Two Families of Lattices

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Abstract

Let X denote a set with q elements. Suppose $\mathcal{L}(n,q)$ denote the set X^n (resp. $X^n \cup \{\Delta\}$) whenever $q = 2$ (resp. $q \geq 3$). For any two elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{L}(n,q)$, define $\alpha \leq \beta$ if and only if $\beta = \Delta$ or $\alpha_i = \beta_i$ whenever $\alpha_i \neq 0$ for $1 \leq i \leq n$. Then $\mathcal{L}(n,q)$ is a lattice, denoted by $\mathcal{L}_O(n,q)$. Reversing above partial order, we obtain the dual of $\mathcal{L}_O(n,q)$, denoted by $\mathcal{L}_R(n,q)$. This paper discusses their geometricity, and computes their characteristic polynomials, determine their full automorphism groups. Moreover, we construct a family of quasi-strongly regular graphs from the lattice $\mathcal{L}_O(n,q)$.

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1 Introduction

We recall some terminology and definitions about finite posets and lattices. For more theory about finite posets and lattices, we would like to refer readers to [1].

Let P denote a finite set. A *partial order* on P is a binary relation $\leq$ on P such that

(i) $\alpha \leq \alpha$ for any $\alpha \in P$.

(ii) $\alpha \leq \beta$ and $\beta \leq \alpha$ implies $\alpha = \beta$.

(iii) $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$.

By a *partial ordered set* (or *poset* for short), we mean a pair $(P, \leq)$ where P is a finite set and $\leq$ is a partial order on P. As usual, we write $\alpha < \beta$ whenever $\alpha \leq \beta$ and $\alpha \neq \beta$. By abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P be a poset and let R be a commutative ring with the identical element. A binary function $\mu(\alpha, \beta)$ on P with values in R is said to be the *Möbius function* of P if

$$\mu(\alpha,\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \not\leq \beta, \\ -\sum\limits_{\alpha\leq\gamma<\beta} \mu(\alpha,\gamma), & \text{if } \alpha < \beta. \end{cases}$$

For any two elements $\alpha, \beta \in P$, we say α *covers* β, denoted by $\beta <\cdot\, \alpha$, if $\beta < \alpha$ and there exists no $\gamma \in P$ such that $\beta < \gamma < \alpha$. If P has the minimum (resp. maximum) element, then we denote it by $\perp$ (resp. $\top$) and say that P is a poset with $\perp$ (resp. $\top$). Let P be a finite poset with $\perp$.

By a *rank function* on P, we mean a function r from P to the set of all the nonnegative integers such that

(i) $r(\perp) = \perp$.

(ii) $r(\alpha) = r(\beta) + 1$ whenever $\beta <\cdot\, \alpha$.

Let P be a finite poset with $\perp$ and $\top$. The polynomial

$$\chi(P, x) = \sum_{\alpha \in P} \mu(\perp, \alpha) x^{r(\top) - r(\alpha)}$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset P is said to be a *lattice* if both $\alpha \vee \beta := \sup\{\alpha, \beta\}$ and $\alpha \wedge \beta := \inf\{\alpha, \beta\}$ exist for any two elements $\alpha, \beta \in P$. Let P be a finite lattice with $\perp$. By an *atom* in P, we mean an element in P covering $\perp$. We say P is *atomic* if any element in $P \backslash \{\perp\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(\alpha \wedge \beta) + r(\alpha \vee \beta) \leq r(\alpha) + r(\beta), \forall\, \alpha,\ \beta \in P.$$

Let P be a lattice. A bijective map f from P to P is an *automorphism* of P if f is *join-preserving* and *meet-preserving*, that is, for all $\alpha, \beta \in P$,

$$f(\alpha \vee \beta) = f(\alpha) \vee f(\beta) \text{ and } f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta).$$

All the automorphisms of P form a group, called the *full automorphism group* of P, denoted by $\mathrm{Aut}(P)$.

Let $X = \{0, 1, \ldots, q-1\}$. Suppose $\mathcal{L}(n, q)$ denote the set X^n (resp. $X^n \cup \{\Delta\}$) whenever $q = 2$ (resp. $q \geq 3$). For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in X^n$, the *weight* of α is the number of its non-zero entries, denoted by δ_α. Note that the number of elements in X^n with weight m is $\binom{n}{m}(q-1)^m$.

