# The existence of block-transitive 2-$(q, 8, 1)$designs with $q$ a prime power *


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**Abstract**

In this paper, we investigate the existence of 2-$(v, 8, 1)$ designs admitting a block-transitive automorphism group $G \leq A\Gamma L(1, q)$. Using Weil's theorem on character sums, the following theorem is proved: if a prime power $q$ is large enough and $q \equiv 57 \pmod{112}$ then there is always a 2-$(v, 8, 1)$ design which has a block-transitive, but non flag-transitive automorphism group $G$.



**2000 Mathematics Subject Classification:** 05B05, 20B25

**Keywords:** Design; block-transitive; Weil's theorem


## 1 Introduction

A 2-$(v, k, 1)$ designs is a pair $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a $v$-set of points and $\mathcal{B}$ is a collection of $k$-subset of $\mathcal{X}$ (call blocks) such that any 2-subset of $\mathcal{X}$ is incident exactly with one block. We consider automorphisms of $\mathcal{D}$ as pairs of permutations on $\mathcal{X}$ and $\mathcal{B}$ which preserve incidence. An automorphism group $G$ of $\mathcal{D}$ is a group whose elements are automorphisms of $\mathcal{D}$ and call it block transitive if it acts transitively on the block set $\mathcal{B}$ of $\mathcal{D}$.

In recent years, different author devotes to the classification of the pair $(G, \mathcal{D})$ where $G$ is block-transitive on a design $\mathcal{D}$ of a given block size $k$ (see[1, 2, 5, 6, 8, 9]). According to this classification, these pair fall into three classes, that in which $G$ is unsolvable and is flag-transitive (such examples are included in [1]), that in which $G \leq A\Gamma L(1.q)$, and that in which that $G$ is solvable and is of small order. However, little is known about latter two classes.

In this paper, we investigate the existence of the pairs $(G, \mathcal{D})$ such that $\mathcal{D}$ is a 2-$(v, k, 1)$ design, $G$ is a one-dimensional affine group acting on $\mathcal{D}$ as an automorphism group block-transitively, but non flag-transitively. Using Weil's theorem on character sums, we prove that for the case that $\mathcal{D}$ is a 2-$(v, 8, 1)$


*Supported by the Nation Nature Science Foundation China (Grant No. 10871205 and 10971252).



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design, a pair $(G, \mathcal{D})$ always exists if $q$ is sufficiently large. The main result is the following theorem.

**Main Theorem:** *Let $q$ be a prime power with $q \equiv 57 \pmod{112}$. Suppose $q$ is sufficiently large, then there exists a 2-$(q, 8, 1)$ design $\mathcal{D}$ which has a block-transitive, but non flag-transitive automorphism group $G$, moreover, $G \leq A\Gamma L(1, q)$.*

# 2 Notation and Preliminaries

In this section, we give some notation and preliminaries which will be used throughout this paper. We always assume that $q$ is a prime power such that $q \equiv k(k-1)+1 \pmod{2k(k-1)}$. Let $GF(q)$ denote the finite field of $q$ elements, $\theta$ a generating element of the multiplicative group $GF(q)^\times$. Let $M = \langle\theta^{k(k-1)/2}\rangle$, $L = \langle\theta^{k(k-1)}\rangle$ be two subgroups of $GF(q)^\times$, then $[GF(q)^\times : M] = k(k-1)/2$ and $[M : L] = 2$.

Given $\alpha \in L$ and $\sigma \in GF(q)^\times$, define a map $g_{\alpha\sigma}$ as follows: $g_{\alpha\sigma} : x \mapsto \alpha x + \sigma$, $\forall x \in GF(q)$. Let $G = GF(q)^+ \rtimes L$ denote the set of such map, then $G$ is a subgroup of $AGL(1, q)$ of order $q(q-1)/k(k-1)$.

Let $B = \{\beta_1, \beta_2, \cdots, \beta_k\}$ be a subset of $GF(q)$ consisting of $k$ different elements. Define $B^- = \{\beta_j - \beta_i | 1 \leq i < j \leq k\}$, clearly $|B^-| = k(k-1)/2$. For an element $g = g_{\alpha\sigma} \in G$, define $B^g = \{\beta_1^g, \beta_2^g, \cdots, \beta_k^g\}$.

