Some α-graphs and odd graceful graphs

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Abstract

We show that if G has an odd graceful labeling f such that $\max\{f(x): f(x)$ is even, $x \in A\} < \min\{f(x): f(x)$ is odd, $x \in B\}$, then G is an α-graph, and if G is an α-graph, then $G \odot \overline{K}_w$ is odd graceful for all $w \geq 1$. Also we show that if G_1 is an α-graph and G_2 is an odd graceful, then $G_1 \cup G_2$ is odd graceful. Finally we show that some families of graphs are α-graphs and odd graceful.

0. Introduction

In 1967, Rosa [6] called a function f a β-valuation of a graph $G = (V(G)\ E(G)$, with q edges if f is an injection from $V(G)$ to the set $\{0,1,2,\ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [4] subsequently called such labelings graceful, and this is now the popular term.

Rosa [6] defined an α-labeling to be a graceful labeling with the additional property, that there exists an integer k so that, for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1.

Gnanajothi [3] defined a graph G with q edges to be odd graceful if there is an injection f from $V(G)$ to $\{0,1,2,\ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1,3,5,\ldots, 2q-1\}$. She proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved the following graphs are odd graceful: P_n, C_n if and only if n is even, $K_{m,n}$, combs $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of P_n), books, crowns and $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of C_n) if and only if n is even.

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Seoud and Abdel-Aal [7] determined all connected odd graceful graphs of order ≤ 6, and they showed that if G is an odd graceful graph, then $G \cup K_{m,n}$ is odd graceful for all $m, n \geq 1$. Seoud, Diab and Elskhawi [8] showed that a connected n-partite graph is odd graceful if and only if $n = 2$ and that the join of any two connected graphs is not odd graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian's survey [2]. Throughout this paper, we use the standard notations and conventions as in [2] and [5]. This paper is divided into two sections. Section 1, in which we show that if G has an odd graceful labeling f such that, $\max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is odd}, x \in B\}$, then G is an α-graph, and if G is an α-graph, then $G \odot \overline{K_w}$ is odd graceful for all $w \geq 1$. We also show that if G_1 is an α-graph and G_2 is an odd graceful, then $G_1 \cup G_2$ is odd graceful. In Section 2, we show that several families of graphs are odd graceful and α-graphs.

1. General Theorems

Theorem 1.1. Gnanajothi [3] proved the following Theorem: Every α-graph is odd graceful and every odd graceful graph is bipartite.

Theorem 1.2 Barrientos [1] proved the following Theorem: Any α-graph is odd graceful.

Proof. Let G be an α-graph of size q, as a consequence G is bipartite with partition $\{A,B\}$. Suppose that f is an α-labeling of G such that $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$. Let g be a labeling of the vertices of G defined by

$$g(x) = \begin{cases} (2f(x)) & , x \in A \\ (2f(x)) - 1 & , x \in B \end{cases}$$

Thus, the labels assigned by g are in the set $\{0,1,2,\ldots, 2q-1\}$; furthermore, the label of the edge xy of G induced by the labeling f, where $x \in A$ and $y \in B$ is $w = f(y) - f(x)$, so its label under the labeling g is $g(y) - g(x) = [2f(y) - 1] - 2f(x) = 2[f(y) - f(x)] - 1 = 2w - 1$. Since $1 \leq w \leq q$, we have that the label induced by g are $\{1,3,5,\ldots,2q-1\}$. Therefore, g is an odd graceful labeling of G and $\max\{g(x) : g(x) \text{ is even}, x \in A\} < \min\{g(x) : g(x) \text{ is odd}, x \in B\}$.

Corollary 1.3. Any α-graph has an odd graceful labeling f such that, $\max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is even}, x \in B\}$.

Now, we give the following Theorems.

Theorem 1.4. If G has an odd graceful labeling f such that, $\max\{f(x) : f(x)$ is even, $x \in A\} < \min\{f(x) : f(x)$ is odd, $x \in B\}$, i.e. $\{A,B\}$ is a partition of the bipartite graph G, then G is an α-graph.

Proof. Let $G = (V(G), E(G))$ be an odd graceful graph with $f : V(G) \to \{0,1,2,\ldots, 2q-1\}$ labeling of size q. Hence, G is a bipartite graph with partition $\{A,B\}$, i.e. $V(G) = \{A,B\}$, such that $f(x)$ is even, if $x \in A$ and $f(x)$ is odd, if $x \in B$, and let $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$. Now, let g be another labeling of G, defined as follows :

$$g(x) = \begin{cases} (1/2) f(x) & , x \in A \\ (1/2)\,[f(x)+1] & , x \in B \end{cases}$$

Thus, the labels assigned by g will be in the set $\{0,1,2,\ldots, q\}$. Note that g is injective, since $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$. Since the label of the edge xy due to f, is $w = |f(y) - f(x)| = f(y) - f(x)$, where $x \in A$ and $y \in B$, the label of the edge xy given by g will be $g(y) - g(x) = 1/2\,[f(y) + 1] - 1/2 f(x) = 1/2\,[f(y) - f(x) + 1] = 1/2\,(w+1)$. Since $1 \le w \le 2q - 1$, the labels of the set of edges of G induced by g will be $\{1,2,3,\ldots,q\}$, i.e. G is a graceful graph. It remains to find an integer k such that for any edge xy $g(x) \le k < g(y)$, where $x \in A$, $y \in B$. Now, since G is now graceful, then there exists the labeling $|g(y) - g(x)| = 1$, and since $\max\{g(x) : x \in A\} < \min\{g(y) : y \in B\}$, then take $k = 1/2 \max\{f(x) : x \in A\}$, which is the required, and the proof is complete.