For any two elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{L}(n, q)$, define $\alpha \leq \beta$ if and only if $\beta = \Delta$ or $\alpha_i = \beta_i$ whenever $\alpha_i \neq 0$ for $1 \leq i \leq n$. Then $\mathcal{L}(n, q)$ is a lattice, denoted by $\mathcal{L}_O(n, q)$. For any two elements $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathcal{L}_R(n, q)$, define $\alpha \leq \beta$ if and only if $\alpha = \Delta$ or $\alpha_i = \beta_i$ whenever $\beta_i \neq 0$ for $1 \leq i \leq n$. Then $\mathcal{L}(n,q)$ is a lattice, denoted by $\mathcal{L}_R(n,q)$.

In a series of papers ([5, 6, 7, 8, 9, 11, 12]), Y. Huo, Y. Liu and Z. Wan et al. constructed lattices from orbits of subspaces under finite classical groups, computed their characteristic polynomials and discussed their geometricity. Very recently, lattices associated with distance-regular graphs have been constructed in [3, 13]. In this paper, we discuss the geometricity of the above two families of lattices, and compute their characteristic polynomials, determine their full automorphism groups are determined. Moreover, we construct a family of quasi-strongly regular graphs from the lattice $\mathcal{L}_O(n,q)$.

2 The lattice $\mathcal{L}_O(n,q)$

Since the set of all the atoms of $\mathcal{L}_O(n,q)$ consists of all the elements with weight 1, $\mathcal{L}_O(n,q)$ is a finite atomic lattice. In this case, $\top = \{\Delta\}$ and $\bot = \{(0,\ldots,0)\}$.

Theorem 2.1 *$\mathcal{L}_O(n,q)$ is a geometric lattice if and only if $n=1$ or $q=2$.*

Proof. In the case $q=2$, for any $\alpha \in \mathcal{L}_O(n,q)$, define $r(\alpha) = \delta_\alpha$. In the case $q \geq 3$, for any $\alpha \in \mathcal{L}_O(n,q)$, define

$$r(\alpha) = \begin{cases} \delta_\alpha, & \text{if}\, \alpha \neq \top, \\ n+1, & \text{if}\, \alpha = \top. \end{cases}$$

Then r is the rank function of $\mathcal{L}_O(n,q)$.

If $n=1$ or $q=2$, it is routine to check that $\mathcal{L}_O(n,q)$ is geometric. Now suppose $n \geq 2$ and $q \geq 3$. Pick $\alpha = (1,0,\ldots,0), \beta = (2,0,\ldots,0)$. Since

$$r(\alpha \vee \beta) + r(\alpha \wedge \beta) = n+1 \geq 3 > 2 = r(\alpha) + r(\beta),$$

$\mathcal{L}_O(n,q)$ is not geometric. □

In order to compute the characteristic polynomial of $\mathcal{L}_O(n,q)$, we need the following lemma.

Lemma 2.2 *The Möbius function of $\mathcal{L}_O(n,q)$ is*

$$\mu(\alpha,\beta)=\begin{cases}1, & \text{if } \alpha=\beta=\top,\\ (-1)^{\delta_\beta-\delta_\alpha}, & \text{if } \alpha\le\beta\ne\top,\\ -(2-q)^n, & \text{if } \bot=\alpha<\beta=\top,\\ -(2-q)^{n-\delta_\alpha}, & \text{if } \bot\ne\alpha<\beta=\top,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The Möbius function of $\mathcal{L}_O(n,q)$ is

$$\mu(\alpha,\beta)=\begin{cases}1, & \text{if } \alpha=\beta=\top,\\ (-1)^{\delta_\beta-\delta_\alpha}, & \text{if } \alpha\le\beta\ne\top,\\ -\sum_{\alpha\le v<\beta}\mu(\alpha,v), & \text{if } \alpha<\beta=\top,\\ 0, & \text{otherwise.}\end{cases}$$

Since

$$\sum_{\bot\le v<\top}(-1)^{\delta_v}=(2-q)^n,$$

and

$$\sum_{\bot\ne\alpha\le v<\top}\mu(\alpha,v)=\sum_{i=0}^{n-\delta_\alpha}(-1)^i\binom{n-\delta_\alpha}{i}(q-1)^i=(2-q)^{n-\delta_\alpha},$$

the desired result follows. □

Theorem 2.3 *The characteristic polynomial of $\mathcal{L}_O(n,q)$ is*

$$\chi(\mathcal{L}_O(n,q),x)=\sum_{i=0}^{n}\binom{n}{i}(q-1)^i(-1)^i x^{n+1-i}-(2-q)^n.$$

