Extremal Zeroth-Order General Randić Index Of Thorn Graphs

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Abstract

Let $G = (V, E)$ be a simple connected graph, d_v is the degree of vertex v. The zeroth-order Randić index of G is defined as $R^0_\alpha(G) = \sum_{v\in V} {d_v}^\alpha$, where α is an arbitrary real number. Let G^* be the thorn graph of G by attaching $d_G(v_i)$ new pendent edges to each vertex $v_i (1 \leq i \leq n)$ of G. In this paper, we investigate the zeroth-order general Randić index of a class thorn tree and determine the extremal zeroth-order general Randić index of the thorn graphs $G^*(n, m)$.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with the vertex set V and the edge set E. For any $v \in V$, $N(v)$ denotes the neighbors of v, and $d_v = |N(v)|$ is the degree of v. The distance between u and v is the smallest length of any $u - v$ path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if the graph G under consideration is clear. Hence if $d(u, v) = k$, then there exists a $u - v$ path

$$P : u = v_0, v_1, \cdots, v_k = v$$

of length k in G, but no $u - v$ path of smaller length exists in G. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by *diam(G)* or denote it simply as D.

The Randić index (or connectivity index) of G was introduced by Randić in 1975 and defined as [1]

$$R(G) = \sum_{uv\in E} (d_u d_v)^{-\frac{1}{2}}.$$

Randić demonstrated that his index is well correlated with a variety of physic-chemical properties of various classes of organic compounds.

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Recently, Li and Zheng in [2] defined the zeroth-order general Randić index of a graph G as

$$R^0_\alpha(G) = \sum_{v \in V(G)} d_v^\alpha$$

for any real number α. Li and Zhao in [3] characterized trees with the first three largest and smallest zeroth-order general Randić index. Wang, Hua and Deng in [4,5] characterized the unicycle graphs with the maximum zeroth-order general Randić index. Y. Hu et al. [6] investigated the zeroth-order general Randić index for molecular (n, m)-graphs. Chen and Deng in [7] characterized the (n,n+1)-graphs with extremal zeroth-order general Randić index for any real number α.

For a graph G with vertices $v_1, \cdots, v_n$, the thorn graph $G^* = G^*(p_1, \cdots, p_n)$ of G is obtained by attaching p_i thorns (pendant edges) to each vertex v_i for $i = 1, \cdots, n$. If G is a tree, then G^* is called a thorn tree. In Refs. [8-9], formulae are reported for the wiener index of several classes of thorn graphs. In Refs. [10], formulae are reported for the Schultz index of several classes of thorn graphs. In this report, we let $p_i = d_G(v_i)$. Let $G(n, m)$ be the graphs with n vertices and m edges, $G^*(n, m)$ be the thorn graphs of $G(n, m)$ obtained from above operation. $\Delta(G)$ be the maximum degree of G.

In this paper, we'll investigate the zeroth-order general Randić index $R^0_\alpha(G)$ of a class thorn tree, and determine the extremal zeroth-order general Randić index the thorn trees for $\alpha > 1$ or $\alpha < 0$ and $0 < \alpha < 1$. Characterize zeroth-order general Randić index of $G^*(n, m)$.

2 Preliminary

Firstly, we need to introduce degree sequence of the graph G.

Denote by $D(G) = [d_1, d_2, \cdots, d_n]$ the degree sequence of the graph G, where d_i stands the degree of the i-th vertex of G, and $d_1 \geq d_2 \geq \cdots \geq d_n$. Furthermore, $D(G) = [d_1^{a_1}, d_2^{a_2}, \cdots, d_t^{a_t}]$ means that G has a_i vertices of degree d_i, where $i = 1, 2, \cdots, t$.

Definition Let $(c_1, c_2, \cdots, c_k)$ be a partition of n, the *starlike tree* is constructed in the following way:

(1) Let $S_1, S_2, \cdots, S_k$ be the stars with edge number $c_1 - 1, c_2 - 1, \cdots, c_k - 1$ respectively, and $v_1, v_2, \cdots, v_k$ be their center vertices;

(2) Add a vertex v_0, which join the center vertices $v_1, v_2, \cdots, v_k$ of $S_1, S_2, \cdots, S_k$ respectively.

Then, we can get a tree T with diameter not more than 4. The degree of $v_1, v_2, \cdots, v_k$ are $c_1, c_2, \cdots, c_k$, resp. $|V(T)| = n + 1$, $|E(T)| = k + (c_1 - 1) + (c_2 - 1) + \cdots + (c_k - 1) = c_1 + c_2 + \cdots + c_k = n$. We denote it as $S(c_1, c_2, \cdots, c_k)$ is shown in Figure 1.