In Figure1, we show an odd graceful labeling of a bipartite graph $K_{3,3}$, followed by the corresponding α-labeling.

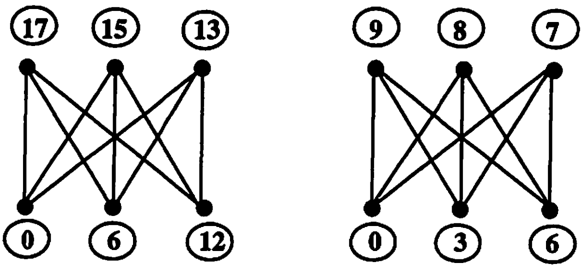

Figure 1.

Corollary 1.5. The graph G is an α-graph if and only if G has an odd graceful labeling f such that, $\max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(y) : f(y) \text{ is even}, y \in B\}$, where $\{A, B\}$ is a partition of the graph G.

Let G_1 and G_2 be two disjoint graphs. The corona ($G_1 \odot G_2$) of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the j^{th} vertex of G_1 to every vertex in the j^{th} copy of G_2.

Theorem 1.6. If G is an α-graph then the graphs $G \odot \overline{K_w}$, for $w \geq 1$, are odd graceful.

Proof. Let G be an α-graph of size q and order p. By Theorem 1.1 every α-graph has an odd graceful labeling $f: V(G) \to \{0,1,2,\ldots, 2q-1\}$. Hence, G is a bipartite graph with partition $\{A, B\}$ such that

$$U_1(G) = \{f(x_1), f(x_2), f(x_3), \ldots, f(x_s)\}, \text{ if } f(x_i) \text{ is even}, x_i \in A, |U_1(G)| = s,$$
$$f(x_1) < f(x_2) < f(x_3) < \ldots < f(x_s),$$
$$U_2(G) = \{f(y_1), f(y_2), f(y_3), \ldots, f(y_t)\}, \text{ if } f(y_j) \text{ is odd}, y_j \in B, |U_1(G)| = t,$$
$$f(y_1) > f(y_2) > f(y_3) > \ldots > f(y_t) \text{ and } |V(G)| = s + t = p.$$

Now, for $\overline{K_w}$, let the copies $z^i_1, z^i_2, z^i_3, \ldots, z^i_s$, $1 \leq i \leq w$ be joined respectively from the smallest even label $f(x_1)$ to the largest even label $f(x_s)$ and the copies $h^i_1, h^i_2, h^i_3, \ldots, h^i_t$, $1 \leq i \leq w$ be joined respectively from the largest odd label $f(y_1)$ to the smallest odd label $f(y_t)$. It is clear that the number of edges of $G \odot \overline{K_w}$ is $q + wp$. We define the labeling function

$\bar{f}: V(G \odot \overline{K_w}) \to \{0,1,2,\ldots, 2(q + wp) - 1\}$ as follows:

$$\bar{f}\,|\,V(G) = \begin{cases} f(x) & , \text{ if } f(x) \text{ is even}, \ x \in A \\ f(y) + 2wp & , \text{ if } f(y) \text{ is odd}, \ y \in B \end{cases}$$

$$\bar{f}(z^i_1) = f(x_1) + 2i - 1 \qquad , \ 1 \leq i \leq w$$
$$\bar{f}(z^i_2) = f(x_2) + 2w + 2i - 1 \qquad , \ 1 \leq i \leq w$$
$$\bar{f}(z^i_3) = f(x_3) + 4w + 2i - 1 \qquad , \ 1 \leq i \leq w$$

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$$\bar{f}(z^i_s) = f(x_s) + 2w(s-1) + 2i - 1 \qquad , \ 1 \leq i \leq w, \ s = |U_1(G)|$$

and

$\bar{f}(h^i_1) = f(y_1) + 2wt - 2i + 1$, $1 \le i \le w$
$\bar{f}(h^i_2) = f(y_2) + 2w(t-1) - 2i + 1$, $1 \le i \le w$
$\bar{f}(h^i_3) = f(y_3) + 2w(t-2) - 2i + 1$, $1 \le i \le w$

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$\bar{f}(h^i_t) = f(y_t) + 2w - 2i + 1$, $1 \le i \le w, t = |U_2(G)|$.

Since G is an α-graph, it is easy to show that $\bar{f}$ is injective.

The edge labels will be as follows :

- The edges $x_j z^i_j$, $1 \le i \le w$, $1 \le j \le s$, $s = |U_1(G)|$ take the labels $\{1, 3, 5, \ldots, 2w-1\}$, $\{2w+1, 2w+3, 2w+5, \ldots, 4w-1\}$, $\{4w+1, 4w+3, 4w+5, \ldots, 6w-1\}$and so on until $\{2w(s-1)+1, 2w(s-1)+3, 2w(s-1)+5, \ldots, 2ws-1\}$. Since $s = p - t$ then the labels are$\{2w(p-t-1)+1, 2w(p-t-1)+3, 2w(p-t-1)+5, \ldots, 2w(p-t)-1\}$.

- The edges $y_j h^i_j$, $1 \le i \le w$, $1 \le j \le t$, $t = |U_2(G)|$ take the labels$\{2w(p-t)+1, 2w(p-t)+3, 2w(p-t)+5, \ldots, 2w(p-t+1)-1\}$,$\{2w(p-t+1)+1, 2w(p-t+1)+3, 2w(p-t+1)+5, \ldots, 2w(p-t+2)-1\}$,$\{2w(p-t+2)+1, 2w(p-t+2)+3, 2w(p-t+2)+5, \ldots, 2w(p-t+3)-1\}$, and so on until$\{2w(p-1)+1, 2w(p-1)+3, 2w(p-1)+5, \ldots, 2wp-1\}$.
- The remaining edge labels$\{2wp+1, 2wp+3, 2wp+5, \ldots, 2wp+2q-1\}$of the graph $G \odot \overline{K_w}$ come from the edge labels of the graph G, since G is an α-graph and we added a constant number on its odd vertex labels .