**Lemma 2.1** (see [3]) *$M = L \cup (-L)$, where $-L = \{-\delta | \delta \in L\}$.*

**Lemma 2.2** (see [3]) *Let $B = \{\beta_1, \beta_2, \cdots, \beta_k\}$ be a $k$-subset of $GF(q)$. If $B^-$ is a system of representative of the cosets of $M$ in $GF(q)^\times$, then $\mathcal{D} = (GF(q), B^G)$ is a 2-$(q, k, 1)$ design, and $G$ is block-transitive, but not flag-transitive on $\mathcal{D}$.*

**Lemma 2.3** *Given a finite number if polynomials $a_{10} + a_{11}x + \cdots + a_{1n_1}x^{n_1}$, $a_{20} + a_{21}x + \cdots + a_{2n_2}x^{n_2}, \cdots, a_{m0} + a_{m1}x + \cdots + a_{mn_m}x^{n_m}$ in $C[x]$, if $c_0 + c_1x + \cdots + c_sx^s$ is the product of those polynomials, then*

$$\sum_{j=0}^{s} |c_j| \leq \prod_{i=1}^{m} (|a_{i0}| + |a_{i1}| + \cdots + a_{in_i}|).$$

Weil's theorem on character sums is very impotent for our proof of the main theorem.

**Lemma 2.4** (see [7]) *Let $GF(r)$ be finite field, and $\Psi$ a multiplicative character of $GF(r)$ of order $m > 1$. Suppose that $f(x) \in GF(r)[x]$ is a monic polynomial of positive degree, and that $f(x)$ is not a $m$th power of a polynomial. Let $d$ denote the number of distinct roots of $f(x)$ in its splitting field over $GF(r)$. Then for any element $\alpha \in GF(r)$,*

$$| \Sigma_{x \in GF(r)} \Psi(\alpha f(x)) | \leq (d-1)\sqrt{r}.$$

# 3 The proof of the main theorem

In this section, we will prove our main theorem, and the methods are similar to [3]. We always assume that $q$ is a prime power such that $q \equiv 57 \pmod{112}$, $\theta$ a generating element of the multiplicative group $GF(q)^{\times}$. Let $M = \langle\theta^{28}\rangle$, $B = \{0,1,\beta,\beta^2,\beta^3,\beta^4,\beta^5,\beta^6\}$, where $\beta \in GF(q)^{\times}$. Now $B^- = \{1,\beta,\beta^2,\beta^3,\beta^4,\beta^5,\beta^6\} \cup \{\beta^j - \beta^i | 0 \le i < j \le 6\}$, and the elements of $B^-$ are listed as follows:

$$
\begin{array}{lllllll}
1 & \beta-1 & \beta^2-1 & \beta^3-1 & \beta^4-1 & \beta^5-1 & \beta^6-1 \\
\beta & \beta(\beta-1) & \beta(\beta^2-1) & \beta(\beta^3-1) & \beta(\beta^4-1) & \beta(\beta^5-1) & \\
\beta^2 & \beta^2(\beta-1) & \beta^2(\beta^2-1) & \beta^2(\beta^3-1) & \beta^2(\beta^4-1) & & \\
\beta^3 & \beta^3(\beta-1) & \beta^3(\beta^2-1) & \beta^3(\beta^3-1) & & & \\
\beta^4 & \beta^4(\beta-1) & \beta^4(\beta^2-1) & & & & \\
\beta^5 & \beta^5(\beta-1) & & & & & \\
\beta^6 & & & & & &
\end{array}
\tag{1}
$$

**Lemma 3.1** *Let $B = \{0,1,\beta,\beta^2,\beta^3,\beta^4,\beta^5,\beta^6\}$, where $\beta \in GF(q)^{\times}$ satisfies the following conditions:*

$$
\begin{cases}
\beta \in M\theta \cup M\theta^- \\
\beta^{21}(\beta-1) \in M \\
\beta^{22}(\beta+1) \in M \\
\beta^{7}(\beta^2-1) \in M \\
\beta^{17}(\beta^2+\beta+1) \in M \\
\beta^{25}(\beta^2-\beta+1) \in M \\
\beta^{10}(\beta^4+\beta^3+\beta^2+\beta+1) \in M
\end{cases}
\tag{2}
$$

*Then $B^-$ is a system of representatives of the cosets of $M$ in $GF(q)^{\times}$.*

**Proof.** Let $\theta$ be a generating element of $GF(q)^{\times}$, then the cosets of $M$ in $GF(q)^{\times}$ are $M\theta^j$, where $j = 1,2,\cdots,27$. If $\beta \in M\theta$ (similiarly, $\beta \in M\theta^-$), then $(\beta-1) \in M\theta^7$,$(\beta+1) \in M\theta^6$,$(\beta^2+1) \in M\theta^8$,$(\beta^2+\beta+1) \in M\theta^{11}$, $(\beta^2-\beta+1) \in M\theta^3$, $(\beta^4+\beta^3+\beta^2+\beta+1) \in M\theta^{18}$.