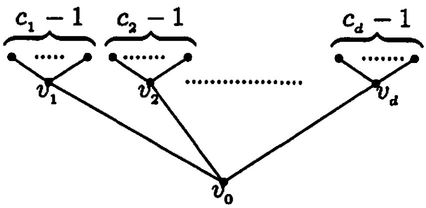

Figure 1. $S(c_1, c_2, \cdots, c_d)$

3 Extremal zeroth-order general Randić index of starlike thorn trees

In [11], the authors investigated zeroth-order general Randić index of starlike trees, in this section, we'll get the zeroth-order general Randić index of starlike thorn trees and their respond extremal graphs.

Lemma 3.1.[11] Let $S(c_1, c_2, \cdots, c_d)$ be graph depicted above. Then

$$R_\alpha^0(c_1, c_2, \cdots, c_d) = n - d + d^\alpha + \sum_{i=1}^{d} c_i^\alpha$$

Let $\mathcal{A}(n, d) = \{S(c_1, c_2, \cdots, c_d) | c_1 + c_2 + \cdots + c_d = n\}$.

Lemma 3.2.[11] Let $G \in \mathcal{A}(n, d)$, then

(i) $S(n-d, 1, 1, \cdots, 1)$; $S(n-d-1, 2, 1, 1, \cdots, 1), 2 \leq d \leq n-3$; $S(n-d-2, 3, 1, 1, \cdots, 1), 2 \leq d \leq n-5$ or $S(2, 2, 2, 1, \cdots, 1), d = n-4$; $S(k+1, k+1, \cdots, k+1, k, k, \cdots, k), k = \lfloor \frac{n}{d} \rfloor$ has the smallest, the second smallest, the third smallest and the largest zeroth-order general Randić index for $\alpha > 1$ or $\alpha < 0$, respectively;

(ii) $S(n-d, 1, 1, \cdots, 1)$; $S(n-d-1, 2, 1, 1, \cdots, 1), 2 \leq d \leq n-3$; $S(n-d-2, 3, 1, 1, \cdots, 1), 2 \leq d \leq n-5$ or $S(2, 2, 2, 1, \cdots, 1), d = n-4$; $S(k+1, k+1, \cdots, k+1, k, k, \cdots, k), k = \lfloor \frac{n}{d} \rfloor$ has the largest, the second largest, the third largest and the smallest zeroth-order general Randić index for $0 < \alpha < 1$, respectively.

Next we shall get the zeroth-order general Randić index of thorn trees of the starlike trees and their respond extremal graphs.

Let $S^*(c_1, c_2, \cdots, c_d)$ be the graph obtained from $S(c_1, c_2, \cdots, c_d)$ by adding each vertex the number of thorns equal to the degree of the vertex in graph $S(c_1, c_2, \cdots, c_d)$. It is suffice to see that the number of vertices of $S^*(c_1, c_2, \cdots, c_d$ is $3(n(S(c_1, c_2, \cdots, c_d))) - 2 = 3n + 1$, where n is the vertices number of $S(c_1, c_2, \cdots, c_d)$ and the graph $S^*(c_1, c_2, \cdots, c_d)$ is shown in Figure 2.

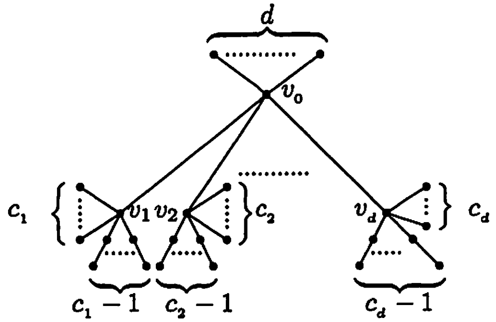

Figure 2. $S^*(c_1, c_2, \cdots, c_d)$

From above description, we have

Theorem 3.3. Let $3c_i - 1 \le 3n + 1 - (3c_i - 1)$. Then

$$R^0_\alpha(S^*(c_1, c_2, \cdots, c_d)) = 2n + 2^\alpha(n - d + d^\alpha) + 2^\alpha \sum_{i=1}^{d} c_i^\alpha$$

Proof. By the definition of zeroth-order general Randić index, we have

$$\begin{aligned} & R^0_\alpha(S^*(c_1, c_2, \cdots, c_d)) \\ = \;& 2n + 2^\alpha[(c_1 - 1) + (c_2 - 1) + \cdots + (c_d - 1)] + \sum_{i=1}^{d}(2c_i)^\alpha + (2d)^\alpha \\ = \;& 2n + 2^\alpha(n - d + d^\alpha) + 2^\alpha \sum_{i=1}^{d} c_i^\alpha \end{aligned}$$

Combine Lemma 3.1 with Theorem 3.3, we shall get

Theorem 3.4. Let $S(c_1, c_2, \cdots, c_d)$, $S^*(c_1, c_2, \cdots, c_d)$ be the graphs depicted above. Then

$$R^0_\alpha(S^*(c_1, c_2, \cdots, c_d)) = 2^\alpha R^0_\alpha(S(c_1, c_2, \cdots, c_d)) + 2n$$

Note that, from Theorem 3.4, it suffice to see that, $R^0_\alpha(S^*(c_1, c_2, \cdots, c_d))$ and $R^0_\alpha(S(c_1, c_2, \cdots, c_d))$ have the same extremal value characteristic.

