On Potentially $K_5 - E_3$-graphic Sequences *

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Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of H where H is a subgraph of K_m. In this paper, we characterize the potentially $K_5 - P_3$, $K_5 - A_3$, $K_5 - K_3$ and $K_5 - K_{1,3}$-graphic sequences where A_3 is $P_2 \cup K_2$. Moreover, we also characterize the potentially $K_5 - 2K_2$-graphic sequences where pK_2 is the matching consisted of p edges.

Key words: graph; degree sequence; potentially $K_5 - H$-graphic sequences
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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. The set of all graphic sequence in NS_n is denoted by GS_n. A graphic sequence π is potentially H-graphic if there is a realization of π containing H as a subgraph. Let C_k

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and P_k denote a cycle on k vertices and a path on $k+1$ vertices, respectively. Let $\sigma(\pi)$ the sum of all the terms of π and let A_3 and Z_4 denote $P_2 \cup K_2$ and $K_4 - P_2$, respectively. We use the symbol E_3 to denote graphs on 5 vertices and 3 edges. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph. Let $G - H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. Gould, Jacobson and Lehel [4] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n-term positive graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph. They proved that $\sigma(pK_2, n) = (p-1)(2n-p) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Erdös, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that the equality holds. In the same paper, they proved the conjecture is true for $k = 3$ and $n \geq 6$. The cases $k = 4$ and 5 were proved separately (see [4] and [17], and [18]). For $k \geq 6$ and $n \geq \binom{k}{2}+3$, Li, Song and Luo [19] proved the conjecture true via linear algebraic techniques. Recently, Ferrara, Gould and Schmitt proved the conjecture [5] and they also determined in [6] $\sigma(F_k, n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex. Yin, Li, and Mao [25] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$ and $r + 1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$, and Yin and Li [24] further determined $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$. Moreover, Yin and Li in [24] also gave two sufficient conditions for a sequence $\pi \epsilon GS_n$ to be potentially $K_{r+1} - e$-graphic. Yin [27] determined $\sigma(K_{r+1} - K_3, n)$ for $r \geq 3$ and $n \geq 3r + 5$. Lai [12-15] determined $\sigma(K_4 - e, n)$ for $n \geq 4$ and $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$, $\sigma(K_5 - P_4, n)$, $\sigma(K_5 - K_3, n)$ for $n \geq 5$. Lai [10-11] proved that $\sigma(C_{2m+1}, n) = m(2n - m - 1) + 2$, for $m \geq 2, n \geq 3m$; $\sigma(C_{2m+2}, n) = m(2n - m - 1) + 4$, for $m \geq 2, n \geq 5m - 2$. Lai and Hu [16] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r + 10$, $r \geq 3$, $r + 1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not contain a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r + 8$, $r \geq 3$.

A harder question is to characterize the potentially H-graphic sequences without zero terms. Luo [21] characterized the potentially C_k-graphic sequences for each $k = 3, 4, 5$. Recently, Luo and Warner [22] characterized the potentially K_4-graphic sequences. Eschen and Niu [23] characterized the potentially $K_4 - e$-graphic sequences. Yin and Chen [26] characterized the potentially $K_{r,s}$-graphic sequences for $r = 2, s = 3$ and $r = 2, s = 4$. Chen [2] characterized the potentially $K_5 - 2K_2$-graphic sequences for $5 \leq n \leq 8$. Hu and Lai [7-8] characterized the potentially $K_5 - C_4$ and $K_5 - Z_4$-graphic sequences.

In this paper, we completely characterize the potentially $K_5 - E_3$ -graphic sequences, that is potentially $K_5 - P_3$, $K_5 - A_3$, $K_5 - K_3$ and $K_5 - K_{1,3}$-graphic sequences. Moreover, we also characterize the potentially $K_5 - 2K_2$-graphic sequences.

2 Preparations

Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n, 1 \leq k \leq n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{if } d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \cdots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is a rearrangement of the $n - 1$ terms of π''_k. Then π'_k is called the residual sequence obtained by laying off d_k from π. In this paper, we denote π'_n by π'.

For a nonincreasing positive integer sequence $\pi = (d_1, d_2, \cdots, d_n)$, we write $m(\pi)$ and $h(\pi)$ to denote the largest positive terms of π and the smallest positive terms of π, respectively. We need the following results.

Theorem 2.1 [4] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [20] If $\pi = (d_1, d_2, \cdots, d_n)$ is a sequence of nonnegative integers with $1 \leq m(\pi) \leq 2$, $h(\pi) = 1$ and even $\sigma(\pi)$, then π is graphic.

Theorem 2.3 [21] Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence. Then π is potentially C_4-graphic if and only if the following conditions hold: (1) $d_4 \geq 2$; (2) $d_1 = n - 1$ implies $d_2 \geq 3$; (3) If $n = 5, 6$, then $\pi \neq (2^n)$.

Lemma 2.4 [2] Let $\pi = (d_1, d_2, \cdots, d_n) \epsilon NS_n$, $1 \leq j \leq n-5$, $0 \leq k \leq [\frac{n-j-i-4}{2}]$. Let

$$\pi = \begin{cases} (n-i, n-j, 3^{n-i-j-2k}, 2^{2k}, 1^{i+j-2}) \\ n-i-j \text{ is even;} \\ (n-i, n-j, 3^{n-i-j-2k-1}, 2^{2k+1}, 1^{i+j-2}) \\ n-i-j \text{ is odd.} \end{cases}$$

Let S_1 be the set consisting of the above sequences and let S_2 be the set of the following sequences: $(n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$. If $\pi \epsilon S_1$ or $\pi \epsilon S_2$, then π is not potentially $K_{1,2,2}$-graphic.

Lemma 2.5 [8] If $\pi = (d_1, d_2, \cdots, d_n)$ is a nonincreasing sequence of positive integers with even $\sigma(\pi)$, $n \geq 4$, $d_1 \leq 3$ and $\pi \neq (3^3, 1), (3^2, 1^2)$, then π is graphic.

Lemma 2.6 (Kleitman and Wang [9]) π is graphic if and only if π' is graphic.

The following corollary is obvious.

Corollary 2.7 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

3 Main Theorems

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - P_3$-graphic if and only if the following conditions hold:

(1) $d_1 \geq 4$, $d_3 \geq 3$ and $d_5 \geq 2$.

(2) $\pi \neq (4, 3^2, 2^3)$, $(4, 3^2, 2^4)$ and $(4, 3^6)$.

Proof: Assume that π is potentially $K_5 - P_3$-graphic. (1) and (2) are obvious. To prove the sufficiency, we use induction on n. Suppose the graphic sequence π satisfies the conditions (1) and (2). We first prove the base case where $n = 5$. In this case, π is one of the following: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(4, 3^2, 2^2)$. It is easy to check that all of these are potentially $K_5 - P_3$-graphic. Now we assume that the sufficiency holds for $n - 1 (n \geq 6)$, we will show that π is potentially $K_5 - P_3$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. Clearly, π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - P_3$-graphic, and hence so is π.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-3} \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - P_3$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_1 = 3$, then $\pi' = (3^k, 2^{n-1-k})$ where $n - 3 \leq k \leq n - 1$. Since $\sigma(\pi')$ is even, k must be even. If $k = n - 3$, then $\pi = (4, 3^{n-1})$ where n is odd. Since $\pi \neq (4, 3^6)$, we have $n \geq 9$. By Lemma 2.5, $\pi_1 = (3^{n-5})$ is graphic. Let G_1 be a realization of π_1, then $K_{1,2,2} \cup G_1$ is a realization of $\pi = (4, 3^{n-1})$. Thus, $\pi = (4, 3^{n-1})$ is potentially $K_5 - P_3$-graphic since $K_5 - P_3 \subseteq K_{1,2,2}$. If $k = n - 2$, then $\pi = (4^2, 3^{n-2})$ where n is even. It is easy to see that $\pi = (4^2, 3^4)$ and $\pi = (4^2, 3^6)$ are potentially $K_5 - P_3$-graphic. Let G_2 be a realization of $(4^2, 3^4)$, which contains $K_5 - P_3$. If $n \geq 10$, then $\pi_2 = (3^{n-6})$ is graphic by Lemma 2.5. Let G_3 be a realization of π_2, then $G_2 \cup G_3$ is a realization of $\pi = (4^2, 3^{n-2})$. In other words, $\pi = (4^2, 3^{n-2})$ is potentially $K_5 - P_3$-graphic. If $k = n - 1$, then $\pi = (4^3, 3^{n-3})$ where n is odd. It is easy to see that $\pi = (4^3, 3^4)$ is potentially $K_5 - P_3$-graphic. If $n \geq 9$, then $K_5 - e \cup G_1$ is a realization of $\pi = (4^3, 3^{n-3})$. Thus, $\pi = (4^3, 3^{n-3})$ is potentially $K_5 - P_3$-graphic since $K_5 - P_3 \subseteq K_5 - e$.