Therefore, the elements in the first column of (1) run over $M\theta^j (j = 0,1,\cdots,6)$, the elements in the second column of (1) run over $M\theta^j (j = 7,\cdots,12)$, the elements in the third column of (1) run over $M\theta^j (j = 13,\cdots,17)$, the elements in the forth column of (1) run over $M\theta^j (j = 18,\cdots,21)$, the elements in the fifth column of (1) run over $M\theta^j (j = 22,23,24)$, the elements in the sixth column of (1) run over $M\theta^j (j = 22,23,24)$, and finally $\beta^6 - 1 = (\beta-1)(\beta^2+\beta+1(\beta-1)(\beta^2-\beta+1) \in M\theta^{27}$.□

Let $\Omega = \{\beta | \beta \in GF(q)\}$ satisfy conditions (2). To prove our main theorem, it suffices to show that if $q$ is larger enough then $|\Omega| > 0$ by Lemma 2.2.

Let $\alpha = e^{2\pi/28}$ be a 28th unity root, for any integer $j$, define $\Psi(\theta^j) = \alpha^j$.

Since $q \equiv 57 \pmod{112}$, $\Psi$ is a character of order 28 on $GF(q)$, and so $\Psi^- = \Psi^{27}$. As usual, define $\Psi(0) = 0, \Psi^0(0) = 1$.

Let $f_1(x) = x^{21}(x-1)$, $f_2(x) = x^{22}(x+1)$, $f_3(x) = x^7(x^2+1)$, $f_4(x) = x^{17}(x^2+x+1)$, $f_5(x) = x^{25}(x^2-x+1)$, $f_6(x) = x^{10}(x^4+x^3+x^2+x+1)$.

For $j \in \{1,2,3,4,5,6\}$, we have

$$1 + \Psi(f_j(x)) + \cdots + \Psi^{27}(f_j(x)) = \begin{cases} 28, & \text{if} \quad f_j(x) \in M \\ 1, & \text{if} \quad f_j(x) = 0 \\ 0, & \qquad \text{otherwise} \end{cases}. \tag{3}$$

Let $\mathbb{I} = \{3,5,7,9,11,13\}$,

$$F(x) = [2 - \Psi^{14}(x) - \Psi^{-14}(x)] \prod_{j \in \mathbf{I}} [\Psi(x) + \Psi^{-1}(x) - \alpha^j - \alpha^{-j}]. \tag{4}$$

Note that if $x \in M\theta^j, 2|j$, then $\Psi^{14}(x) = \Psi^{-14}(x) = 1$, and if $x \in M\theta^j \cup M\theta^{-j}$ then $\Psi(x) + \Psi^{-1}(x) = \alpha^j + \alpha^{-j}$. Therefore,

$$F(x) = \begin{cases} F(\theta), & \text{if} \quad x \in M\theta \cup M\theta^- \\ F(0), & \text{if} \quad x = 0 \\ 0, & \qquad \text{otherwise} \end{cases}. \tag{5}$$

Write $b = F(\theta)$. A direct calculation shows that

$$b = 2 \prod_{j \in \mathbf{I}} (2\cos\frac{\pi}{14} - 2\cos\frac{j\pi}{14}) \approx 62.9154.$$

Let

$$H(x) = F(x) \prod_{j=1}^{6} [1 + \Psi(f_j(x)) + \cdots + \Psi^{27}(f_j(x))], \tag{6}$$

and consider the sum $S = \sum_{x \in GF(q)} H(x)$. We partition the set $GF(q)$ into three disjoint parts $GF(q) = \Omega \cup \Omega_1 \cup \Omega_2$, $\Omega_1 = \{\beta | f_j(\beta) = 0 \text{ for some } j\}$, $\Omega_2 = GF(q) - (\Omega \cup \Omega_1)$. Clearly, $\Omega_1 = \{0, \pm 1, \beta | \beta^2 + 1 = 0, \beta^2 + \beta + 1 = 0, \beta^2 - \beta + 1 = 0, \textit{ or } \beta^4 + \beta^3 + \beta^2 + 1 = 0\}$. So $|\Omega| \leq 13$. Now

$$S = \sum_{x \in \Omega} H(x) + \sum_{x \in \Omega_1} H(x) + \sum_{x \in \Omega_2} H(x). \tag{7}$$

