximum degree Δ, then $\Delta + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2) \leqslant \Delta + 2$ if $\Delta \geqslant 3$, and $\mathrm{ch}(G^2) = \chi(G^2) = \Delta + 1$ if $\Delta \geqslant 6$. All inequalities here are sharp, even for outerplanar graphs.

Keywords: Choosability; Outerplanar graph; Minor-free graph; List square colouring

1 Introduction

We use standard terminology, as defined in the references: for example [5] or [9]. The *square* G^2 of a graph G has the same vertex-set as G, and two vertices are adjacent in G^2 if they are within distance two of each other in G.

There is great interest in discovering classes of graphs G for which the choosability or list chromatic number $\mathrm{ch}(G)$ is equal to the chromatic number $\chi(G)$. The *list-square-colouring conjecture* (*LSCC*) [5] is that, for every graph G, $\mathrm{ch}(G^2) = \chi(G^2)$. It is clear that this conjecture holds when the maximum degree $\Delta(G)$ of G is 0 or 1, and it can be deduced from the results of [7] when $\Delta(G) = 2$: see [4]. In general, it is easy to see that $\Delta(G) + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2)$.

It is well known that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. Squares of K_4-minor-free graphs were considered in [4]. For $K_{2,3}$-minor-free graphs we have the following result, which is the same as for the slightly smaller class of outerplanar graphs.

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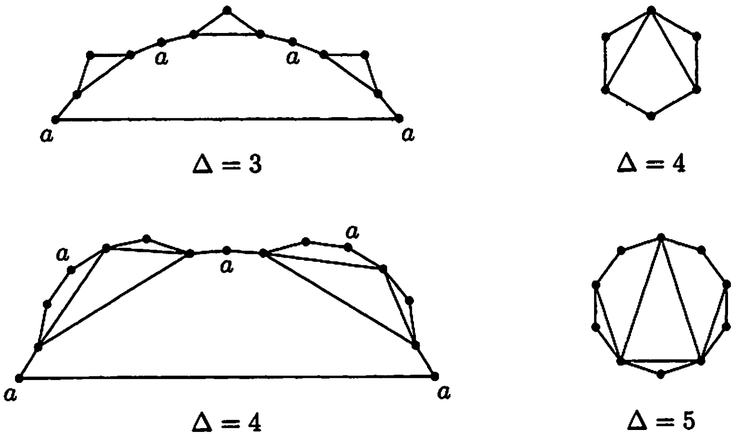

Fig. 1

Theorem 1. *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then* $\Delta + 1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2) \leqslant \Delta + 2$ *if* $\Delta \geqslant 3$, *and* $\mathrm{ch}(G^2) = \chi(G^2) = \Delta + 1$ *if* $\Delta \geqslant 6$.

We are indebted to the referee for telling us about reference [6], which led us indirectly to [1]. These papers contain alternative proofs of parts of Theorem 1 when G is outerplanar: [6] proves most of the results for $\chi(G^2)$, and [1] proves all of the results for $\chi(G^2)$ and also ('as a bonus') that $\mathrm{ch}(G^2) = \Delta + 1$ if $\Delta \geqslant 7$. Both of these papers were motivated by the conjecture of Wegner [8] that if G is a planar graph with maximum degree Δ then $\chi(G^2) \leqslant \Delta + 5$ if $4 \leqslant \Delta \leqslant 7$ and $\chi(G^2) \leqslant 3\Delta/2 + 1$ if $\Delta \geqslant 8$. Our motivation, the LSCC, is somewhat different.

When $3 \leqslant \Delta \leqslant 5$, the upper bound on $\mathrm{ch}(G^2)$ in Theorem 1 is sharp even for $\chi(G^2)$, and even for the smaller class of outerplanar graphs, as shown by the graphs in Fig. 1. For each of the cases $\Delta = 3$ and $\Delta = 4$ there is an infinite family of minimal (under subgraph-inclusion) extremal examples. One member of each family is shown in Fig. 1; in each case, if only $\Delta + 1$ colours are available, then all the vertices labelled a have to have the same colour, which gives a contradiction on the bottom edge. Fig. 1 also shows the smallest extremal example with $\Delta = 4$ and a smallest known extremal example with $\Delta = 5$; in fact, for $\Delta = 5$ we know of only two minimal extremal examples, both of order 10.