In Figure 2 , we use the odd graceful labeling obtained in Figure 1 to find an odd graceful labeling of the graph $K_{3,3} \odot \overline{K_3}$.

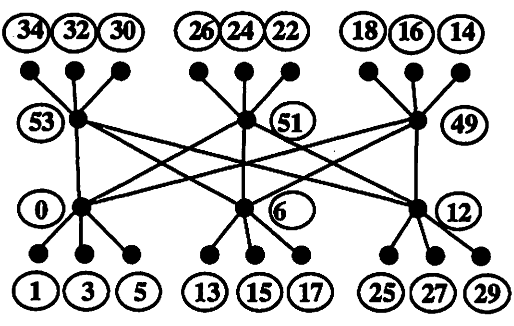

Figure 2 .

Let G_1 and G_2 be two disjoint graphs. The union $(G_1 \cup G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Theorem 1.7. The graphs $G_1 \cup G_2$ are odd graceful if G_1 is an α-graph and G_2 is an odd graceful graph.

Proof. Let G_1 be an α-graph of size q_1 and order p_1. By the Theorem 1.1 every α-graph has an odd graceful labeling $f: V(G_1) \to \{0,1,2,\ldots,2q_1 - 1\}$, and G is a bipartite graph with partition $\{A,B\}$ such that, $\max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is odd}, x \in B\}$, there exists an integer k such that, $f(x) \leq k < f(y)$, where $x \in A$ and $y \in B$, $k = \max\{f(x) : f(x) \text{ is even}, x \in A\} = \min\{f(x) : f(x) \text{ is odd}, x \in B\} - 1$.

Let G_2 be an odd graceful graph of size q_2 and order p_2 with labeling $f^*: V(G_2) \to \{0,1,2,\ldots, 2q_2 - 1\}$. Hence, G_2 is also bipartite with partition $\{C, D\}$ such that, $f^*(x)$ is even, if $x \in C$ and $f^*(x)$ is odd, if $x \in D$. Now, the number of edges of $G_1 \cup G_2$ is $q_1 + q_2$.

We define the labeling function $\bar{f}: V(G_1 \cup G_2) \to \{0,1,2,\ldots, 2(q_1 + q_2) - 1\}$ as follows :

$$\bar{f}(x) = \begin{cases} f(x) & \text{, if } f(x) \text{ is even}, x \in A \\ f(x) + 2q_2 & \text{, if } f(x) \text{ is odd}, x \in B \\ f^* + (k+1) & \text{, if } x \in C \cup D \end{cases}$$

It is easy to check that $\bar{f}$ is injective.

The edge labels will be as follows :

- The edge labels of the edge xy of G_1 under the labeling $\bar{f}$ will be $[f(y) + 2q_2] - f(x) = [f(y) - f(x)] + 2q_2 = w + 2q_2$, where w is odd and $1 \leq w \leq 2q_1 - 1$. Hence, the set of edge labels of G_1 induced by $\bar{f}$ will be $\{1+2q_2, 3+2q_2, 5+2q_2, \ldots, 2q_1 + 2q_2 - 1\}$.
- The remaining edge labels $\{1,3,5,\ldots, 2q_2 - 1\}$ come from the edge labels of the graph G_2, since G_2 is odd graceful and adding an integer $k+1$ to all the vertex labels does not change the edge labels.

In Figure 3, we show an α-labeling and an odd graceful labeling of the ladder $(P_5 \times P_2)$.

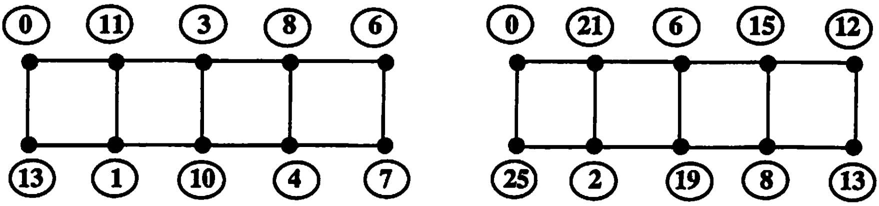

Figure 3.

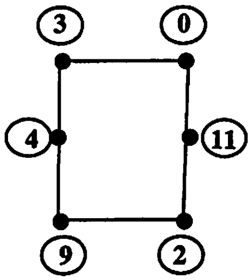

Figure 4. odd graceful labeling of the cycle C_6.

In Figure 5, we use the odd graceful labeling obtained in Figure 3 and the odd graceful labeling in Figure 4 to find an odd graceful labeling of the graph $(P_5 \times P_2) \cup C_6$.

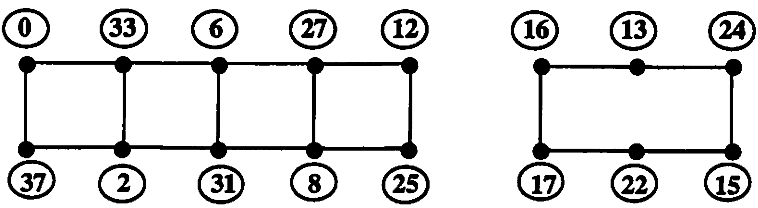

Figure 5.

Corollary1.8. According to Theorem 1.7 numerous families of disconnected graphs are odd graceful, e.g.