If π' does not satisfy (2), then π' is just $(4, 3^6)$, and hence $\pi = (5, 4^2, 3^5)$ or $(4^4, 3^4)$. It is easy to see that these sequences are potentially $K_5 - P_3$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_2 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - P_3$-graphic, and hence so is π.

If π' does not satisfy (1), there are two subcases:

Subcase 1: $d'_1 \geq 4$ and $d'_3 = 2$. Then $\pi = (d_1, 3^2, 2^{n-3})$ where $d_1 \geq 5$. Since $\sigma(\pi)$ is even, d_1 must be even. We will show that π is potentially $K_5 - P_3$-graphic. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5})$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-4})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Subcase 2: $d'_1 = 3$. Then $d_1 = 4$, $d_3 = 3$, $d_2 = 4$ or $d_2 = 3$.

If $d_2 = 4$, then $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 1$ and $n - 2 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - P_3$-graphic. First, we consider $\pi = (4^2, 3^2, 2^{n-4})$. It is enough to show $\pi_1 = (2^{n-5}, 1^2)$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic. Then we consider $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 4$. It is easy to see that $(4^2, 3^4)$ is potentially $K_5 - P_3$-graphic. Let G_1 be a realization of $(4^2, 3^4)$, which contains $K_5 - P_3$. If $n \geq 10$, then $\pi_2 = (3^{k-4}, 2^{n-2-k})$ is graphic by Lemma 2.5. Let G_2 be a realization of π_2, then $G_1 \cup G_2$ is a realization of $\pi = (4^2, 3^k, 2^{n-2-k})$. If $n \leq 9$, then π is one of the following: $(4^2, 3^4, 2)$, $(4^2, 3^4, 2^2)$, $(4^2, 3^4, 2^3)$, $(4^2, 3^6, 2)$. It is easy to check that all of these are potentially $K_5 - P_3$-graphic. In other words, $\pi = (4^2, 3^k, 2^{n-2-k})$ is potentially $K_5 - P_3$-graphic.

If $d_2 = 3$, then $\pi = (4, 3^k, 2^{n-1-k})$ where $k \geq 2$ and $n - 1 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - P_3$-graphic. First, we consider $\pi = (4, 3^2, 2^{n-3})$. Since $\pi \neq (4, 3^2, 2^3)$ and $(4, 3^2, 2^4)$, we have $n \geq 8$. It is enough to show $\pi_1 = (2^{n-5})$ is graphic. Clearly, C_{n-5} is a realization of π_1. Second, we consider $\pi = (4, 3^4, 2^{n-5})$. It is enough to show $\pi_2 = (2^{n-5}, 1^2)$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic. Then we consider $\pi = (4, 3^k, 2^{n-1-k})$ where $k \geq 6$. If $n \geq 9$, then $\pi_3 = (3^{k-4}, 2^{n-1-k})$ is graphic by Lemma 2.5. Let G_1 be a realization of π_3, then $K_{1,2,2} \cup G_1$ is a realization of $\pi = (4, 3^k, 2^{n-1-k})$. Hence, $\pi = (4, 3^k, 2^{n-1-k})$ is potentially $K_5 - P_3$-graphic since $K_5 - P_3 \subseteq K_{1,2,2}$. If $n \leq 8$, then $\pi = (4, 3^6, 2)$. It is easy to see that π is potentially $K_5 - P_3$-graphic. In other words, $\pi = (4, 3^k, 2^{n-1-k})$ is potentially $K_5 - P_3$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4, 3^2, 2^3)$, $(4, 3^2, 2^4)$, $(4, 3^6)$. Hence π is one of the following: $(5, 4, 3, 2^4)$, $(5, 3^3, 2^3)$, $(4^3, 2^4)$, $(5, 4, 3, 2^5)$, $(5, 3^3, 2^4)$, $(4^3, 2^5)$, $(5, 4, 3^5, 2)$, $(4^3, 3^4, 2)$. It is easy to check that all of these are potentially $K_5 - P_3$-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_3 \geq 3$ and $d'_5 \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - P_3$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_1 = 3$, then $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ where $k \geq 2$, $k + t \geq 4$ and $n - 1 - k - t \geq 1$. Since $\sigma(\pi)$ is even, $n - 1 - t$ must be even. We will show that π is potentially $K_5 - P_3$-graphic. First, we consider $\pi = (4, 3^2, 2^t, 1^{n-3-t})$. It is enough to show $\pi_1 = (2^{t-2}, 1^{n-3-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic. Second, we consider $\pi = (4, 3^3, 2^t, 1^{n-4-t})$. It is enough to show $\pi_2 = (2^{t-1}, 1^{n-3-t})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic. Third, we consider $\pi = (4, 3^4, 2^t, 1^{n-5-t})$. It is enough to show $\pi_3 = (2^t, 1^{n-3-t})$ is graphic. By $\sigma(\pi_3)$ being even and Theorem 2.2, π_3 is graphic. Then we consider $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ where $k \geq 5$. Let $\pi_4 = (3^{k-4}, 2^t, 1^{n-1-k-t})$. If $n \geq 9$ and $\pi_4 \neq (3^3, 1)$ or $(3^2, 1^2)$, then π_4 is graphic by Lemma 2.5. Let G_1 be a realization of π_4, then $K_{1,2,2} \cup G_1$ is a realization of $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$. Hence, $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ is potentially $K_5 - P_3$-graphic since $K_5 - P_3 \subseteq K_{1,2,2}$. If $n = 9$ and $\pi_4 = (3^3, 1)$ or $(3^2, 1^2)$, then $\pi = (4, 3^7, 1)$ or $(4, 3^6, 1^2)$. If $n \leq 8$, then $\pi = (4, 3^5, 1)$ or $(4, 3^5, 2, 1)$. It is easy to check that all of these are potentially $K_5 - P_3$-graphic. In other words, $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ is potentially $K_5 - P_3$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4, 3^2, 2^3)$, $(4, 3^2, 2^4)$, $(4, 3^6)$. Hence π is one of the following: $(5, 3^2, 2^3, 1)$, $(4^2, 3, 2^3, 1)$, $(5, 3^2, 2^4, 1)$, $(4^2, 3, 2^4, 1)$, $(5, 3^6, 1)$, $(4^2, 3^5, 1)$. It is easy to check that all of these are potentially $K_5 - P_3$-graphic.

Theorem 3.2 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - A_3$-graphic if and only if the following conditions hold:

(1) $d_4 \geq 3$ and $d_5 \geq 2$.

(2) $\pi \neq (n-1, 3^3, 2^{n-k}, 1^{k-4})$ where $n \geq 6$ and $k = 4, 5, \cdots, n-2$, n and k have the same parity.

(3) $\pi \neq (3^4, 2^2), (3^6), (3^4, 2^3), (3^6, 2), (4, 3^6), (3^7, 1), (3^8), (n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$.

Proof: First we show the conditions (1)-(3) are necessary conditions for π to be potentially $K_5 - A_3$-graphic. Assume that π is potentially $K_5 - A_3$-graphic. (1) is obvious. If $\pi = (n-1, 3^3, 2^{n-k}, 1^{k-4})$ is potentially $K_5 - A_3$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_5 - A_3$ as a subgraph so that the vertices of $K_5 - A_3$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-4, 2^{n-1-k}, 1^{k-4})$ obtained from $G - (K_5 - A_3)$ must be graphic, which is impossible since $G - (K_5 - A_3)$ has only $n-4$ vertices, $\Delta(G - (K_5 - A_3)) \leq n-5$. Hence, (2) holds. Now it is easy to check that $(3^4, 2^2), (3^6), (3^4, 2^3), (3^6, 2), (4, 3^6), (3^7, 1)$ and (3^8) are not potentially $K_5 - A_3$-graphic. If $\pi = (n-1, 3^5, 1^{n-6})$ is potentially $K_5 - A_3$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_5 - A_3$ as a subgraph so that the vertices of $K_5 - A_3$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-4, 3, 1^{n-5})$ obtained from $G - (K_5 - A_3)$ must be graphic. It follows that the sequence $\pi_1 = (2)$ must be graphic, a contradiction. Hence, $\pi \neq (n-1, 3^5, 1^{n-6})$. If $\pi = (n-1, 3^6, 1^{n-7})$ is potentially $K_5 - A_3$-graphic, then according to Theorem 2.1, there exists a realization G of π containing

$K_5 - A_3$ as a subgraph so that the vertices of $K_5 - A_3$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-4, 3^2, 1^{n-6})$ obtained from $G - (K_5 - A_3)$ must be graphic. It follows that the sequence $\pi_2 = (2^2)$ must be graphic, a contradiction. Hence, $\pi \neq (n-1, 3^6, 1^{n-7})$. In other words, (3) holds.