We know if $x \in \Omega$, then $H(x) = b \cdot 28^6$, if $x \in \Omega_2$, then $H(x) = 0$. Therefore,

$$S = 28^6 b|\Omega| + \sum_{x \in \Omega_1} H(x). \tag{8}$$

On the other hand,

$$S = H(0) + \sum_{x \in GF(q)^\times} H(x). \tag{9}$$

For $x \neq 0, \Psi(x)\Psi^-(x) \equiv 1$ holds. Hence $F(x)$ can be written as

$$F(x) = c_0 + c_1\Psi(x) + c_2\Psi^2(x) + \cdots + c_{27}\Psi^{27}(x) \tag{10}$$

and

$$H(x) = c_0 + \sum_{(n,n_1,\cdots,n_6)} c_j\Psi^n\Psi^{n_1}(f_1)\cdots\Psi^{n_6}(f_6) = c_0 + \sum_{(n,n_1,\cdots,n_6)} c_j\Psi(x^n f_1^{n_1}\cdots f_6^{n_6})$$

then the sum in (8) becomes that

$$S = H(0) + \sum_{x\in GF(q)^\times} c_0 + \sum_{(n,n_1,\cdots,n_6)} \sum_{x\in GF(q)^\times} c_j\Psi(x^n f_1^{n_1}\cdots f_6^{n_6}). \tag{11}$$

where $\Psi, f_1, \Psi(f_1)$ denote $\Psi(x), f_1(x), \Psi(f_1(x))$ respectively, etc, and $(n, n_1, \cdots, n_6)$ runs over $\{0,1,\cdots,27\}^7 - \{0,\cdots,0\}$. Equating (8) and (11), we get that

$$28^6 b|\Omega| = c_0(q-1) + S_1 + S_2 \tag{12}$$

where $S_1 = H(0) - \sum_{x\in\Omega_1} H(x)$, and

$$S_2 = \sum_{x\in GF(q)^\times} c_0 + \sum_{(n,n_1,\cdots,n_6)} \sum_{x\in GF(q)^\times} c_j\Psi(x^n f_1^{n_1}\cdots f_6^{n_6}).$$

Note that $|\Psi(x)| \leq 1$, so from (6) follows that $|H(x)| \leq 4^{11}\cdot 28^6$, hence

$$|S_1| = |H(0) - \sum_{x\in\Omega_1} H(x)| \leq (|\Omega_1|+1)4^{10}\cdot 28^6. \tag{13}$$

By applying Lemma 2.3 to (4), the coefficients in (10) satisfy that $|c_j| \leq 4^{10}, j = 0,1,\cdots,27$. Notice $c_0 \neq F(0)$, it follows from (4) that

$$\begin{aligned} F(x) &= [2 - \Psi^{14}(x) - \Psi^{-14}(x)]\textstyle\prod_{j\in I}[\Psi(x) + \Psi^{-1}(x) - \alpha^j - \alpha^{-j}] \\ &= [2 - \Psi^{14}(x) - \Psi^{-14}(x)]\textstyle\prod_{j\in I}[\Psi(x) + \Psi^{-1}(x) - 2\cos j\pi/14] \\ &= [2 - \Psi^{14}(x) - \Psi^{-14}(x)]\cdot[e_0 + \textstyle\sum_{i=1}^6 e_i(\Psi + \Psi^-)^i] \end{aligned}$$

where

$e_0 = \prod_{j\in I} 2\cos\pi j/14 \approx -1.92388\cdot 10^{-7}$
$e_2 = \sum_{j_1<\cdots<j_4; j_1,\cdots,j_4\in I} 16\cos\pi j_1/14 \times \cdots \times \cos\pi j_4/14 \approx 1.84117$
$e_4 = \sum_{j_1<j_2; j_1,j_2\in I} 4\cos\pi j_1/14 \times \cos\pi j_2/14 \approx -3.19806$
$e_6 = 1.$