Next, we shall discuss the extremal zeroth-order general Randić index of $R^0_\alpha(S^*(c_1, c_2, \cdots, c_d))$.

Let $\mathcal{A}^*(n, d) = \{S^*(c_1, c_2, \cdots, c_d) | c_1 + c_2 + \cdots + c_d = n\}$ be the thorn graphs of $\mathcal{A}(n, d)$.

From Lemma 3.2 and Theorem 3.4, we have

Theorem 3.5. Let $G \in \mathcal{A}^*(n, d)$, then

(i) $S^*(n - d + 1, 1, 1, \cdots, 1)$; $S^*(n - d, 2, 1, 1, \cdots, 1), 2 \le d \le n - 2$; $S^*(n - d - 1, 3, 1, 1, \cdots, 1), 2 \le d \le n - 4$ or $S^*(2, 2, 2, 1, \cdots, 1), d = n - 3$; $S^*(k + 1, k + 1, \cdots, k + 1, k, k, \cdots, k), k = \lfloor \frac{n}{d} \rfloor$ has the smallest, the second smallest, the third smallest and the largest zeroth-order general Randić index, respectively;

(ii) $S^*(n-d+1,1,1,\cdots,1)$; $S^*(n-d,2,1,1,\cdots,1), 2 \le d \le n-2$; $S^*(n-d-1,3,1,1,\cdots,1), 2 \le d \le n-4$ or $S^*(2,2,2,1,\cdots,1), d=n-3$; $S^*(k+1,k+1,\cdots,k+1,k,k,\cdots,k), k = \lfloor \frac{n}{d} \rfloor$ has the largest, the second largest, the third largest and the smallest zeroth-order general Randić index, respectively.

4 The zeroth-order general Randić index of thorn graphs

Next, we shall discuss zeroth-order general Randić index of $G^*(n,m)$ and their extremal values.

Let $G(n,m)$ be the graph described above, $\Delta_G = \Delta$, and let $D(G) = [1^{k_1}, 2^{k_2}, \cdots, i^{k_i}, \cdots, \Delta^{k_\Delta}]$ be the degree sequence of $G(n,m)$, and $1 \le i \le \Delta$ may be vacant for some i. Then, we have $\sum_{i=1}^{\Delta} ik_i = 2m$. $G^*(n,m)$ be the thorn graph of $G(n,m)$ by attaching $d_G(v_j)$ pendent edges in each vertex of $v_j (1 \le j \le n)$ in $G(n,m)$.

By the definition of $G^*(n,m)$, the degree sequence of $G^*(n,m)$ should be

$$D(G^*) = [1^{k_1+2k_2+\cdots+ik_i+\cdots+\Delta k_\Delta}, 2^{k_1}, 4^{k_2}, \cdots, (2i)^{k_i}, \cdots, (2\Delta)^{k_\Delta}]$$

$1 \le i \le \Delta$ may be vacant for some i.

Theorem 4.1. Let $G(n,m)$, $G^*(n,m)$ be the graphs described above, then

$$R^0_\alpha(G^*(n,m)) = 2^\alpha R^0_\alpha(G(n,m)) + 2m$$

Proof. By the definition of zeroth-order general Randić index, we have

$$\begin{aligned}
& R^0_\alpha(G(n,m)) \\
= & \textstyle\sum_{v\in V(G)} {d_v}^\alpha \\
= & k_1 1^\alpha + k_2 2^\alpha + \cdots + k_i i^\alpha \\
& R^0_\alpha(G^*(n,m)) \\
= & \textstyle\sum_{v\in V(G)} {d_v}^\alpha \\
= & (k_1 + 2k_2 + \cdots + ik_i)1^\alpha + k_1 2^\alpha + \cdots + k_i(2i)^\alpha \\
= & 2m + 2^\alpha(k_1 1^\alpha + k_2 2^\alpha + \cdots + k_i i^\alpha) \\
= & 2m + 2^\alpha R^0_\alpha(G(n,m))
\end{aligned}$$

The proof of theorem is completed.

Remark: From Theorem 4.1, it suffice to see that, if $G(n,m)$ attains the extremal(maximum or minimum) zeroth-order general Randić index, then its thorn graph $G^*(n,m)$ has the same extremal zeroth-order general

Randić index as well. Then we can item the extremal zeroth-order general Randić index of the thorn graphs of trees, unicyclic graphs, bicyclic graphs, etc., but we omit it here.

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