Now we turn to show the conditions (1)-(3) are sufficient conditions for π to be potentially $K_5 - A_3$-graphic. Suppose the graphic sequence π satisfies the conditions (1)-(3). Our proof is by induction on n. We first prove the base case where $n = 5$. In this case, π is one of the following: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(3^4, 2)$. It is easy to check that all of these are potentially $K_5 - A_3$-graphic. Now suppose that the sufficiency holds for $n - 1 (n \geq 6)$, we will show that π is potentially $K_5 - A_3$-graphic in terms of the following cases:

Case 1: $d_n \geq 3$. Clearly, π' satisfies (1). If π' also satisfies (2) and (3), then by the induction hypothesis, π' is potentially $K_5 - A_3$-graphic, and hence so is π.

If π' does not satisfy (2), then π' is just $(5, 3^3, 2^2)$, and hence $\pi = (6, 3^6)$ which is impossible by (3).

If π' does not satisfy (3), since $\pi \neq (4, 3^6)$ and (3^8), then π' is only one of the following: $(3^6), (3^6, 2), (4, 3^6)$, (3^8), $(5, 3^5)$, $(6, 3^6)$. Hence, π is one of the following: $(4^3, 3^4), (4^2, 3^6), (5, 4^2, 3^5), (4^4, 3^4), (4^3, 3^6)$, $(6, 4^2, 3^4)$, $(7, 4^2, 3^5)$. It is easy to check that all of these are potentially $K_5 - A_3$-graphic.

Case 2: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_2 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_5 - A_3$-graphic, and hence so is π.

If π' does not satisfy (1), then $d'_4 = 2$. Hence $\pi = (d_1, 3^3, 2^{n-4})$. Since $\sigma(\pi)$ is even, d_1 must be odd. We will show that π is potentially $K_5 - A_3$-graphic. If $d_1 = 3$, then $\pi = (3^4, 2^{n-4})$. Since $\pi \neq (3^4, 2^2)$ and $(3^4, 2^3)$, we have $n \geq 8$. It is enough to show $\pi_1 = (2^{n-5})$ is graphic. Clearly, C_{n-5} is a realization of π_1. If $d_1 \geq 5$, since $\pi \neq (n-1, 3^3, 2^{n-4})$, we have $d_1 \leq n-2$. It is enough to show $\pi_2 = (d_1 - 3, 2^{n-5})$ is graphic. It clearly suffices to show $\pi_3 = (2^{n-2-d_1}, 1^{d_1-3})$ is graphic. By $\sigma(\pi_3)$ being even and Theorem 2.2, π_3 is graphic. Thus, $\pi = (d_1, 3^3, 2^{n-4})$ is potentially $K_5 - A_3$-graphic.

If π' does not satisfy (2), i.e., $\pi' = (n-2, 3^3, 2^{n-5})$. Since $\sigma(\pi')$ is even, n must be odd. Hence $\pi = (n-1, 4, 3^2, 2^{n-4})$ or $(n-1, 3^4, 2^{n-5})$. We will show that both of them are potentially $K_5 - A_3$-graphic. It is enough to show $\pi_1 = (n-4, 2^{n-5}, 1)$ is graphic. It clearly suffices to show $\pi_2 = (1^{n-5})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

If π' does not satisfy (3), then π' is one of the following: $(3^4, 2^2)$, (3^6), $(3^4, 2^3)$, $(3^6, 2)$, $(4, 3^6)$, (3^8), $(5, 3^5)$, $(6, 3^6)$. Since $\pi \neq (3^6, 2)$, then π is one of the following: $(4^2, 3^2, 2^3)$, $(4, 3^4, 2^2)$, $(4^2, 3^4, 2)$, $(4^2, 3^2, 2^4)$, $(4, 3^4, 2^3)$, $(3^6, 2^2)$, $(4^2, 3^4, 2^2)$, $(4, 3^6, 2)$, $(5, 4, 3^5, 2)$, $(4^3, 3^4, 2)$, $(4^2, 3^6, 2)$, $(6, 4, 3^4, 2)$, $(7, 4, 3^5, 2)$. It is easy to check that all of these are potentially $K_5 - A_3$-graphic.

Case 3: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_3 \geq 3$ and $d'_5 \geq 2$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_5 - A_3$-graphic, and hence so is π.

If π' does not satisfy (1), then $d'_4 = 2$. Hence $\pi = (3^4, 2^k, 1^{n-4-k})$ where $k \geq 1$ and $n-4-k \geq 1$. Since $\sigma(\pi)$ is even, $n-4-k$ must be even. We will show that π is potentially $K_5 - A_3$-graphic. It is enough to show $\pi_1 = (2^{k-1}, 1^{n-4-k})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

If π' does not satisfy (2), i.e., $\pi' = (n-2, 3^3, 2^{n-1-k}, 1^{k-4})$. Hence $\pi = (n-1, 3^3, 2^{n-1-k}, 1^{k-3})$ which contradicts condition (2).

If π' does not satisfy (3), then by $\pi \neq (n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$, π' is only one of the following: $(3^4, 2^2)$, (3^6), $(3^4, 2^3)$, $(3^6, 2)$, $(4, 3^6)$, $(3^7, 1)$, (3^8). Since $\pi \neq (3^7, 1)$, then π is one of the following: $(4, 3^3, 2^2, 1)$, $(3^5, 2, 1)$, $(4, 3^5, 1)$, $(4, 3^3, 2^3, 1)$, $(3^5, 2^2, 1)$, $(4, 3^5, 2, 1)$, $(5, 3^6, 1)$, $(4^2, 3^5, 1)$, $(4, 3^6, 1^2)$, $(4, 3^7, 1)$. It is easy to check that all of these are potentially $K_5 - A_3$-graphic.

Theorem 3.3 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - K_3$-graphic if and only if the following conditions hold:

(1) $d_2 \geq 4$ and $d_5 \geq 2$.

(2) $\pi \neq (4^2, 2^4)$, $(4^2, 2^5)$, $(4^3, 2^3)$ and (4^6).

Proof: Assume that π is potentially $K_5 - K_3$-graphic. (1) and (2) are obvious. To prove the sufficiency, we use induction on n. Suppose the graphic sequence π satisfies the conditions (1) and (2). We first prove the base case where $n = 5$. In this case, π is one of the following: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4^2, 2^3)$. It is easy to check that all of these are potentially $K_5 - K_3$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 6)$, we will show that π is potentially $K_5 - K_3$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. Clearly, π' satisfies (1). If π' also satisfies (2), then by the induction hypothesis, π' is potentially $K_5 - K_3$-graphic, and hence so is π. If π' does not satisfy (2), then π' is just (4^6), and hence $\pi = (5^4, 4^3)$. It is easy to see that π is potentially $K_5 - K_3$-graphic.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-2} \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_3$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 3$, then $d_2 = 4$ and $3 \leq d_4 \leq d_3 \leq 4$. There are three subcases:

Subcase 1: $d_4 = 4$. Then $\pi = (4^4, 3^{n-4})$. Since $\sigma(\pi)$ is even, n must be even. We will show that π is potentially $K_5 - K_3$-graphic. It is easy to see that $(4^4, 3^2)$ and $(4^4, 3^4)$ are potentially $K_5 - K_3$-graphic. Let G_1 be a realization of $(4^4, 3^2)$, which contains $K_5 - K_3$. If $n \geq 10$, then $\pi_1 = (3^{n-6})$ is graphic by Lemma 2.5. Let G_2 be a realization of π_1, then $G_1 \cup G_2$ is a realization of $\pi = (4^4, 3^{n-4})$. In other words, $\pi = (4^4, 3^{n-4})$ is potentially $K_5 - K_3$-graphic.

Subcase 2: $d_4 = 3$ and $d_3 = 4$. Then $\pi = (d_1, 4^2, 3^{n-3})$. Since $\sigma(\pi)$ is even, d_1 and n have different parities. We will show that π is potentially $K_5 - K_3$-graphic. It is enough to show $\pi_1 = (d_1 - 4, 3^{n-5}, 2, 1^2)$ is graphic and the vertex with degree $d_1 - 4$ is not adjacent to the vertices with degree 2 or 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (3^{n-1-d_1}, 2^{d_1-3}, 1^2)$ is graphic. By Lemma 2.5, π_2 is graphic. Thus, $\pi = (d_1, 4^2, 3^{n-3})$ is potentially $K_5 - K_3$-graphic.

Subcase 3: $d_3 = 3$. then $\pi = (d_1, 4, 3^{n-2})$. Since $\sigma(\pi)$ is even, d_1 and n have the same parity. We will show that π is potentially $K_5 - K_3$-graphic. It is enough to show $\pi_1 = (d_1 - 4, 3^{n-5}, 1^3)$ is graphic and the vertex with degree $d_1 - 4$ is not adjacent to the vertices with degree 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (3^{n-1-d_1}, 2^{d_1-4}, 1^3)$ is graphic. By Lemma 2.5, π_2 is graphic.

If π' does not satisfy (2), then π' is just (4^6), and hence $\pi = (5^3, 4^3, 3)$. It is easy to check that π is potentially $K_5 - K_3$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_2 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_3$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 3$, then $d_2 = 4$. There are two subcases: $d_1 = 4$ and $d_1 \geq 5$.