1. $G_1 \cup G_2$, if G_1 and G_2 are α-graphs.
2. $C_n \cup G$, if G is odd graceful, $n \equiv 0 \pmod 4$.

3. $T \cup G$, if G is odd graceful and T is a tree has an α-labeling.
4. $G \cup C_n$, if G is an α-graph and C_n is odd graceful for an even integer n.
(C_n is an α-graph if and only if $n \equiv 0 \pmod 4$) and C_n is odd graceful for every even integer $n \geq 4$.)

2. Some Odd Graceful Graphs

A **gear** graph G_m is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle.

Theorem 2.1. A gear graph G_m for all $m \geq 3$ is an α-graph.
Proof. By Theorem 1.2, we want to show that a gear graph G_m for all $m \geq 3$ has an odd graceful labeling f such that, $\max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is odd}, x \in B\}$, when $\{A,B\}$ is a partition of the bipartite graph G_m.

- **If m is even, $m \geq 4$**

Let G_m be described as indicated in Figure 6.

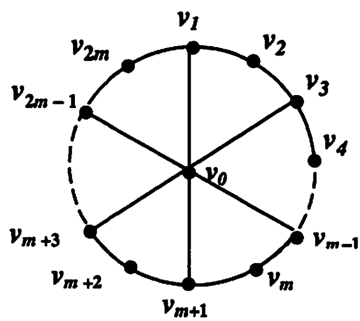

Figure 6 .

The number of edges of the graph G_m is $3m$.We define the labeling function $f: V(G_m) \to \{0,1,2,\ldots, 6m-1\}$ as follows :
$f(v_0) = 6m-1$, $f(v_{2m}) = 2m-1$

$$f(v_i) = \begin{cases} i-1 & , \ i = 1,3,5,\ldots, 2m-1 \\ 4m-i+1 & , \ i = 2,4,6,\ldots, m \\ 4m-i-1 & , \text{ if } i \text{ is even and } m+2 \leq i \leq 2m-2 \end{cases}$$

The edge labels will be as follows :

- The edges v_0v_i , $i = 1,3,5,\ldots, 2m-1$ take the labels$\{6m-1, 6m-3, 6m-5, \ldots, 4m+1\}$.
- The edges of the path $v_1v_2v_3v_4\ldots v_mv_{m+1}$ take the labels$\{4m-1, 4m-3, 4m-5, \ldots, 2m+1\}$.
- The edge v_1v_{2m} takes the label $2m-1$.
- The edges of the path $v_{m+1}v_{m+2}v_{m+3}v_{m+4}\ldots v_{2m-1}v_{2m}$ take the labels $\{2m-3, 2m-5, 2m-7, \ldots, 5, 3, 1\}$.

Hence, G_m for m is even, $m \geq 4$ has an odd graceful labeling f and there exists $k = f(v_{2m-1}) = 2m-1$ such that, $k = \max\{f(x) : f(x)$ is even , $x \in A\} < \min\{f(x) : f(x)$ is odd, $x \in B\}$, $A = \{v_1, v_3, v_5, \ldots, v_{2m-1}\}$, $B = \{v_2, v_4, v_6, \ldots, v_{2m}\}$.
By Theorem 1.4, the graph G_m for m is even, $m \geq 4$ is an α-graph.

- **If m is odd, $m \geq 3$**

Let G_m be described as indicated in Figure 7.

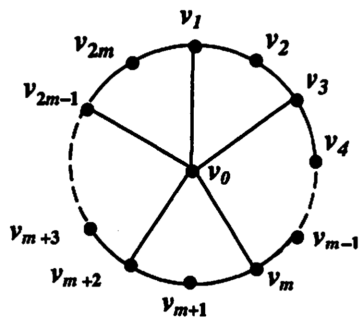

Figure7.

The number of edges of the graph G_m is $3m$.We define the labeling function $f : V(G_m) \to \{0,1,2,\ldots, 6m-1\}$ as follows :
$f(v_0) = 6m-3$, $f(v_1) = 0$, $f(v_2) = 6m-1$, $f(v_{2m}) = 2m+1$

$$f(v_i) = \begin{cases} i+1 & , \; i = 3,5,7,\ldots, 2m-1 \\ 4m-i+3 & , \; i = 4,6,8,\ldots, m-1 \\ 4m-i+1 & , \; \text{if } i \text{ is even and } m+1 \leq i \leq 2m-2 \end{cases}$$

The edge labels will be as follows :

- The edges v_1v_2, v_0v_1, v_2v_3 take the labels $6m-1$, $6m-3$, $6m-5$ respectively.

• The edges v_0v_i, $i = 3,5,7,\ldots, 2m-1$ take the labels $\{6m-7, 6m-9, 6m-11,\ldots, 4m-3\}$.
• The edges of the path $v_3v_4v_5v_6\ldots v_{m-1}v_m$ take the labels $\{4m-5, 4m-7, 4m-9,\ldots, 2m+3\}$.
• The edge v_1v_{2m} takes the label $2m+1$.
• The edges of the path $v_mv_{m+1}v_{m+2}v_{m+3}\ldots v_{2m-1}v_{2m}$ take the labels $\{2m-1, 2m-3, 2m-5,\ldots, 5,3,1\}$.

Hence G_m for m is odd, $m \geq 3$ has an odd graceful labeling f, and there exists $k = f(v_{2m-1})$, similar to the case m is even, $m \geq 4$, the result follows by Theorem 1.4, so the graph G_m for m is odd, $m \geq 3$ is an α-graph.

The **dragon** (or ballon) $D_{m,n}$ is a graph formed by identifying the end vertex of the path of m edges ($m \geq 1$) and any vertex in the cycle C_n ($n \geq 3$).