Therefore

$$c_0 = 2\times[e_0 + e_2C_2^1 + e_4C_4^2 + e_6C_6^3] \approx 8.98792, \tag{14}$$

Now $x^n f_1^{n_1}\cdots f_6^{n_6}$ has at most 14 distinct roots in any extension field of $GF(q)$. Applying Lemma 2.4, for $(n, n_1, \cdots, n_6) \in \{0,1,\cdots,27\}^7 - \{0,\cdots,0\}$, we have

$$|\sum_{x\in GF(q)^\times} c_j\Psi(x^n f_1^{n_1}\cdots f_6^{n_6})| \leq |c_j|(14-11)\sqrt{q} \leq 13\cdot 4^{10}\sqrt{q}.$$

Hence

$$|S_2| = |\sum_{(n,n_1,\cdots,n_6)} \sum_{x\in GF(q)^\times} c_j\Psi(x^n f_1^{n_1}\cdots f_6^{n_6})| \leq 13\cdot 4^{10}\cdot 28^7\sqrt{q}. \qquad (15)$$

From (12)-(15), we get

$$\begin{aligned} 28^6 b|\Omega| &\geq c_0(q-1) - 15\cdot 4^{10}\cdot 28^6 - 13\cdot 4^{10}\cdot 28^7\sqrt{q} \\ &> c_0(q-1) - 13\cdot 4^{10}\cdot 28^7(\sqrt{q}+1) \\ &= c_0(\sqrt{q}+1)(\sqrt{q}-1-\tfrac{13\cdot 4^{10}\cdot 28^7}{c_0}) \end{aligned}$$

Therefore, if $q > 5.2856\cdot 10^{32}$, then $|\Omega| > 0$, which implies that there is $\beta\in GF(q)^\times$ satisfying (2), as required. □

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# On chromatic number of graphs without certain induced subgraphs

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**Abstract**

Gyárfás conjectured that for a given forest $F$, there exists an integer function $f(F, \omega(G))$ such that $\chi(G) \leq f(F, \omega(G))$ for any $F$-free graph $G$, where $\chi(G)$ and $\omega(G)$ are respectively, the chromatic number and the clique number of $G$. Let $G$ be a $C_5$-free graph and $k$ be a positive integer. We show that if $G$ is $(kP_1 + P_2)$-free for $k \geq 2$, then $\chi(G) \leq 2\omega^{k-1}\sqrt{\omega}$; if $G$ is $(kP_1 + P_3)$-free for $k \geq 1$, then $\chi(G) \leq \omega^k\sqrt{\omega}$. A graph $G$ is $k$-divisible if for each induced subgraph $H$ of $G$ with at least one edge, there is a partition of the vertex set of $H$ into $k$ sets $V_1, \cdots, V_k$ such that no $V_i$ contains a clique of size $\omega(G)$. We show that a $(2P_1+P_2)$-free and $C_5$-free graph is 2-divisible.

**Keywords:** $F$-free graph, Perfect graph, Divisibility.

## 1 Introduction

All graphs considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notation. Let $G = (V(G), E(G))$ be a graph, and let $S$ be a nonempty subset of $V(G)$. The subgraph of $G$ induced by $S$, denoted $G[S]$, is the subgraph of $G$ with vertex set $S$, in

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which two vertices are adjacent if and only if they are adjacent in $G$. If $G[S]$ has no edge, $S$ is called an independent set; if $G[S]$ is a complete graph, then $S$ is called a clique. The maximum cardinality of an independent set is called the independence number of $G$, denoted $\alpha(G)$; the maximum cardinality of a clique is called the clique number of $G$, denoted $\omega(G)$. The chromatic number of $G$, denoted $\chi(G)$, is the minimum number $k$ such that the vertices of $G$ can be partitioned into $k$ independent sets. In general, there is no upper bound on the chromatic number of a graph in terms of its clique number, since there are graphs containing no triangle, but having arbitrarily large chromatic number.

A graph $G$ is called perfect if $\chi(H) = \omega(H)$ for each induced subgraph $H$ of $G$. There are graph classes which can be characterized by forbidden induced subgraphs, e.g. cographs (i.e. $P_4$-free graphs), chordal graphs, split graphs, threshold graphs. Berge conjectured that a graph $G$ is perfect if and only if neither $G$ nor its complement $\overline{G}$ contains an induced odd cycle of order at least five. This famous conjecture, known as Strong Perfect Graph Conjecture, has recently been solved by Chudnovsky, Robertson, Seymour and Thomas [2].