Subcase 1: $d_1 = 4$.

If $d_3 = 4$, then $\pi = (4^3, 3^k, 2^{n-3-k})$ where $n - 3 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - K_3$-graphic. First, we consider $\pi = (4^3, 2^{n-3})$. Since $\pi \neq (4^3, 2^3)$, we have $n \geq 7$. It is enough to show $\pi_1 = (2^{n-4})$ is graphic. Clearly, C_{n-4} is a realization of π_1. Second, we consider $\pi = (4^3, 3^2, 2^{n-5})$. It is easy to see that $\pi = (4^3, 3^2, 2)$ and $\pi = (4^3, 3^2, 2^2)$ are potentially $K_5 - K_3$-graphic. If $n \geq 8$, then $K_5 - e \cup C_{n-5}$ is a realization of $\pi = (4^3, 3^2, 2^{n-5})$. Thus, $\pi = (4^3, 3^2, 2^{n-5})$ is potentially $K_5 - K_3$-graphic since $K_5 - K_3 \subseteq K_5 - e$. Then we consider $\pi = (4^3, 3^k, 2^{n-3-k})$ where $k \geq 4$. If $n \geq 9$, then $\pi_1 = (3^{k-2}, 2^{n-3-k})$ is graphic by Lemma 2.5. Let G_1 be a realization of π_1, then $K_5 - e \cup G_1$ is a realization of $\pi = (4^3, 3^k, 2^{n-3-k})$. Thus, π is potentially $K_5 - K_3$-graphic since $K_5 - K_3 \subseteq K_5 - e$. If $n \leq 8$, then $\pi = (4^3, 3^4, 2)$. It is easy to see that $(4^3, 3^4, 2)$ is potentially $K_5 - K_3$-graphic. In other words, $\pi = (4^3, 3^k, 2^{n-3-k})$ is potentially $K_5 - K_3$-graphic.

If $d_3 \leq 3$, then $\pi = (4^2, 3^k, 2^{n-2-k})$ where $n - 2 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - K_3$-graphic. First, we consider $\pi = (4^2, 2^{n-2})$. Since $\pi \neq (4^2, 2^4)$ and $(4^2, 2^5)$, we have $n \geq 8$. It is enough to show $\pi_1 = (2^{n-5})$ is graphic. Clearly, C_{n-5} is a realization of π_1. Second, we consider $\pi = (4^2, 3^2, 2^{n-4})$. It is enough to show $\pi_2 = (2^{n-5}, 1^2)$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic. Then we consider $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 4$. It is easy to check that $\pi = (4^2, 3^4)$ is potentially $K_5 - K_3$-graphic. Let G_1 be a realization of $(4^2, 3^4)$, which contains $K_5 - K_3$. If $n \geq 10$, then $\pi_3 = (3^{k-4}, 2^{n-2-k})$ is graphic by Lemma 2.5. Let G_2 be a realization of π_3, then $G_1 \cup G_2$ is a realization of $\pi = (4^2, 3^k, 2^{n-2-k})$. If $n \leq 9$, then π is one of the following: $(4^2, 3^4, 2)$, $(4^2, 3^4, 2^2)$, $(4^2, 3^4, 2^3)$, $(4^2, 3^6, 2)$. It is easy to check that all of these are potentially $K_5 - K_3$-graphic. In other words, $\pi = (4^2, 3^k, 2^{n-2-k})$ is potentially $K_5 - K_3$-graphic.

Subcase 2: $d_1 \geq 5$. Then $\pi = (d_1, 4, 3^k, 2^{n-2-k})$ where $n - 2 - k \geq 1$. Since $\sigma(\pi)$ is even, d_1 and k have the same parity. We will show that π is potentially $K_5 - K_3$-graphic.

First, we consider $\pi = (d_1, 4, 2^{n-2})$. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5})$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-4})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Second, we consider $\pi = (d_1, 4, 3, 2^{n-3})$. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5}, 1)$ is graphic and there exists no edge between two vertices with degree $d_1 - 4$ and 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-3})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Third, we consider $\pi = (d_1, 4, 3^2, 2^{n-4})$. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5}, 1^2)$ is graphic and the vertex with degree $d_1 - 4$ is not adjacent to the vertices with degree 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-2})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Fourth, we consider $\pi = (d_1, 4, 3^3, 2^{n-5})$. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5}, 1^3)$ is graphic and the vertex with degree $d_1 - 4$ is not adjacent to the vertices with degree 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-1})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Then we consider $\pi = (d_1, 4, 3^k, 2^{n-2-k})$ where $k \geq 4$. It is enough to show $\pi_1 = (d_1 - 4, 3^{k-3}, 2^{n-2-k}, 1^3)$ is graphic and the vertex with degree $d_1 - 4$ is not adjacent to the vertices with degree 1 in the realization of π_1. Assume that the vertex with degree $d_1 - 4$ is adjacent to $t (t \leq k-3)$ vertices with degree 3 and $d_1 - 4 - t$ vertices with degree 2 in the realization of π_1. Hence, it suffices to show $\pi_2 = (3^{k-3-t}, 2^{n+2-d_1-k+2t}, 1^{d_1-1-t})$ is graphic. By Lemma 2.5, π_2 is graphic. Thus, $\pi = (d_1, 4, 3^k, 2^{n-2-k})$ is potentially $K_5 - K_3$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4^2, 2^4)$, $(4^2, 2^5)$, $(4^3, 2^3)$, (4^6). Hence π is one of the following: $(5^2, 2^5)$, $(5^2, 2^6)$, $(5^2, 4, 2^4)$, $(5^2, 4^4, 2)$. It is easy to check that all of these are potentially $K_5 - K_3$-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_1 \geq 4$, $d'_2 \geq 3$ and $d'_5 \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_3$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 3$, then $\pi = (4^2, 3^k, 2^t, 1^{n-2-k-t})$ where $k + t \geq 3$ and $n - 2 - k - t \geq 1$. Since $\sigma(\pi)$ is even, $n - 2 - t$ must be even. We will show that π is potentially $K_5 - K_3$-graphic.

First, we consider $\pi = (4^2, 2^t, 1^{n-2-t})$. It is enough to show $\pi_1 = (2^{t-3}, 1^{n-2-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

Second, we consider $\pi = (4^2, 3, 2^t, 1^{n-3-t})$. It is enough to show $\pi_1 = (2^{t-2}, 1^{n-2-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

Third, we consider $\pi = (4^2, 3^2, 2^t, 1^{n-4-t})$. It is enough to show $\pi_1 = (2^{t-1}, 1^{n-2-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

Fourth, we consider $\pi = (4^2, 3^3, 2^t, 1^{n-5-t})$. It is enough to show $\pi_1 = (2^t, 1^{n-2-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

Then we consider $\pi = (4^2, 3^k, 2^t, 1^{n-2-k-t})$ where $k \geq 4$ and $n - 2 - k - t \geq 1$. It is easy to see that $\pi = (4^2, 3^4)$ is potentially $K_5 - K_3$-graphic. Let G_1 be a realization of $(4^2, 3^4)$, which contains $K_5 - K_3$. Let $\pi_1 = (3^{k-4}, 2^t, 1^{n-2-k-t})$. If $n \geq 10$ and $\pi_1 \neq (3^3, 1)$, $(3^2, 1^2)$, then π_1 is graphic by Lemma 2.5. Let G_2 be a realization of π_1, then $G_1 \cup G_2$ is a realization of $\pi = (4^2, 3^k, 2^t, 1^{n-2-k-t})$. If $n = 10$ and $\pi_1 = (3^3, 1)$ or $(3^2, 1^2)$, then $\pi = (4^2, 3^7, 1)$ or $(4^2, 3^6, 1^2)$. If $n \leq 9$, then $\pi = (4^2, 3^4, 1^2)$, $(4^2, 3^4, 2, 1^2)$, $(4^2, 3^5, 1)$ or $(4^2, 3^5, 2, 1)$. It is easy to check that all of these are potentially $K_5 - K_3$-graphic. In other words, $\pi = (4^2, 3^k, 2^t, 1^{n-2-k-t})$ is potentially $K_5 - K_3$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4^2, 2^4)$, $(4^2, 2^5)$, $(4^3, 2^3)$, (4^6). Hence π is one of the following: $(5, 4, 2^4, 1)$, $(5, 4, 2^5, 1)$, $(5, 4^2, 2^3, 1)$, $(5, 4^5, 1)$. It is easy to check that all of these are potentially $K_5 - K_3$-graphic.

Theorem 3.4 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - K_{1,3}$-graphic if and only if the following conditions hold:

(1) $d_1 \geq 4$ and $d_4 \geq 3$.

(2) $\pi \neq (4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^5, 1)$, $(n-1, 3^4, 1^{n-5})$ and $(n-1, 3^5, 1^{n-6})$.