Proof. By Lemma 2.2, we obtain

$$\begin{aligned}&\chi(\mathcal{L}_O(n,q),x)\\ =&\sum_{\bot\le\beta\le\top}\mu(\bot,\beta)x^{r(\top)-r(\beta)}\\ =&\sum_{\bot\le\beta<\top}(-1)^{\delta_\beta}x^{n+1-\delta_\beta}+\mu(\bot,\top)\\ =&\sum_{i=0}^{n}(-1)^i\binom{n}{i}(q-1)^i x^{n+1-i}-(2-q)^n,\end{aligned}$$

as desired. □

3 The lattice $\mathcal{L}_R(n,q)$

Since the set of all the atoms of $\mathcal{L}_R(n,q)$ consists of all the elements with weight n, $\mathcal{L}_R(n,q)$ is a finite atomic lattice. In this case, $\top = \{(0,\dots,0)\}$ and $\bot = \{\Delta\}$.

Theorem 3.1 *$\mathcal{L}_R(n,q)$ is a geometric lattice if and only if $n=1$ or $q=2$.*

Proof. In the case $q=2$, for any $\alpha \in \mathcal{L}_O(n,q)$, define $r(\alpha) = n - \delta_\alpha$. In the case $q \geq 3$, For any $\alpha \in \mathcal{L}_R(n,q)$, define

$$r(\alpha) = \begin{cases} n+1-\delta_\alpha, & \text{if } \alpha \neq \bot, \\ 0, & \text{if } \alpha = \bot. \end{cases}$$

Then r is the rank function of $\mathcal{L}_R(n,q)$.

If $n=1$ or $q=2$, $\mathcal{L}_R(n,q)$ is geometric. Now suppose $n \geq 2$ and $q \geq 3$. Pick $\alpha = (1,1,\dots,1), \beta = (2,2,\dots,2)$. Since

$$r(\alpha \vee \beta) + r(\alpha \wedge \beta) = n+1 \geq 3 > 2 = r(\alpha) + r(\beta),$$

$\mathcal{L}_R(n,q)$ is not geometric. □

In order to compute the characteristic polynomial of $\mathcal{L}_R(n,q)$, we need the following lemma.

Lemma 3.2 *The Möbius function of $\mathcal{L}_R(n,q)$ is*

$$\mu(\alpha,\beta) = \begin{cases} 1, & \text{if } \alpha = \beta = \bot, \\ (-1)^{\delta_\alpha - \delta_\beta}, & \text{if } \bot \neq \alpha \leq \beta, \\ -(2-q)^n, & \text{if } \bot = \alpha < \beta = \top, \\ -(2-q)^{n-\delta_\beta}, & \text{if } \bot = \alpha < \beta \neq \top, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The Möbius function of $\mathcal{L}_R(n,q)$ is

$$\mu(\alpha,\beta) = \begin{cases} 1, & \text{if } \alpha = \beta = \bot, \\ (-1)^{\delta_\alpha - \delta_\beta}, & \text{if } \bot \neq \alpha \leq \beta, \\ -\sum\limits_{\bot < v \leq \beta} (-1)^{\delta_v - \delta_\beta}, & \text{if } \bot = \alpha < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\sum_{\perp < v \leq \top} (-1)^{\delta_\gamma} = (2-q)^n,$$

and

$$\sum_{\perp < v \leq \beta \neq \top} (-1)^{\delta_v - \delta_\beta} = \sum_{i=0}^{n-\delta_\beta} (-1)^i \binom{n-\delta_\beta}{i} (q-1)^i = (2-q)^{n-\delta_\beta},$$

the desired result follows. □

Theorem 3.3 *The characteristic polynomial of $\mathcal{L}_R(n,q)$ is*

$$\chi(\mathcal{L}_R(n,q),x) = x^{n+1} - \sum_{i=0}^{n} \binom{n}{i}(q-1)^i(2-q)^{n-i}x^i.$$