For the case $\Delta = 6$, the proof that $\mathrm{ch}(G^2) = \Delta + 1$ in Theorem 1 is exceptionally long and involved, and so we omit it from this paper, instead proving only that $\mathrm{ch}(G^2) \leqslant \Delta + 2 = 8$; the proof that $\mathrm{ch}(G^2) = \Delta + 1 = 7$ is included in the first author's doctoral thesis [3]. Since $\Delta(G)+1 \leqslant \chi(G^2) \leqslant \mathrm{ch}(G^2)$ for every graph G, in order to prove this weaker version of Theorem 1 it suffices to prove the following.

Theorem 2. *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then $\mathrm{ch}(G^2) \leqslant \Delta+2$ if $\Delta \geqslant 3$, and $\mathrm{ch}(G^2) \leqslant \Delta+1$ if $\Delta \geqslant 7$.*

The rest of this paper is devoted to a proof of Theorem 2. We will need the following simple lemma.

Lemma 1. *Let G be a $K_{2,3}$-minor-free graph. Then each block of G is either K_4-minor-free (and hence outerplanar) or else isomorphic to K_4.*

Proof. Suppose B is a block of G that has a K_4 minor. Since $\Delta(K_4) = 3$, it follows that B has a subgraph H homeomorphic to K_4. Since any graph obtained by subdividing an edge of K_4, or by adding a path joining two vertices of K_4, has a $K_{2,3}$ minor, it follows that $H \cong K_4$ and $B = H$. □

As usual, $d(v) = d_G(v)$ will denote the degree of vertex v in graph G.

2 The start of the proof

Fix the value of $\Delta \geqslant 3$, and suppose if possible that G is a $K_{2,3}$-minor-free graph with maximum degree *at most* Δ and with as few vertices as possible such that $\mathrm{ch}(G^2) > \Delta+2$ or $\Delta+1$ as appropriate. By Lemma 1, every block of G is outerplanar or isomorphic to K_4. Clearly G is connected and is not K_2. If G is 2-connected, let $B := G$ and let z_0 be an arbitrary vertex of G; otherwise, let B be an endblock of G with cutvertex z_0. Assume that every vertex v of G is given a list $L(v)$ of $\Delta+2$ or $\Delta+1$ colours, as appropriate, such that G^2 has no proper colouring from these lists.

Claim 2.1. *Not every vertex of $B - z_0$ is adjacent to z_0.*

Proof. Suppose it is. Then every vertex of $B - z_0$ has degree at most Δ in G^2, since all its neighbours in G^2 are in the closed neighbourhood of z_0 in G. Thus we can colour $(G-(B-z_0))^2$ from its lists by the minimality of G, and then colour all the remaining vertices. This contradiction proves Claim 2.1. □

Claim 2.2. *G does not contain three vertices u, v, w of degree 2 such that $uv, vw \in E(G)$.*

Proof. Suppose it does. Then $d_{G^2}(v) \leqslant 4$. Let $H := G - v + uw$, so that H is $K_{2,3}$-minor-free and $G^2 - v \subseteq H^2$. Then we can colour G^2 from its lists by first colouring $G^2 - v$ and then colouring v. This is the required contradiction. □

It follows from Claim 2.1 that $B \not\cong K_2$, K_3 or K_4; thus B is an outerplanar graph that is 2-connected but not complete, and consists of a cycle C with chords. (A *chord* is an edge that joins two nonconsecutive vertices of the cycle.) Claim 2.2 shows that C has at least one chord.

Assume that B is embedded in the plane with C bounding the outside face. In [2], a *cap* is defined to be a region R of the plane that is bounded by a segment of C and one chord u_1u_2. We modify the definition slightly here by insisting also that z_0 is not in the interior of this segment; so z_0 is either u_1 or u_2 or is not in R. We call u_1 and u_2 the *endvertices* of R. By an abuse of terminology, the subgraph of B induced by all vertices in R will also be referred to as a *cap*. We will refer to an edge of C as a *trivial cap* or a *0-cap*. For $i \geqslant 1$, an *i-cap* is a cap that properly contains an $(i-1)$-cap and is minimal with this property.