Proof: Assume that π is potentially $K_5 - K_{1,3}$-graphic. (1) is obvious. Now it is easy to check that $(4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^5, 1)$ are not potentially $K_5 - K_{1,3}$-graphic. If $\pi = (n-1, 3^4, 1^{n-5})$ is potentially $K_5 - K_{1,3}$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_5 - K_{1,3}$ as a subgraph so that the vertices of $K_5 - K_{1,3}$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-5, 2, 1^{n-5})$ obtained from $G - (K_5 - K_{1,3})$ must be graphic and there must be no edge between two vertices with degree $n-5$ and 2 in the realization of π^*. Thus, π^* satisfies: $(n-5) + 2 \leq n - 5$, a contradiction. Hence, $\pi \neq (n-1, 3^4, 1^{n-5})$. If $\pi = (n-1, 3^5, 1^{n-6})$ is potentially $K_5 - K_{1,3}$-graphic, then according to Theorem 2.1, there exists a realization G of π

containing $K_5 - K_{1,3}$ as a subgraph so that the vertices of $K_5 - K_{1,3}$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-5, 3, 2, 1^{n-6})$ obtained from $G - (K_5 - K_{1,3})$ must be graphic and there must be no edge between two vertices with degree $n-5$ and 2 in the realization of π^*. It follows that the sequence $\pi_1 = (2^2)$ must be graphic, a contradiction. Hence, $\pi \neq (n-1, 3^5, 1^{n-6})$. In other words, (2) holds.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1) and (2). Our proof is by induction on n. We first prove the base case where $n = 5$. Since $\pi \neq (4, 3^4)$, then π is one of the following: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^3, 1)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 6)$, we will show that π is potentially $K_5 - K_{1,3}$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. Clearly, π' satisfies (1). If π' also satisfies (2), then by the induction hypothesis, π' is potentially $K_5 - K_{1,3}$-graphic, and hence so is π. If π' does not satisfy (2), since $\pi \neq (4^6)$ and (4^7), then π' is just (4^6) or (4^7), and hence $\pi = (5^4, 4^3)$ or $(5^4, 4^4)$. It is easy to check that these sequences are potentially $K_5 - K_{1,3}$-graphic.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-3} \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_{1,3}$-graphic, and hence so is π.

If π' does not satisfy (1), there are two subcases:

Subcase 1: $d'_1 \geq 4$ and $d'_4 = 2$. Then $\pi' = (4, 3^2, 2^2)$, and hence $\pi = (5, 3^5)$ which contradicts condition (2).

Subcase 2: $d'_1 = 3$. Then $\pi' = (3^k, 2^{n-1-k})$ where $n-3 \leq k \leq n-1$. Since $\sigma(\pi')$ is even, k must be even. If n is odd, then $k = n-3$ or $n-1$. If $k = n-3$, then $\pi = (4, 3^{n-1})$. Since $\pi \neq (4, 3^6)$, we have $n \geq 9$. It is easy to check that $(4, 3^8)$ and $(4, 3^{10})$ are potentially $K_5 - K_{1,3}$-graphic. Let G_1 be a realization of $(4, 3^8)$, which contains $K_5 - K_{1,3}$. If $n \geq 13$, then $\pi_1 = (3^{n-9})$ is graphic by Lemma 2.5. Let G_2 be a realization of π_1, then $G_1 \cup G_2$ is a realization of $\pi = (4, 3^{n-1})$. In other words, $\pi = (4, 3^{n-1})$ is potentially $K_5 - K_{1,3}$-graphic. If $k = n-1$, then $\pi = (4^3, 3^{n-3})$. It is easy to see that $\pi = (4^3, 3^4)$ is potentially $K_5 - K_{1,3}$-graphic. If $n \geq 9$, then $\pi_2 = (3^{n-5})$ is graphic by Lemma 2.5. Let G_3 be a realization of π_2, then $K_5 - e \cup G_3$ is a realization of $\pi = (4^3, 3^{n-3})$. Hence, $\pi = (4^3, 3^{n-3})$ is potentially $K_5 - K_{1,3}$-graphic since $K_5 - K_{1,3} \subseteq K_5 - e$. If n is even, then $k = n-2$, thus $\pi = (4^2, 3^{n-2})$. Since $\pi \neq (4^2, 3^4)$, we have $n \geq 8$. It is easy to see that $(4^2, 3^6)$ and $(4^2, 3^8)$ are potentially $K_5 - K_{1,3}$-graphic. Let G_4 be a realization of $(4^2, 3^6)$, which contains $K_5 - K_{1,3}$. If $n \geq 12$, then $\pi_3 = (3^{n-8})$ is graphic by Lemma 2.5. Let G_5 be a realization of π_3, then $G_4 \cup G_5$ is a realization of $\pi = (4^2, 3^{n-2})$. In other words, $\pi = (4^2, 3^{n-2})$ is potentially $K_5 - K_{1,3}$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^4)$, $(5, 3^5)$. Hence π is one of the following: $(5, 4, 3^5)$, $(5^3, 4^3, 3)$, $(5^2, 4, 3^4)$, $(5, 4^3, 3^3)$, $(4^5, 3^2)$, $(5, 4^2, 3^5)$, $(4^4, 3^4)$, $(5^3, 4^4, 3)$, $(5, 4^2, 3^3)$, $(4^4, 3^2)$, $(6, 4^2, 3^4)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_3 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_{1,3}$-graphic, and hence so is π.

If π' does not satisfy (1), there are two subcases:

Subcase 1: $d'_1 \geq 4$ and $d'_4 = 2$. Then $\pi = (d_1, 3^3, 2^{n-4})$ where $d_1 \geq 5$. Since $\sigma(\pi)$ is even, d_1 must be odd. We will show that π is potentially $K_5 - K_{1,3}$-graphic. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5}, 1)$ is graphic and there exists no edge between two vertices with degree $d_1 - 4$ and 1 in the realization of π_1. Hence, it suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-3})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Subcase 2: $d'_1 = 3$. Then $d_1 = 4$, $d_3 = d_4 = 3$, $d_2 = 4$ or $d_2 = 3$.

If $d_2 = 4$, then $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 2$ and $n - 2 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - K_{1,3}$-graphic. First, we consider $\pi = (4^2, 3^2, 2^{n-4})$. It is enough to show $\pi_1 = (2^{n-5}, 1^2)$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic. Second, we consider $\pi = (4^2, 3^4, 2^{n-6})$. It is easy to see that $(4^2, 3^4, 2)$, $(4^2, 3^4, 2^2)$ and $(4^2, 3^4, 2^3)$ are potentially $K_5 - K_{1,3}$-graphic. Let G_1 be a realization of $(4^2, 3^4, 2)$, which contains $K_5 - K_{1,3}$. If $n \geq 10$, then $G_1 \cup C_{n-7}$ is a realization of $\pi = (4^2, 3^4, 2^{n-6})$. In other words, $\pi = (4^2, 3^4, 2^{n-6})$ is potentially $K_5 - K_{1,3}$-graphic. Then we consider $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 6$. It is easy to see that $\pi = (4^2, 3^6)$ is potentially $K_5 - K_{1,3}$-graphic. Let G_2 be a realization of $(4^2, 3^6)$, which contains $K_5 - K_{1,3}$. If $n \geq 12$, then $\pi_2 = (3^{k-6}, 2^{n-2-k})$ is graphic by Lemma 2.5. Let G_3 be a realization of π_2, then $G_2 \cup G_3$ is a realization of $\pi = (4^2, 3^k, 2^{n-2-k})$. If $n \leq 11$, then π is one of the following: $(4^2, 3^6, 2)$, $(4^2, 3^6, 2^2)$, $(4^2, 3^6, 2^3)$, $(4^2, 3^8, 2)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic. In other words, $\pi = (4^2, 3^k, 2^{n-2-k})$ is potentially $K_5 - K_{1,3}$-graphic.

If $d_2 = 3$, then $\pi = (4, 3^k, 2^{n-1-k})$ where $k \geq 3$ and $n-1-k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - K_{1,3}$-graphic. First, we consider $\pi = (4, 3^4, 2^{n-5})$. Since $\pi \neq (4, 3^4, 2)$, we have $n \geq 7$. It is enough to show $\pi_1 = (2^{n-4})$ is graphic. Clearly, C_{n-4} is a realization of π_1. Second, we consider $\pi = (4, 3^6, 2^{n-7})$. It is easy to see that $(4, 3^6, 2)$, $(4, 3^6, 2^2)$ and $(4, 3^6, 2^3)$ are potentially $K_5 - K_{1,3}$-graphic. Let G_1 be a realization of $(4, 3^6, 2)$, which contains $K_5 - K_{1,3}$. If $n \geq 11$, then $G_1 \cup C_{n-8}$ is a realization of $\pi = (4, 3^6, 2^{n-7})$. In other words, $\pi = (4, 3^6, 2^{n-7})$ is potentially $K_5 - K_{1,3}$-graphic. Then we consider $\pi = (4, 3^k, 2^{n-1-k})$ where $k \geq 8$. It is easy to see that $\pi = (4, 3^8)$ is potentially $K_5 - K_{1,3}$-graphic. Let G_2 be a realization of $(4, 3^8)$, which contains $K_5 - K_{1,3}$. If $n \geq 13$, then $\pi_2 = (3^{k-8}, 2^{n-1-k})$ is graphic by Lemma 2.5. Let G_3 be a realization of π_2, then $G_2 \cup G_3$ is a realization of $\pi = (4, 3^k, 2^{n-1-k})$. If $n \leq 12$, then π is one of the following: $(4, 3^8, 2)$, $(4, 3^8, 2^2)$, $(4, 3^8, 2^3)$, $(4, 3^{10}, 2)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic. In other words, $\pi = (4, 3^k, 2^{n-1-k})$ is potentially $K_5 - K_{1,3}$-graphic.