Theorem 2.2. The dragon is odd graceful when n is even, $n \geq 4$ and $m \geq 1$, and it is an α-graph if $n \equiv 0 \pmod 4$, $m \geq 1$.
Proof. Let $D_{m,n}$ be described as indicated in Figure 8.

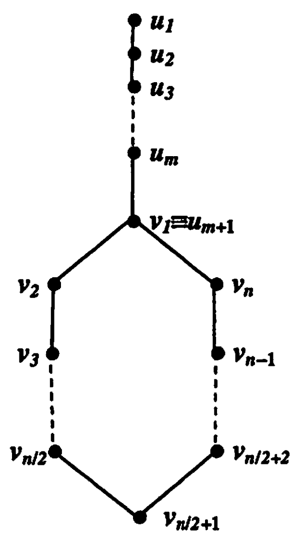

Figure 8.

The number of edges of the dragon is $m + n$. We define the following function $f : V(D_{m,n}) \rightarrow \{0,1,2,\ldots, 2(m+n)-1\}$ as follows :

Case I.

- **If m is odd and n is even, $n \geq 4$**

$$f(u_i) = \begin{cases} i-1 & , \quad i = 1,3,5,\ldots, m \\ 2(m+n) - i + 1 & , \quad i = 2,4,6,\ldots, m+1 \end{cases}$$

- **If $n \equiv 0 \pmod 4$**

$$f(v_j) = \begin{cases} m+j-1 & , \quad j = 2,4,6,\ldots, n/2 \\ m+2n-j+1 & , \quad j = 3,5,7,\ldots, n-1 \\ m+j+1 & , \quad \text{if } j \text{ is even and } n/2+2 \leq j \leq n \end{cases}$$

- **If $n \equiv 2 \pmod 4$**

$$f(v_j) = \begin{cases} m+j-1 & , \quad j = 2,4,6,\ldots, n-2 \\ m+2n-j+1 & , \quad j = 3,5,7,\ldots, n/2 \\ m+2n-j-1 & , \quad \text{if } j \text{ is odd and } n/2+2 \leq j \leq n-1 \end{cases}$$

$$f(v_n) = m+n+1$$

Case II.

- **If m is even and n is even, $n \geq 4$**

$$f(u_i) = \begin{cases} i-1 & , \quad i = 1,3,5,\ldots, m+1 \\ 2(m+n) - i + 1 & , \quad i = 2,4,6,\ldots, m \end{cases}$$

- **If $n \equiv 0 \pmod 4$**

$$f(v_j) = \begin{cases} m+j-1 & , \quad j = 3,5,7,\ldots, n-1 \\ m+2n-j+1 & , \quad j = 2,4,6,\ldots, n/2 \\ m+2n-j-1 & , \quad \text{if } j \text{ is even and } n/2+2 \leq j \leq n \end{cases}$$

- **If $n \equiv 2 \pmod 4$**

$$f(v_j) = \begin{cases} m+2n-j+1 & , \quad j = 2,4,6,\ldots, n-2 \\ m+j-1 & , \quad j = 3,5,7,\ldots, n/2 \\ m+j+1 & , \quad \text{if } j \text{ is odd and } n/2+2 \leq j \leq n-1 \end{cases}$$

$$f(v_n) = m+n-1$$

The edge labels will be as follows :

- The edges of the path $u_1 u_2 u_3 \ldots u_{m+1}$ take the labels $\{2(m+n)-1, 2(m+n)-3, 2(m+n)-5, \ldots, 2n+1\}$.
- The edges of the path $v_1 v_2 v_3 \ldots v_{n/2+1}$ take the labels $\{2n-1, 2n-3, 2n-5, \ldots, n+1\}$.
- The edge $v_1 v_n$ takes the label $\{n-1\}$.
- Finally, the edges of the path $v_{n/2+1} v_{n/2} v_{n/2-1} \ldots v_{n-1} v_n$ take the labels $\{n-3, n-5, \ldots, 5,3,1\}$.

So we obtain the edge labels. Hence $D_{m,n}$ is odd graceful for $m \geq 1$, n is even, $n \geq 4$. In the case n ≡ 0 (mod 4), m is odd, $m \geq 1$ there exists $k = f(v_n)$ such that, $k = \max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is odd}, x \in B\}$.
By Theorem 1.4 the graph $D_{m,n}$ in case n ≡ 2 (mod 4) is an α-graph similar the case n ≡ 0 (mod 4), m is even, $m \geq 2$, there exists $k = f(v_{n-1})$ = m+n − 2, the result follows by Theorem 1.4, i.e. the graph $D_{m,n}$ in this case is an α-graph .

In the following Theorems we mention only the vertex labels, the reader can fulfill the proof as we did in the previous Theorems.

The graph obtained from a gear graph G_m by attaching n pendant points to each vertex between the vertices of the rim of the wheel, will be denoted by $G_{m,n}$.

Theorem 2.3. The graph $G_{m,n}$ is odd graceful when $m \geq 3$, $n \geq 1$.
Proof.
• If m is even, m ≥ 4
Let $G_{m,n}$ be described as indicated in Figure 9.

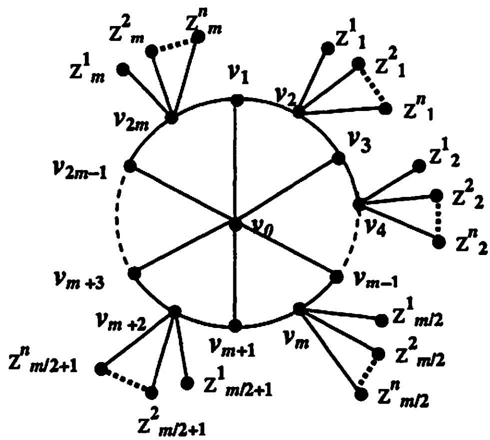

Figure 9.