The proof now divides into two cases.

3 Proof that $\mathrm{ch}(G^2) \leqslant \Delta + 2$

In this section we assume that every vertex v of G has a list $L(v)$ of $\Delta + 2$ colours, and G^2 is not colourable from these lists, but if H is any $K_{2,3}$-minor-free graph with maximum degree at most Δ and fewer vertices than G then $\mathrm{ch}(H^2) \leqslant \Delta + 2$.

Claim 3.1. *Every* 1*-cap in* B *is a triangle* xu_1u_2 *where* $d_G(x) = 2$ *and* $d_G(u_i) \geqslant 4$ $(i = 1, 2)$.

Proof. By definition, a 1-cap is a region bounded by a chord u_1u_2 and a segment $u_1x_1 \ldots x_ru_2$ of C, where $d_G(x_i) = 2$ for each i. By Claim 2.2, $r \leqslant 2$. So if Claim 3.1 is false then either $r = 2$, or $r = 1$ and $d_G(u_j) \leqslant 3$ for some $j \in \{1, 2\}$. But in either case $G^2 - x_1 = (G - x_1)^2$ and $d_{G^2}(x_1) \leqslant \Delta + 1$, and so we can colour G^2 from its lists by first colouring $(G - x_1)^2$ (by the minimality of G) and then colouring x_1. This contradiction proves Claim 3.1. □

Claim 3.2. *B has a cap that is not a* 1*-cap.*

Proof. Suppose that every cap in B is a 1-cap. Then z_0 is not the endvertex of a chord, since a chord z_0y bounds two caps, and if both of these caps are 1-caps then $d_G(y) = 3$, contrary to Claim 3.1. Also, at most two chords of C are incident with any one vertex, since if there were three (or more) chords incident with the same vertex then the middle one (or more) of these chords would bound a cap that is not a 1-cap. It follows from this and Claim 3.1 that the endvertices of every chord have degree exactly 4 in G. The chords therefore form a cycle inside C, every edge of

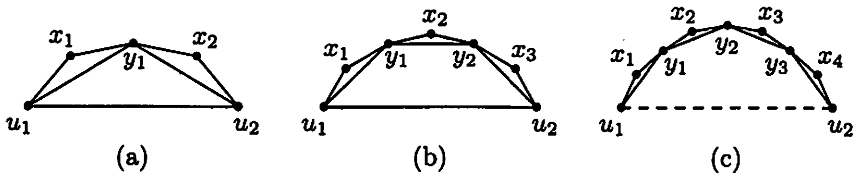

Fig. 2

which joins vertices that are distance 2 apart around C, except possibly for the edge e of the cycle that bounds a face of B with z_0 in its boundary. Now, a cap cannot contain z_0 by definition, except as an endvertex of its chord, which we have already shown to be impossible. Thus there is a unique cap bounded by e, and this cap contains all the 1-caps in B and so is not a 1-cap itself. This contradiction proves Claim 3.2. □

Claim 3.3. *Every 2-cap in B looks like one of the caps in the sequence of which Figs* 2(a) *and* 2(b) *are the first two members.*

Proof. Let R be a 2-cap that is bounded by a chord u_1u_2 and a segment of C. Since R properly contains a 1-cap and is minimal with this property, there is at least one chord inside R, and *every* such chord cuts off a 1-cap. So the chords inside R can be enumerated as $l_1r_1, \ldots, l_kr_k$, where the vertices

$$u_1, l_1, r_1, \ldots, l_k, r_k, u_2$$

occur in that order round C, but possibly $u_1 = l_1$, or $r_i = l_{i+1}$ for some i, or $r_k = u_2$. In fact, since $d(l_i) \geqslant 4$ and $d(r_i) \geqslant 4$ for each i by Claim 3.1, necessarily $u_1 = l_1$, and $r_i = l_{i+1}$ for *every* i, and $r_k = u_2$, since otherwise $d(l_i) = 3$ or $d(r_i) = 3$ for some i. Since, by Claim 3.1 again, every chord l_ir_i cuts off a triangle from R, the proof of Claim 3.3 is complete. □

It follows from Claims 3.2 and 3.3 that $\Delta \geqslant 4$ and B contains one of the configurations H shown in Fig. 2, where the dashed edge may or may not be present in Fig. 2(c), and z_0 is either u_1 or u_2 or is not in H.