If π' does not satisfy (2), then π' is one of the following: $(4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^4)$, $(5, 3^5)$. Hence π is one of the following: $(5, 4, 3^3, 2^2)$, $(5, 3^5, 2)$, $(4^3, 3^2, 2^2)$, $(5^2, 4^4, 2)$, $(5^2, 3^4, 2)$, $(5, 4^2, 3^3, 2)$, $(4^4, 3^2, 2)$, $(5, 4, 3^5, 2)$, $(4^3, 3^4, 2)$, $(5^2, 4^5, 2)$, $(5, 4, 3^3, 2)$, $(4^3, 3^2, 2)$, $(6, 4, 3^4, 2)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_4 \geq 3$. If π' satisfies (1) and (2), then by the induction hypothesis, π' is potentially $K_5 - K_{1,3}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_1 = 3$, then $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ where $k \geq 3$ and $n - 1 - k - t \geq 1$. Since $\sigma(\pi)$ is even, $n - 1 - t$ must be even. We will show that π is potentially $K_5 - K_{1,3}$-graphic.

First, we consider $\pi = (4, 3^3, 2^t, 1^{n-4-t})$. If $t = 0$, it is enough to show $\pi_1 = (1^{n-5})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic. If $t \geq 1$, it is enough to show $\pi_2 = (2^{t-1}, 1^{n-3-t})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Second, we consider $\pi = (4, 3^4, 2^t, 1^{n-5-t})$. It is enough to show $\pi_1 = (2^{t+1}, 1^{n-5-t})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

Then we consider $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ where $k \geq 5$. Since $\pi \neq (4, 3^5, 1)$, we have $n \geq 8$. It is enough to show $\pi_1 = (3^{k-4}, 2^{t+1}, 1^{n-1-k-t})$

is graphic. By Lemma 2.5, π_1 is graphic.

If π' does not satisfy (2), since $\pi \neq (n-1, 3^4, 1^{n-5})$ and $(n-1, 3^5, 1^{n-6})$, then π' is one of the following: $(4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^5, 1)$. Hence, π is one of the following: $(5, 3^4, 2, 1)$, $(4^2, 3^3, 2, 1)$, $(5, 4^5, 1)$, $(5, 4, 3^4, 1)$, $(4^3, 3^3, 1)$, $(5, 3^6, 1)$, $(4^2, 3^5, 1)$, $(5, 4^6, 1)$, $(5, 3^5, 1^2)$, $(4^2, 3^4, 1^2)$. It is easy to check that all of these are potentially $K_5 - K_{1,3}$-graphic.

Theorem 3.5 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - 2K_2$-graphic if and only if the following conditions hold:

(1) $d_1 \geq 4$ and $d_5 \geq 3$;

(2)

$$\pi \neq \begin{cases} (n-i, n-j, 3^{n-i-j-2k}, 2^{2k}, 1^{i+j-2}) \\ n-i-j \text{ is even;} \\ (n-i, n-j, 3^{n-i-j-2k-1}, 2^{2k+1}, 1^{i+j-2}) \\ n-i-j \text{ is odd.} \end{cases}$$

where $1 \leq j \leq n-5$ and $0 \leq k \leq [\frac{n-j-i-4}{2}]$.

(3) $\pi \neq (4^2, 3^4)$, $(4, 3^4, 2)$, $(5, 4, 3^5)$, $(5, 3^5, 2)$, (4^7), $(4^3, 3^4)$, $(4^2, 3^4, 2)$, $(4, 3^6)$, $(4, 3^5, 1)$,$(4, 3^4, 2^2)$, $(5, 3^7)$, $(5, 3^6, 1)$, (4^8), $(4^2, 3^6)$, $(4^2, 3^5, 1)$, $(4, 3^6, 2)$, $(4, 3^5, 2, 1)$, $(4, 3^7, 1)$, $(4, 3^6, 1^2)$, $(n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$.

Proof: Assume that π is potentially $K_5 - 2K_2$-graphic. (1) is obvious. According to Lemma 2.4, (2) holds. Now it is easy to check that $(4^2, 3^4)$, $(4, 3^4, 2)$, $(5, 4, 3^5)$, $(5, 3^5, 2)$, (4^7), $(4^3, 3^4)$, $(4^2, 3^4, 2)$, $(4, 3^6)$, $(4, 3^5, 1)$, $(4, 3^4, 2^2)$, $(5, 3^7)$, $(5, 3^6, 1)$, (4^8), $(4^2, 3^6)$, $(4^2, 3^5, 1)$, $(4, 3^6, 2)$, $(4, 3^5, 2, 1)$, $(4, 3^7, 1)$, $(4, 3^6, 1^2)$ are not potentially $K_5 - 2K_2$-graphic and by Lemma 2.4, $\pi \neq (n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$. Hence, (3) holds.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1)-(3). Our proof is by induction on n. We first prove the base case where $n = 5$. In this case, π is one of the following: (4^5), $(4^3, 3^2)$, $(4, 3^4)$. It is easy to check that all of these are potentially $K_5 - 2K_2$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 6)$, we will show that π is potentially $K_5 - 2K_2$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. Clearly, $\pi' = (d_1', d_2', \cdots, d_n')$ satisfies (1) and (2). If π' also satisfies (3), then by the induction hypothesis, π' is potentially $K_5 - 2K_2$-graphic, and hence so is π. If π' does not satisfy (3), since $\pi \neq (4^7)$

and (4^8), then π' is just (4^7) or (4^8), and hence $\pi = (5^4, 4^4)$ or $(5^4, 4^5)$. It is easy to check that these sequences are potentially $K_5 - 2K_2$-graphic.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-3} \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_5 - 2K_2$-graphic, and hence so is π.

If π' does not satisfy (1), there are three subcases:

Subcase 1: $d'_1 = d'_5 = 3$. Then $\pi' = (3^k, 2^{n-1-k})$ where $n - 3 \leq k \leq n - 1$. Since $\sigma(\pi')$ is even, k must be even. If $k = n - 3$, then $\pi = (4, 3^{n-1})$ where n is odd. Since $\pi \neq (4, 3^6)$, we have $n \geq 9$. By Lemma 2.5, $\pi_1 = (3^{n-5})$ is graphic. Let G_1 be a realization of π_1, then $K_{1,2,2} \cup G_1$ is a realization of $\pi = (4, 3^{n-1})$. In other words, $\pi = (4, 3^{n-1})$ is potentially $K_5 - 2K_2$-graphic. If $k = n - 2$, then $\pi = (4^2, 3^{n-2})$ where n is even. Since $\pi \neq (4^2, 3^4)$ and $(4^2, 3^6)$, we have $n \geq 10$. It is easy to see that $(4^2, 3^8)$ and $(4^2, 3^{10})$ are potentially $K_5 - 2K_2$-graphic. Let G_2 be a realization of $(4^2, 3^8)$, which contains $K_5 - 2K_2$. If $n \geq 14$, then $\pi_2 = (3^{n-10})$ is graphic by Lemma 2.5. Let G_3 be a realization of π_2, then $G_2 \cup G_3$ is a realization of $\pi = (4^2, 3^{n-2})$. In other words, $\pi = (4^2, 3^{n-2})$ is potentially $K_5 - 2K_2$-graphic. If $k = n - 1$, then $\pi = (4^3, 3^{n-3})$ where n is odd. Since $\pi \neq (4^3, 3^4)$, we have $n \geq 9$. Clearly, $K_5 - e \cup G_1$ is a realization of $\pi = (4^3, 3^{n-3})$. Thus, $\pi = (4^3, 3^{n-3})$ is potentially $K_5 - 2K_2$-graphic since $K_5 - 2K_2 \subseteq K_5 - e$.

Subcase 2: $d'_1 \geq 4$ and $d'_5 = 2$. Since $d'_{n-3} \geq 3$, we have $n = 6$ or $n = 7$. Then π is $(5^2, 3^4)$, $(5, 3^5)$ or $(6, 3^6)$, which is impossible by condition (2) and (3).

Subcase 3: $d'_1 = 3$ and $d'_5 = 2$. Then $\pi = (4^2, 3^4)$ or $(4, 3^6)$, which is impossible by condition (3).