The number of edges of the graph $G_{m,n}$ is $3m + mn$. We define the labeling function $f : V(G_{m,n}) \rightarrow \{0,1,2,\dots, 2(3m+mn) - 1\}$ as follows :

$$f(v_i) = \begin{cases} i - 1 & , i = 1,3,5,\dots, 2m-1 \\ 4m+2mn - i + 1 & , i = 2,4,6,\dots, m \\ 4m + 2mn - i - 1 & , \text{if } i \text{ is even and } m+2 \leq i \leq 2m-2 \end{cases}$$

$f(v_0) = 6m + 2mn - 1$, $f(v_{2m}) = 2m + 2mn - 1$

$f(z^j_1) = f(v_2) - 2j + 1$, $1 \le j \le n$
$f(z^j_2) = f(v_4) - 2n - 2j + 1$, $1 \le j \le n$
$f(z^j_3) = f(v_6) - 4n - 2j + 1$, $1 \le j \le n$
$f(z^j_4) = f(v_8) - 6n - 2j + 1$, $1 \le j \le n$

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$f(z^j_m) = f(v_{2m}) - 2n(m-1) - 2j + 1$, $1 \le j \le n$
$f(z^j_i) = f(v_{2i}) - 2n(i-1) - 2j + 1$, $1 \le i \le m,\ 1 \le j \le n$

• **If m is odd, m ≥ 3**

Let $G_{m,n}$ be described as indicated in Figure 10.

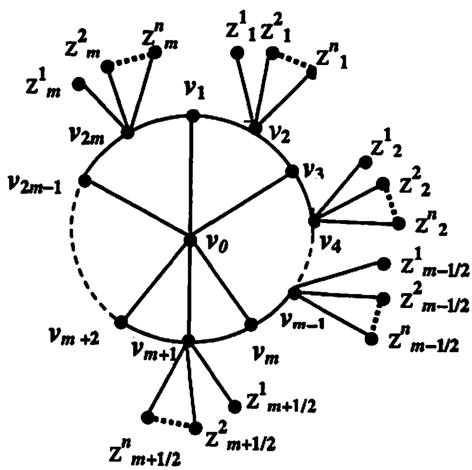

Figure 10.

The number of edges of the graph $G_{m,n}$ is $3m + mn$. We define the labeling function $f : V(G_{m,n}) \to \{0,1,2,\ldots, 2(3m+mn) - 1\}$ as follows :

$f(v_0) = 6m + 2mn - 3,\ f(v_1) = 0,\ f(v_2) = 6m + 2mn - 1,$

$$f(v_i) = \begin{cases} i+1 & ,\ i = 3,5,7,\ldots, 2m-1 \\ 4m + 2mn - i + 3 & ,\ i = 4,6,8,\ldots, m-1 \\ 4m + 2mn - i + 1 & ,\ \text{if } i \text{ is even and } m+1 \le i \le 2m \end{cases}$$

$f(z^j_1) = f(v_2) - 2j + 1$, $1 \le j \le n$
$f(z^j_2) = f(v_4) - 2n - 2j + 1$, $1 \le j \le n$
$f(z^j_3) = f(v_6) - 4n - 2j + 1$, $1 \le j \le n$

$f(z^j_4) = f(v_8) - 6n - 2j + 1 \qquad , 1 \le j \le n$

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$f(z^j_m) = f(v_{2m}) - 2n\,(m-1) - 2j + 1 \qquad , 1 \le j \le n$

$f(z^j_i) = f(v_{2j}) - 2n\,(i-1) - 2j + 1 \qquad , 1 \le i \le m \ , 1 \le j \le n$

Theorem 2.4. The graphs $C_n \odot \overline{K_w}$ are odd graceful for $w \ge 1$, n is even, $n \ge 4$.
Proof. Let $C_n \odot \overline{K_w}$ be described as indicated in Figure 10.

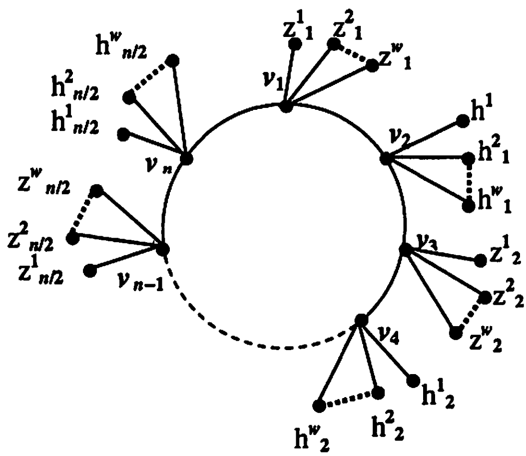

Figure 10.