Suppose first that B contains H as in Fig. 2(a). Then we can colour $(G - x_1)^2$ from its lists by the minimality of G, and then colour x_1, since $d_{G^2}(x_1) = d_G(u_1) + 1 \leqslant \Delta + 1$. This is the required contradiction.

Suppose next that B contains H as in Fig. 2(b). Colour the graph $(G - \{x_1, x_2, x_3, y_1, y_2\})^2$ from its lists, and for each uncoloured vertex w let $L'(w)$ denote the 'residual list' of colours in $L(w)$ that are not used on any G^2-neighbour of w and so are still available for use on w. Then $|L'(w)| \geqslant (\Delta + 2) - (\Delta - 1) = 3$ if $w \in \{x_1, y_1, y_2, x_3\}$, and $|L'(x_2)| \geqslant (\Delta + 2) - 2 \geqslant 4$. So if we try to colour the vertices in the order

$$x_1, y_1, y_2, x_3, x_2, \tag{1}$$

(a) (b)

Fig. 3

it is only at x_2 that we may fail. If $L'(x_1) \cap L'(x_3) \neq \emptyset$, give x_1 and x_3 the same colour; then y_1, y_2 and x_2 can be coloured in the same order as in (1). If however $L'(x_1) \cap L'(x_3) = \emptyset$, then either $|L'(x_2)| \geqslant 6$, or else x_1, say, has a usable colour c_1 not in $L'(x_2)$; in either case, the vertices can be coloured in the order (1), with x_1 receiving colour c_1 if it exists.

Suppose finally that B contains H as in Fig. 2(c). Colour the graph $(G-(V(H)\backslash\{u_1,u_2\}))^2$ from its lists, and let each uncoloured vertex w have residual list $L'(w)$. Then $|L'(w)| \geqslant 3$ if $w \in \{x_1, y_1, y_3, x_4\}$, $|L'(y_2)| \geqslant 4$, and $|L'(w)| \geqslant 5$ if $w \in \{x_2, x_3\}$. So if we try to colour the vertices in the order

$$y_1, x_4, y_3, y_2, x_1, x_2, x_3, \tag{2}$$

it is only at x_3 that we may fail. If $L'(y_1) \cap L'(x_4) \neq \emptyset$, give y_1 and x_4 the same colour, then colour the remaining vertices in the order (2). If however $L'(y_1) \cap L'(x_4) = \emptyset$, then either $|L'(x_3)| \geqslant 6$, or else y_1 or x_4 has a usable colour c_1 not in $L'(x_3)$; in either case, the vertices can be coloured in the order (2), with y_1 or x_4 receiving colour c_1 if it exists.

In every case we have obtained a contradiction, and so we have proved that $\mathrm{ch}(G^2) \leqslant \Delta + 2$ for all $\Delta \geqslant 3$.

4 Proof that $\mathrm{ch}(G^2) \leqslant \Delta + 1$ when $\Delta \geqslant 7$

In this section we assume that every vertex v of G has a list $L(v)$ of $\Delta + 1$ colours, and G^2 is not colourable from these lists, but if H is any $K_{2,3}$-minor-free graph with maximum degree at most Δ and fewer vertices than G then $\mathrm{ch}(H^2) \leqslant \Delta + 1$. To begin with we assume only that $\Delta \geqslant 6$; we will not use the fact that $\Delta \geqslant 7$ until Claim 4.4.

Claim 4.1. *Every vertex of degree* 2 *in* G *has degree at least* $\Delta + 1$ *in* G^2.