If π' does not satisfy (2), then $\pi' = ((n-2)^2, 3^{n-3})$ or $((n-2)^2, 3^{n-4}, 2)$. Hence, $\pi = ((n-1)^2, 4, 3^{n-3})$ or $((n-1)^2, 3^{n-2})$. But $\pi = ((n-1)^2, 3^{n-2})$ contradicts condition (2), thus $\pi = ((n-1)^2, 4, 3^{n-3})$. Since $\pi'_1 = (n - 2, 3, 2^{n-3})$ is potentially C_4-graphic by Theorem 2.3, thus $\pi = ((n-1)^2, 4, 3^{n-3})$ is potentially $K_5 - 2K_2$-graphic.

If π' does not satisfy (3), since $\pi \neq (5, 4, 3^5)$ and $(5, 3^7)$, then π' is one of the following: $(4^2, 3^4)$, $(5, 4, 3^5)$, $(5, 3^5, 2)$, (4^7), $(4^3, 3^4)$, $(4^2, 3^4, 2)$, $(4, 3^6)$, $(5, 3^7)$, (4^8), $(4^2, 3^6)$, $(4, 3^6, 2)$, $(5, 3^5)$, $(6, 3^6)$. Hence, π is one of the following: $(5^2, 4, 3^4)$, $(5, 4^3, 3^3)$, $(4^5, 3^2)$, $(6, 5, 4, 3^5)$, $(6, 4^3, 3^4)$, $(6, 4, 3^6)$, $(5^3, 4^4, 3)$, $(5^3, 3^5)$, $(5^2, 4^2, 3^4)$, $(5, 4^4, 3^3)$, $(4^6, 3^2)$, $(5^2, 3^6)$, $(5, 4^2, 3^5)$, $(4^4, 3^4)$, $(6, 4^2, 3^6)$, $(5^3, 4^5, 3)$, $(5^2, 4, 3^6)$, $(5, 4^3, 3^5)$, $(4^5, 3^4)$, $(5, 4, 3^7)$, $(6, 4^2, 3^4)$,

$(7, 4^2, 3^5)$. It is easy to check that all of these are potentially $K_5 - 2K_2$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_4 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_5 - 2K_2$-graphic, and hence so is π.

If π' does not satisfy (1), there are three subcases:

Subcase 1: $d'_1 = d'_5 = 3$. Then $d_1 = 4$, $d_3 = d_4 = d_5 = 3$ and $3 \leq d_2 \leq 4$. If $d_2 = 4$, then $\pi = (4^2, 3^k, 2^{n-2-k})$ where $k \geq 3$ and $n - 2 - k \geq 1$. Since $\sigma(\pi)$ is even, k must be even. We will show that π is potentially $K_5 - 2K_2$-graphic. It is enough to show $\pi_1 = (3^{k-3}, 2^{n-2-k}, 1)$ is graphic. If $n \geq 8$, then π_1 is graphic by Lemma 2.5. If $n \leq 7$, then $\pi = (4^2, 3^4, 2)$, which is impossible by (3). If $d_2 = 3$, then $\pi = (4, 3^k, 2^{n-1-k})$ where $k \geq 6$, $n - 1 - k \geq 1$ and k is even. Since $\pi \neq (4, 3^6, 2)$, we have $n \geq 9$. We will show that π is potentially $K_5 - 2K_2$-graphic. It is enough to show $\pi_2 = (3^{k-4}, 2^{n-1-k})$ is graphic. By Lemma 2.5, π_2 is graphic.

Subcase 2: $d'_1 \geq 4$ and $d'_5 = 2$. Then $d_1 \geq 5$, $d_2 = d_3 = d_4 = d_5 = 3$ and $d_6 = \cdots = d_{n-1} = 2$. Hence, $\pi = (d_1, 3^4, 2^{n-5})$. Since $\sigma(\pi)$ is even, d_1 must be even. We will show that π is potentially $K_5 - 2K_2$-graphic. It is enough to show $\pi_1 = (d_1 - 4, 2^{n-5})$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-1-d_1}, 1^{d_1-4})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

Subcase 3: $d'_1 = 3$ and $d'_5 = 2$. Then $\pi = (4, 3^4, 2^{n-5})$. Since $\pi \neq (4, 3^4, 2)$ and $(4, 3^4, 2^2)$, we have $n \geq 8$. Clearly, $K_{1,2,2} \cup C_{n-5}$ is a realization of π. In other words, π is potentially $K_5 - 2K_2$-graphic.

If π' does not satisfy (2), i.e.,

$$\pi' = \begin{cases} ((n-2)^2, 3^{n-3-2k}, 2^{2k}), & n \text{ is odd;} \\ ((n-2)^2, 3^{n-4-2k}, 2^{2k+1}), & n \text{ is even.} \end{cases}$$

If $n \geq 7$, then

$$\pi = \begin{cases} ((n-1)^2, 3^{n-3-2k}, 2^{2k+1}), & n \text{ is odd;} \\ ((n-1)^2, 3^{n-4-2k}, 2^{2k+2}), & n \text{ is even.} \end{cases}$$

which contradicts condition (2). If $n = 6$, then $\pi' = (4^2, 3^2, 2)$ and hence $\pi = (5^2, 3^2, 2^2)$ or $(4^4, 2^2)$, which is impossible by (1).

If π' does not satisfy (3), then π' is one of the following: $(4^2, 3^4)$, $(4, 3^4, 2)$, $(5, 4, 3^5)$, $(5, 3^5, 2)$, (4^7), $(4^3, 3^4)$, $(4^2, 3^4, 2)$, $(4, 3^6)$, $(4, 3^4, 2^2)$, $(5, 3^7)$, (4^8), $(4^2, 3^6)$, $(4, 3^6, 2)$, $(5, 3^5)$, $(6, 3^6)$. Since $\pi \neq (5, 3^5, 2)$, then

π is one of the following: $(5^2,3^4,2)$, $(5,4^2,3^3,2)$, $(4^4,3^2,2)$, $(5,4,3^3,2^2)$, $(4^3,3^2,2)$, $(6,5,3^5,2)$, $(6,4^2,3^4,2)$, $(6,4,3^4,2^2)$, $(6,3^6,2)$, $(5^2,4^5,2)$, $(5^2,4,3^4,2)$, $(5,4^3,3^3,2)$, $(4^5,3^2,2)$, $(5^2,3^4,2^2)$, $(5,4^2,3^3,2^2)$, $(4^4,3^2,2^2)$, $(5,4,3^5,2)$, $(4^3,3^4,2)$, $(5,4,3^3,2^3)$, $(5,3^5,2^2)$, $(4^3,3^2,2^3)$, $(6,4,3^6,2)$, $(5^2,4^6,2)$, $(5^2,3^6,2)$, $(5,4^2,3^5,2)$, $(4^4,3^4,2)$, $(5,4,3^5,2^2)$, $(5,3^7,2)$, $(4^3,3^4,2^2)$, $(6,4,3^4,2)$, $(7,4,3^5,2)$. It is easy to check that all of these are potentially K_5-2K_2-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$ where $d_5' \geq 3$. If π' satisfies (1)-(3), then by the induction hypothesis, π' is potentially K_5-2K_2-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d_1' = 3$, then $d_1 = 4$ and $d_2 = \cdots = d_5 = 3$. Hence, $\pi = (4, 3^k, 2^t, 1^{n-1-k-t})$ where $k \geq 4$ and $n-1-k-t \geq 1$. Since $\sigma(\pi)$ is even, $n-1-t$ must be even. We will show that π is potentially K_5-2K_2-graphic. It is enough to show $\pi_1 = (3^{k-4}, 2^t, 1^{n-1-k-t})$ is graphic. Since $\pi \neq (4,3^7,1)$ and $(4,3^6,1^2)$, we have $\pi_1 \neq (3^3,1)$ and $(3^2,1^2)$. If $n \geq 9$, then π_1 is graphic by Lemma 2.5. If $n \leq 8$, since $\pi \neq (4,3^5,1)$ and $(4,3^5,2,1)$, then $\pi = (4,3^4,1^2)$ or $(4,3^4,2,1^2)$. It is easy to see that π is potentially K_5-2K_2-graphic.