The number of edges of the graph $C_n \odot \overline{K_w}$ is $n + wn$. We define the labeling function $f: V(C_n \odot \overline{K_w}) \to \{0,1,2,\ldots, 2(n+wn) - 1\}$ as follows :

- **If n is even, n ≡ 2 (mod 4)**

$$f(v_i) = \begin{cases} i-1 & , i = 1,3,5,\ldots, n/2 \\ i+1 & , \text{if } i \text{ is odd and } n/2 + 2 \le i \le n-1 \\ 2n+2wn - i + 1 & , i = 2,4,6,\ldots, n-2 \end{cases}$$

$f(v_n) = n + 2wn - 1, \ f(z^1_1) = 2wn + 1, \quad f(z^j_1) = 2j - 3 \ , 2 \le j \le w$

$f(z^j_2) = f(v_3) + 2w + 2j - 3$, $1 \le j \le w$
$f(z^j_3) = f(v_5) + 4w + 2j - 3$, $1 \le j \le w$
$f(z^j_4) = f(v_7) + 6w + 2j - 3$, $1 \le j \le w$

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$f(z^j_{n/2}) = f(v_{n-1}) + 2w\,(n/2-1) + 2j - 3$, $1 \le j \le w$
$f(z^j_i) = f(v_{2i-1}) + 2w\,(i-1) + 2j - 3$, $1 \le j \le w$, $2 \le i \le n/2$

and
$f(h^j_1) = f(v_2) - wn - 2j + 3$, $1 \le j \le w$
$f(h^j_2) = f(v_4) - wn - 2w - 2j + 3$, $1 \le j \le w$
$f(h^j_3) = f(v_6) - wn - 4w - 2j + 3$, $1 \le j \le w$
$f(h^j_4) = f(v_8) - wn - 6w - 2j + 3$, $1 \le j \le w$

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$f(h^j_{n/2}) = f(v_n) - wn - 2w\,(n/2-1) - 2j + 3$, $1 \le j \le w$

• **If n is even, n ≡ 0 (mod 4)**
Rosa [6] showed that the n-cycle has an α -labeling if and only if $n \equiv 0 \pmod 4$.
By Theorem 1.6 $C_n \odot \overline{K}_w$ is odd graceful, when $n \equiv 0 \pmod 4$.

Theorem 2.5. The graph C_m^n is shown in Figure 11. It is odd graceful, when m is even and $m \ge 4$.

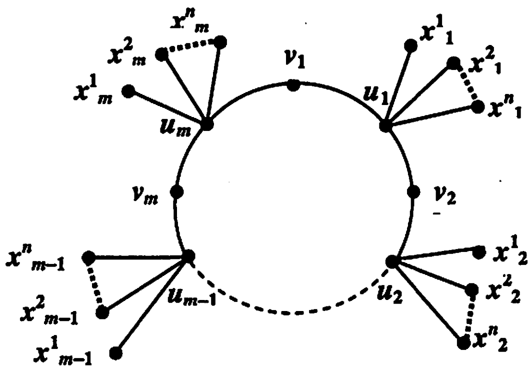

Figure 11.

Proof. The number of edges of the graph C_m is $2m + mn$. We define the following function $f: V(C_m^n) \rightarrow \{0, 1,2,3,\ldots, 2(2m+mn)-1\}$ as follows :

$$f(v_i) = 2i-2 \quad , i = 1,2,3,\ldots, m$$

$$f(u_i) = \begin{cases} 4m+2mn-2i+1 & , i = 1,2,3,\ldots, m/2 \\ 4m+2mn-2i-1 & , m/2+1 \le i \le m \end{cases}$$

$$f(x_1^i) = 4m+2mn-2i \quad , 1 \le i \le n$$
$$f(x_2^i) = 4m+2mn-2i-2(n+1) \quad , 1 \le i \le n$$

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$$f(x_{m/2}^i) = 4m+2mn-2i-2(m/2-1)(n+1) \quad , 1 \le i \le n$$

and

$$f(x_{m/2+1}^i) = 3m+mn-2(i+1) \quad , 1 \le i \le n$$
$$f(x_{m/2+2}^i) = 3m+mn-2(i+1)-2(n+1) \quad , 1 \le i \le n$$
$$f(x_{m/2+3}^i) = 3m+mn-2(i+1)-4(n+1) \quad , 1 \le i \le n$$

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$$f(x_m^i) = 4m+2mn+2n-2i-2m(n+1) \quad , 1 \le i \le n$$

The graph $C^*_{n,m}$ is a graph obtained by identifying an endpoint of a star S_m with a vertex of a cycle C_n.

Theorem 2.6. The graph $C^*_{n,m}$ is odd graceful when n is even, $n \ge 4$, $m \ge 3$ and it is α-graph if $n \equiv 0 \pmod 4$, $m \ge 3$.
Proof. Let $C^*_{n,m}$ be described as indicated in Figure 12.

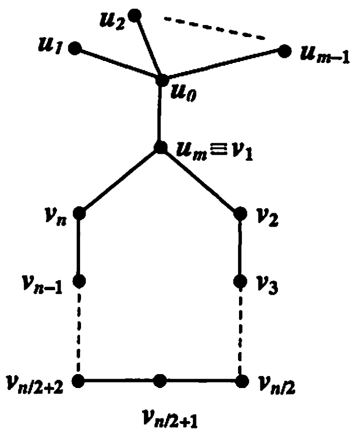

Figure 12.

The number of edges of the graph $C^*_{n,m}$ is $m + n$. We define the labeling function $f : V(C^*_{n,m}) \to \{0,1,2,\ldots, 2(m+n)-1\}$ as follows:
$f(u_0) = 0,\ f(u_i) = 2m + 2n - 2i + 1 \ ,\ 1 \le i \le m.$

- **If n ≡ 0 (mod 4)**

$$f(v_j) = \begin{cases} j & \text{, if } j \text{ is even and } 2 \le j \le n/2 \\ j+2 & \text{, if } j \text{ is even and } n/2+2 \le j \le n \\ 2n-j+2 & \text{, if } j \text{ is odd and } 3 \le j \le n-1 \end{cases}$$

- **If n ≡ 2 (mod 4)**

$$f(v_j) = \begin{cases} j & \text{, if } j \text{ is even and } 2 \le j \le n-2 \\ 2n-j+2 & \text{, if } j \text{ is odd and } 3 \le j \le n/2 \\ 2n-j & \text{, if } j \text{ is odd and } n/2+2 \le j \le n-1 \end{cases}$$

$f(v_n) = n+2$

In the case $n \equiv 0$ (mod 4), $m \ge 3$ there exists $k = f(v_n)$ such that, $k = \max\{f(x) : f(x) \text{ is even}, x \in A\} < \min\{f(x) : f(x) \text{ is odd}, x \in B\}$. By Theorem 1.4, the graph $C^*_{n,m}$ is an α- graph. Similarly, for the case $n \equiv 2$ (mod 4).