Proof. By Lemma 3.2, we obtain

$$\begin{aligned}
& \chi(\mathcal{L}_R(n,q),x) \\
= & \sum_{\perp \leq \beta \leq \top} \mu(\perp,\beta)x^{r(\top)-r(\beta)} \\
= & \sum_{\perp < \beta < \top} (-(2-q)^{n-\delta_\beta})x^{\delta_\beta} + \mu(\perp,\perp)x^{n+1} + \mu(\perp,\top) \\
= & x^{n+1} - \sum_{i=0}^{n} \binom{n}{i}(q-1)^i(2-q)^{n-i}x^i,
\end{aligned}$$

as desired. □

4 The full automorphism group

Let S_q be the symmetric group on the set $X = \{0,1,\ldots,q-1\}$. The stabilizer of 0 is isomorphic to S_{q-1}. Let S_n be the symmetric group on $\{1,2,\ldots,n\}$. Let $S_{q-1} \wr S_n$ denote the wreath product of S_{q-1} and S_n. Then $S_{q-1} \wr S_n$ acts on $\mathcal{L}_O(n,q)$ as the following:

$$\begin{aligned}
(\alpha_1,\alpha_2,\ldots,\alpha_n)^{(\rho_1,\rho_2,\ldots,\rho_n;\theta)} &= ((\alpha_{1^{\theta^{-1}}})^{\rho_1},(\alpha_{2^{\theta^{-1}}})^{\rho_2},\ldots,(\alpha_{n^{\theta^{-1}}})^{\rho_n}), \\
\Delta^{S_{q-1}\wr S_n} &= \Delta.
\end{aligned}$$

Theorem 4.1 $\mathrm{Aut}(\mathcal{L}_O(n,q)) = S_{q-1} \wr S_n$.

Proof. It is routine to check that $S_{q-1} \wr S_n \leq \mathrm{Aut}(\mathcal{L}_O(n,q))$.

Conversely, suppose f is any automorphism of $L_O(n,q))$. Then, for any element α of $L_O(n,q))$, we obatin $\delta_\alpha = \delta_{\alpha^f}$. It follows that $f \in S_{q-1} \wr S_n$; therefore, $\mathrm{Aut}(\mathcal{L}_O(n,q)) \leq S_{q-1} \wr S_n$. □

5 A family of quasi-strongly regular graphs

In this section we shall construct a family of quasi-strongly regular graphs from the lattice $\mathcal{L}_O(n, q)$. We first recall some concepts.

Let $\Gamma = (X, R)$ be a connected regular graph. For any two vertices u, v at distance i, define

$$c_i(u,v) = |\Gamma_{i-1}(u) \cap \Gamma(v)|, b_i(u,v) = |\Gamma_{i+1}(u) \cap \Gamma(v)|.$$

A connected regular graph of diameter d is said to be *distance-regular* if $c_i(u,v)$ and $b_i(u,v)$ depend only on i. For more information, the reader may consult [2].

As a generalization of distance-regular graphs, F. Goldberg [4] introduced the concept of quasi-strongly regular graphs.

Definition 5.1 ([4]) A *quasi-strongly regular graph* with parameters $(n, k, a; c_1, \ldots, c_p)$ is a k-regular graph on n vertices such that any two adjacent vertices have a common neighbours and any two non-adjacent vertices have c_i common neighbours for some $1 \leq i \leq p$.

Let Γ be a graph with the vertex set X^n such that two vertices α and β are adjacent if and only if $\alpha < \cdot\beta$ or $\beta < \cdot\alpha$ in $\mathcal{L}_O(2i+1, 2)$. Then Γ is a Hamming graph, which is distance-regular.

For $1 \leq i \leq n-1$, suppose $L_i = \{\alpha \in \mathcal{L}_O(2i+1, 2) | \delta_\alpha = i\}$ and $L_{i+1} = \{\beta \in \mathcal{L}_O(2i+1, 2) | \delta_\beta = i+1\}$. Let Δ_i be a graph with the vertex set $L_i \cup L_{i+1}$ such that two vertices α and β are adjacent if and only if $\alpha < \beta$ or $\beta < \alpha$. Then Δ_i is a doubled Odd graph, which is distance-regular.

Let Γ_1 be a graph with the vertex set L_1 such that two vertices α and β are adjacent if and only if the distance between α and β in Δ_1 is 2. Then Γ_1 is a strongly-regular graph.

For $2 \leq i \leq n-1$. Let Γ_i be a graph with the vertex set L_i such that two vertices α and β are adjacent if and only if the distance between α and

β in Δ_i is 2. Then Γ is a quasi-strongly regular graph with parameters $(\binom{n}{i}(q-1)^i, i(n-i)(q-1), (n-i-1)(q-1); (n-i)(q-1), 4, 1, 0)$.

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