If π' does not satisfy (2), i.e.,

$$\pi' = \begin{cases} (n-1-i, n-1-j, 3^{(n-1)-i-j-2k}, 2^{2k}, 1^{i+j-2}), \\ \quad n-1-i-j \text{ is even;} \\ (n-1-i, n-1-j, 3^{(n-1)-i-j-2k-1}, 2^{2k+1},, 1^{i+j-2}), \\ \quad n-1-i-j \text{ is odd.} \end{cases}$$

where $1 \leq j \leq (n-1)-5$ and $0 \leq k \leq [\frac{(n-1)-j-i-4}{2}]$. If $n-i > n-j+1$ or $n-i = n-j$, then

$$\pi = \begin{cases} (n-i, n-(j+1), 3^{n-i-(j+1)-2k}, 2^{2k}, 1^{i+(j+1)-2}), \\ \quad n-i-(j+1) \text{ is even;} \\ (n-i, n-(j+1), 3^{n-i-(j+1)-2k-1}, 2^{2k+1}, 1^{i+(j+1)-2}), \\ \quad n-i-(j+1) \text{ is odd.} \end{cases}$$

which contradicts condition (2). If $n-i = n-j+1$, i.e.,

$$\pi' = \begin{cases} (n-1-i, n-2-i, 3^{n-2i-2k-2}, 2^{2k}, 1^{2i-1}), \\ \quad n \text{ is even;} \\ (n-1-i, n-2-i, 3^{n-2i-2k-3}, 2^{2k+1}, 1^{2i-1}), \\ \quad n \text{ is odd.} \end{cases}$$

Then

$$\pi = \begin{cases} (n-i, n-i-2, 3^{n-2i-2k-2}, 2^{2k}, 1^{2i}), \\ \quad n \text{ is even;} \\ (n-i, n-i-2, 3^{n-2i-2k-3}, 2^{2k+1}, 1^{2i}), \\ \quad n \text{ is odd.} \end{cases}$$

or

$$\pi = \begin{cases} ((n-1-i)^2, 3^{n-2i-2k-2}, 2^{2k}, 1^{2i}), \\ \quad n \text{ is even;} \\ ((n-1-i)^2, 3^{n-2i-2k-3}, 2^{2k+1}, 1^{2i}), \\ \quad n \text{ is odd.} \end{cases}$$

which contradicts condition (2).

If π' does not satisfy (3), since $\pi \neq (5,3^6,1), (4^2,3^5,1), (n-1,3^5,1^{n-6})$ and $(n-1,3^6,1^{n-7})$, then π' is one of the following: $(4^2,3^4)$, $(4,3^4,2)$, $(5,4,3^5)$, $(5,3^5,2)$, (4^7), $(4^3,3^4)$, $(4^2,3^4,2)$, $(4,3^5,1)$, $(4,3^4,2^2)$, $(5,3^7)$, $(5,3^6,1)$, (4^8), $(4^2,3^6)$, $(4^2,3^5,1)$, $(4,3^6,2)$, $(4,3^5,2,1)$, $(4,3^7,1)$, $(4,3^6,1^2)$. Hence, π is one of the following:

$(5,4,3^4,1)$, $(4^3,3^3,1)$, $(5,3^4,2,1)$, $(4^2,3^3,2,1)$, $(6,4,3^5,1)$, $(5^2,3^5,1)$, $(6,3^5,2,1)$, $(5,4^6,1)$, $(5,4^2,3^4,1)$, $(4^4,3^3,1)$, $(5,4,3^4,2,1)$, $(4^3,3^3,2,1)$, $(5,3^5,1^2)$, $(4^2,3^4,1^2)$, $(5,3^4,2^2,1)$, $(4^2,3^3,2^2,1)$, $(6,3^7,1)$, $(6,3^6,1^2)$, $(5,4^7,1)$, $(5,4,3^6,1)$, $(4^3,3^5,1)$, $(5,4,3^5,1^2)$, $(4^3,3^4,1^2)$, $(5,3^6,2,1)$, $(4^2,3^5,2,1)$, $(5,3^5,2,1^2)$, $(4^2,3^4,2,1^2)$, $(5,3^7,1^2)$, $(4^2,3^6,1^2)$, $(5,3^6,1^3)$, $(4^2,3^5,1^3)$. It is easy to check that all of these are potentially K_5-2K_2-graphic.

4 Application

Using Theorem 3.1 and Theorem 3.3, we give simple proofs of the following theorems due to Lai:

Theorem 4.1 (Lai [14]) For $n \geq 5$, $\sigma(K_5-P_3,n) = 4n-4$.

Proof: First we claim that for $n \geq 5, \sigma(K_5-P_3,n) \geq 4n-4$. It is enough to show that there exists π_1 with $\sigma(\pi_1) = 4n-6$, such that π_1 is not potentially K_5-P_3-graphic. Take $\pi_1 = ((n-1)^2, 2^{n-2})$, then $\sigma(\pi_1) = 4n-6$, and it is easy to see that π_1 is not potentially K_5-P_3-graphic by Theorem 3.1.

Now we show that if π is an n-term ($n \geq 5$) graphical sequence with $\sigma(\pi) \geq 4n-4$, then there exists a realization of π containing K_5-P_3. Hence, it suffices to show that π is potentially K_5-P_3-graphic.

If $d_5 = 1$, then $\sigma(\pi) = d_1+d_2+d_3+d_4+(n-4)$ and $d_1+d_2+d_3+d_4 \leq 12+(n-4) = n+8$. Therefore, $\sigma(\pi) \leq 2n+4 < 4n-4$, a contradiction. Thus, $d_5 \geq 2$.

If $d_3 \leq 2$, then $\sigma(\pi) \leq d_1+d_2+2(n-2) \leq 2(n-1)+2(n-2) = 4n-6 < 4n-4$, a contradiction. Thus, $d_3 \geq 3$.

If $d_1 \leq 3$, then $\sigma(\pi) \leq 3n < 4n-4$, a contradiction. Thus, $d_1 \geq 4$.

Since $\sigma(\pi) \geq 4n-4$, then π is not one of the following: $(4,3^2,2^3)$, $(4,3^2,2^4)$, $(4,3^6)$. Thus, π satisfies the conditions (1) and (2) in Theorem 3.1. Therefore, π is potentially K_5-P_3-graphic.

Theorem 4.2 (Lai [13]) For $n \geq 5$, $\sigma(K_5-C_4, n) = 4n-4$.

Proof: Obviously, for $n \geq 5$, $\sigma(K_5-C_4,n) \leq \sigma(K_5-P_3,n) = 4n-4$. Now we claim $\sigma(K_5-C_4,n) \geq 4n-4$ for $n \geq 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n-6$, such that π_1 is not potentially K_5-C_4-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n-6$ and the only realization of π_1 does not contain K_5-C_4. Thus, $\sigma(K_5-C_4,n) = 4n-4$.

Theorem 4.3 (Lai [10], Luo[21]) $\sigma(C_5,n) = 4n-4$ for $n \geq 5$.

Proof: Obviously, for $n \geq 5$, $\sigma(K_5-C_5,n) \leq \sigma(K_5-P_3,n) = 4n-4(K_5-C_5 = C_5)$. Now we claim $\sigma(C_5,n) \geq 4n-4$ for $n \geq 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n-6$, such that π_1 is not potentially C_5-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n-6$ and the only realization of π_1 does not contain C_5. Thus, $\sigma(C_5,n) = 4n-4$.

Theorem 4.4 (Lai [15]) For $n = 5$ and $n \geq 7$,

$$\sigma(K_{3,1,1}, n) = 4n-2.$$

For $n = 6$, if π is a 6-term graphical sequence with $\sigma(\pi) \geq 22$, then either there is a realization of π containing $K_{3,1,1}$ or $\pi = (4^6)$. (Thus $\sigma(K_{3,1,1},6) = 26$.)

Proof: First we claim that for $n \geq 5, \sigma(K_5-K_3,n) \geq 4n-2(K_{3,1,1} = K_5-K_3)$. It is enough to show that there exists π_1 with $\sigma(\pi_1) = 4n-4$, such that π_1 is not potentially K_5-K_3-graphic. Take $\pi_1 = (n-1, 3^{n-1})$, then $\sigma(\pi_1) = 4n-4$, and it is easy to see that π_1 is not potentially K_5-K_3-graphic by Theorem 3.3.

Now we show that if π is an n-term ($n \geq 5$) graphical sequence with $\sigma(\pi) \geq 4n-2$, then there exists a realization of π containing $K_5 - K_3$(unless $\pi = (4^6)$). Hence, it suffices to show that π is potentially $K_5 - K_3$-graphic.

If $d_5 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + d_4 + (n-4)$ and $d_1 + d_2 + d_3 + d_4 \leq 12 + (n-4) = n+8$. Therefore, $\sigma(\pi) \leq 2n+4 < 4n-2$, a contradiction. Thus, $d_5 \geq 2$.

If $d_2 \leq 3$, then $\sigma(\pi) \leq d_1 + 3(n-1) \leq n-1+3(n-1) = 4n-4 < 4n-2$, a contradiction. Thus, $d_2 \geq 4$.

Since $\sigma(\pi) \geq 4n-2$, then $\pi \neq (4^2, 2^5)$. Hence, for $n = 5$ and $n \geq 7$, π satisfies the conditions (1) and (2) in Theorem 3.3. Therefore, π is potentially $K_5 - K_3$-graphic. For $n = 6$, since $\sigma(\pi) \geq 4 \times 6 - 2 = 22$, then π is not one of the following: $(4^2, 2^4)$, $(4^3, 2^3)$. Thus, by Theorem 3.3, either there is a realization of π containing $K_{3,1,1}$ or $\pi = (4^6)$.

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