A **butterfly** $BF_{n,m}$,consists of two even cycles of the same order n sharing a common vertex with an arbitrary number of pendant edge m attached at the common vertex as indicated in Figure 13.

Theorem 2.7. The butterfly graphs are odd graceful.

Proof.

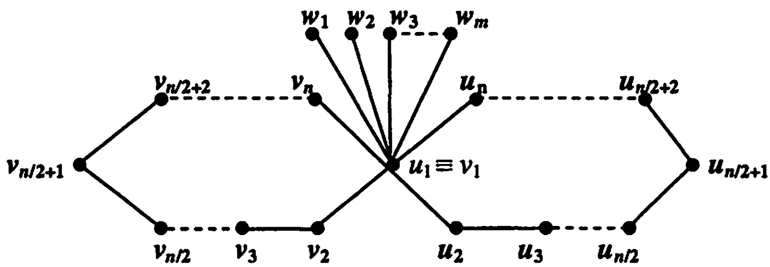

Figure 13.

The number of edges of the butterfly graph is $2n + m$. We define the following function $f: V(BF_{n,m}) \rightarrow \{0,1,2,3,\ldots, 2(2n+m)-1\}$ as follows :

$f(v_1) \equiv f(u_1) = 0$, $f(w_j) = 4n + 2m - 2j + 1$, $j = 1,2,3,\ldots m$

- **If $n \equiv 0 \pmod 4$**

$$f(u_i) = \begin{cases} i-1 & , i = 2,4,6,\ldots, n/2 \\ i+1 & , \text{if } i \text{ is even and } n/2+2 \le i \le n \\ 2n-i+3 & , i = 3,5,7,\ldots, n-1 \end{cases}$$

$$f(v_i) = \begin{cases} i-1 & , i = 3,5,7,\ldots, n-1 \\ 4n-i+1 & , i = 2,4,6,\ldots, n/2 \\ 4n-i-1 & , \text{if } i \text{ is even and } n/2+2 \le i \le n \end{cases}$$

- **If $n \equiv 2 \pmod 4$**

$f(u_2) = 1$, $f(v_n) = 3n - 3$

$$f(u_i) = \begin{cases} i+1 & , \text{if } i \text{ is even and } 4 \le i \le n \\ 2n-i+5 & , \text{if } i \text{ is odd and } 3 \le i \le n/2 \\ 2n-i+3 & , \text{if } i \text{ is odd and } n/2+2 \le i \le n-1 \end{cases}$$

$$f(v_i) = \begin{cases} i-1 & , i = 3,5,7,\ldots, n-1 \\ 4n-i+1 & , i = 2,4,6,\ldots, n/2+1 \\ 4n-i-1 & , \text{if } i \text{ is even and } n/2+2 \le i \le n-2 \end{cases}$$

The graph $C_m^{*}{*}$ consists of two cycles of the same order m joined by a bridge as shown in Figure 14.

Theorem 2.8. The graphs $C_m^{*}{*}$ are odd graceful when m is even and $m \geq 4$.
Proof.

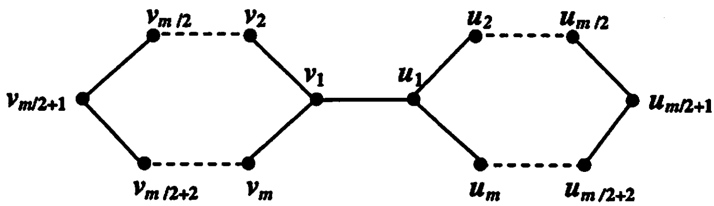

Figure 14.

The number of edges of the graph $C_m^{*}{*}$ is $2m + 1$. We define the following function $f : V(C_m^{*}{*}) \rightarrow \{0,1,2,3,\ldots, 4m+1\}$ as follows :

- **If $m \equiv 0 \pmod 4$**

$f(v_1) = 0$

$$f(v_i) = \begin{cases} i-1 & , i = 2,4,6,\ldots, m/2 \\ i+1 & , \text{if } i \text{ is even and } m/2+2 \leq i \leq m \\ 2m-i+3 & , i = 3,5,7,\ldots, m-1 \end{cases}$$

$$f(u_i) = \begin{cases} i & , i = 2,4,6,\ldots, m/2 \\ i+2 & , \text{if } i \text{ is even and } m/2+2 \leq i \leq m \\ 4m-i+2 & , i = 1,3,5,7,\ldots, m-1 \end{cases}$$

- **If $m \equiv 2 \pmod 4$**

$$f(u_i) = \begin{cases} i & , i = 2,4,6,\ldots, m-2 \\ 4m-i+2 & , i = 1,3,5,\ldots, m/2 \\ 4m-i & , \text{if } i \text{ is odd and } m/2+2 \leq i \leq m-1 \end{cases}$$

$f(u_m) = m+2, \ f(v_1) = 0, \ f(v_2) = 1$

$$f(v_i) = \begin{cases} i+1 & , i = 4,6,8,\ldots, m \\ 2m+5-i & , i = 3,5,7,\ldots, m/2 \\ 2m+3-i & , \text{if } i \text{ is odd and } m/2+2 \leq i \leq m-1 \end{cases}$$

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