Proof. Let v be a vertex of degree 2 in G with neighbours u, w, and suppose that $d_{G^2}(v) \leqslant \Delta$. Let $H := G - v$ if $uw \in E(G)$ and let $H := G - v + uw$ otherwise. Then H is $K_{2,3}$-minor-free and $G^2 - v \subseteq H^2$ and $\mathrm{ch}(G^2 - v) \leqslant \mathrm{ch}(H^2) \leqslant \Delta + 1$ by the minimality of G. So we can colour G^2 from its lists by first colouring $G^2 - v$ and then colouring v. This is the required contradiction. □

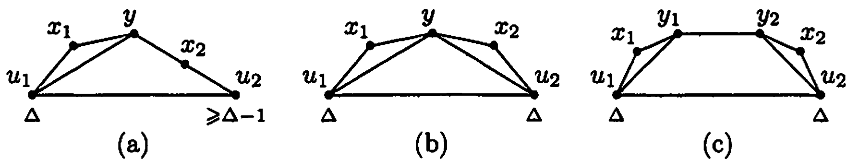

Fig. 4

Claim 4.2. *Every 1-cap in B has the form shown in Fig.* 3(a) *or* 3(b), *where $d_G(u_1)+d_G(u_2)\geqslant\Delta+3$ in Fig.* 3(a), *and $d_G(u_1)=d_G(u_2)=\Delta$ in Fig.* 3(b).

Proof. The first part of the statement follows immediately from Claim 2.2 and the definition of a 1-cap. To prove the second part, note that, by Claim 4.1,

$$\Delta+1\leqslant d_{G^2}(x)\leqslant d_G(u_1)+d_G(u_2)-2$$

in Fig. 3(a), and

$$\Delta+1\leqslant d_{G^2}(x_i)=d_G(u_i)+1 \qquad (i=1,2)$$

in Fig. 3(b). □

Claim 4.3. *Every 2-cap in B has one of the forms shown in Fig.* 4, *where the degrees of u_1 and u_2 are restricted as specified.*

Proof. Let R be a 2-cap that is bounded by a chord u_1u_2 and a segment of C. As in the proof of Claim 3.3, the chords inside R can be enumerated as $l_1r_1,\dots,l_kr_k$, where the vertices

$$u_1,l_1,r_1,\dots,l_k,r_k,u_2$$

occur in that order round C, but possibly $u_1=l_1$, or $r_i=l_{i+1}$ for some i, or $r_k=u_2$. Thus every vertex of R other than u_1 and u_2 has degree at most 4 in G. It follows from the degree conditions in Claim 4.2 that R contains no 1-cap of the type in Fig. 3(b), and also, since $\Delta+3\geqslant 9>2\cdot 4$, any 1-cap in R of the type in Fig. 3(a) must share an endvertex with R. Thus $k=1$ or 2.

If $k=1$ then R is as in Fig. 4(a) (or its reflection). Note that if there were no vertex x_2, just a single edge yu_2, then we would have $d_{G^2}(x_1)=d_G(u_1)\leqslant\Delta$, and if there were a further vertex x_3 subdividing the edge x_2u_2 then we would have $d_{G^2}(x_2)=5<\Delta$, contradicting Claim 4.1 in each case. The degree conditions in Fig. 4(a) also follow from Claim 4.1, because $d_{G^2}(x_1)=d_G(u_1)+1$ and $d_{G^2}(x_2)=d_G(u_2)+2$.

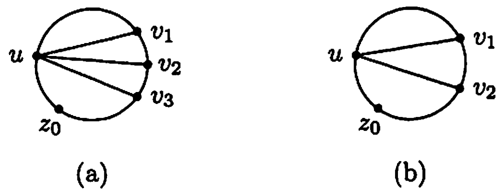

Fig. 5

So suppose $k = 2$. Then R is as in Fig. 4(b) or 4(c). Note that if there were a further vertex w subdividing the edge y_1y_2 in Fig. 4(c) then we would have $d_{G^2}(w) = 6 \leqslant \Delta$, contrary to Claim 4.1. The degree conditions in the figures again follow from Claim 4.1, because $d_{G^2}(x_i) = d_G(u_i) + 1$ $(i = 1, 2)$ in each case. □

From now on, we will assume that $\Delta \geqslant 7$.

Claim 4.4. *Every nontrivial cap in B is a 1-cap or a 2-cap.*

Proof. Suppose this is not true. Then B contains a 3-cap. Let R be a 3-cap in B, with endvertices u_1, u_2. The chords inside R divide R into faces. Let f be the face with u_1u_2 in its boundary. There are three possible types for every other edge of f: it may be an edge of C, or a chord cutting off a 1-cap, or a chord cutting off a 2-cap. There must be at least one edge of f that is a chord cutting off a 2-cap, since otherwise R would itself be a 1-cap or a 2-cap. So let u, v, w be three consecutive vertices in the boundary of f, where uv is a chord cutting off a 2-cap. Then $d_G(v) \leqslant 6$, since the cap cut off by uv, and the cap (possibly a 0-cap) cut off by vw, each contribute at most 3 to the degree of v. Since $\Delta \geqslant 7$, and in view of the degrees indicated in Fig. 4, the only possibility is that uv cuts off a 2-cap of the type in Fig. 4(a), and $d_G(v) = \Delta - 1$. But then this cap contributes only 2 to the degree of v, so that $d_G(v) \leqslant 5 < \Delta - 1$. This contradiction completes the proof of Claim 4.4. □

Claim 4.5. *$\Delta(B) \leqslant 6$, and if u is a vertex of B that is adjacent to z_0 then $d_G(u) \leqslant 5$.*

Proof. Suppose that $u \in V(B)$ and $d_B(u) \geqslant 7$, or $uz_0 \in E(B)$ and $d_B(u) = 6$. Then there are chords uv_1, uv_2 and uv_3 as shown in Fig. 5(a), where z_0 lies in the closed segment of C between u and v_3 that does not contain v_1 and v_2 ('closed' meaning that possibly $z_0 = u$ or $z_0 = v_3$). The chord uv_1 cuts off a cap R_1 which, by Claim 4.4, is a 1-cap or a 2-cap. The chord uv_2 cuts off a cap R_2 that properly contains R_1 and so must be a 2-cap. The chord uv_3 cuts off a cap that properly contains R_2 and so is neither a 1-cap nor a 2-cap. This contradicts Claim 4.4. □

Claim 4.6. *Every nontrivial cap in B is a 1-cap.*

Proof. Suppose this is not true. Then, by Claim 4.4, B contains a 2-cap. Suppose there is a 2-cap in B with endvertices u_1, u_2, where w.l.o.g. $u_2 \neq z_0$. Then $d_G(u_2) = d_B(u_2) \leqslant 6$ by Claim 4.5, while $d_G(u_2) \geqslant \Delta - 1 \geqslant 6$ by the degree constraints in Fig. 4. The only possibility is that $d_G(u_2) = \Delta - 1 = 6$. This is impossible if $u_1 = z_0$, since then Claim 4.5 implies that $d_G(u_2) \leqslant 5$. So $u_1 \neq z_0$. But then the same argument as for u_2 shows that $d_G(u_1) = \Delta - 1$, which is impossible since every 2-cap in Fig. 4 has at least one endvertex with degree Δ. □

Claim 4.7. $\Delta(B) \leqslant 4$.

Proof. Suppose that $u \in V(B)$ and $d_B(u) \geqslant 5$. Then there are chords uv_1 and uv_2 as shown in Fig. 5(b), where z_0 lies in the closed segment of C between u and v_2 that does not contain v_1. The chord uv_1 cuts off a cap R_1 which, by Claim 4.6, is a 1-cap. The chord uv_2 cuts off a cap that properly contains R_1 and so is not a 1-cap. This contradicts Claim 4.6. □

It is now easy to finish the proof. It follows from Claims 4.6 and 4.7 and the degree conditions in Claim 4.2 that every nontrivial cap in B is a 1-cap of the type in Fig. 3(a) with z_0 as one endvertex. But then B consists of a quadrilateral z_0xyzz_0 with one chord z_0y, and this contradicts Claim 2.1. This finally completes the proof of Theorem 2.

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