Gyárfás [4] has introduced the concept of $\chi$-bound functions. Here, a family $\mathbf{G}$ of graphs is called $\chi$-bound with $\chi$-binding function $f$, if $\chi(H) \leq f(\omega(H))$ holds whenever $H$ is an induced subgraph of $G \in \mathbf{G}$. For a given graph $F$, a graph $G$ is $F$-free if it does not contain an induced subgraph isomorphic to $F$. Gyárfás proposed a conjecture: if $F$ is a forest, there exist an integer $f(F, \omega)$ such that every $F$-free graph with maximum clique size $\omega$ is $f(F, \omega)$-colorable. This conjecture is only proved in special cases.

Hoàng and McDiarmid [5] recently introduced the notion of $k$-divisible graphs. A $k$-division of a graph $G = (V, E)$ is a partition of the vertex set $V$ into $k$ sets $V_1, \cdots, V_k$ such that no $V_i$ contains a clique of size $\omega(G)$. A graph $G$ is $k$-divisible if each induced subgraph of $G$ with at least one edge has a $k$-division. The least such $k$ is the divisibility number $div(G)$. A strong $k$-division of a graph $G$ is a partition of the vertex set $V$ into $k$ sets $V_1, \cdots, V_k$ such that no $V_i$ contains a maximal clique of G. We shall say that a graph is strongly $k$-divisible if each induced subgraph with no isolated vertices has a strong $k$-division. Obviously, every strongly $k$-divisible graph is $k$-divisible.

We denote the path on $k$ vertices by $P_k$. The graph with vertices $a, b, c, d$

and edges $ab$,$ac$ will be called a co-paw. The graph with vertices $a, b, c, d$ and edges $ab$ and $cd$ is called $2K_2$. The graph with vertices $a, b, c, d$ and edge $ab$ will be called a $2P_1 + P_2$. Wagon [6] proved that for any $2K_2$-free graph $G$, $\chi(G) \leq \frac{1}{2}(\omega(G)+1)\omega(G)$. Hoàng and McDiarmid [5] showed that for any $C_5$-free and co-paw free graph $G$, $\chi(G) \leq \omega(G)^{\frac{3}{2}}$.

Let $G$ be a $C_5$-free graph and $k$ be a positive integer. We show that if $G$ is $(kP_1 + P_2)$-free for $k \geq 2$, then $\chi(G) \leq 2\omega^{k-1}\sqrt{\omega}$; if $G$ is $(kP_1 + P_3)$-free for $k \geq 1$, then $\chi(G) \leq \omega^k\sqrt{\omega}$. Gravier, Hoàng and Maffray [3] showed that any $(2P_1 + P_2)$-free graph is 3-divisible. But we are interested in the 2-divisible case. Accordingly, we show that a $(2P_1 + P_2)$-free and $C_5$-free graph is 2-divisible in this paper.

# 2 Colorability

Let $R(p, q)$ be the Ramsey function, that is the smallest $m = m(p, q)$ such that all graphs on $m$ vertices contain either an independent set of $p$ vertices or a clique of $q$ vertices. It was pointed out in [4] that for a $(2P_1 + P_2)$-free graph $G$, $\chi(G) \geq \frac{R(3,\omega+1)-1}{2}$. Accordingly, $(2P_1 + P_2)$-free graphs have not linear $\chi$-binding function.

Let $G$ be a graph. For a vertex $x \in V(G)$, $N(x)$ denote the set of neighbors of $x$ and $M(x) = V(G) \setminus (\{x\} \cup N(x))$.

**Theorem 2.1.** *Suppose a graph $G$ contains no induced 5-cycle and no induced $2P_1 + P_2$. Then*
*(1) if $\omega(G) = 2$, $\chi(G) = \omega(G)$,*
*(2) if $\omega(G) \geq 3$, $\chi(G) \leq 2\omega(G)^{\frac{3}{2}}$.*

**Proof.** We can assume that $G$ is connected. First, we prove (1). Since $\omega(G) = 2$, $G$ contains no triangle. Note that the existence of an induced odd cycle of length greater than five would imply that of $2P_1 + P_2$. Combining these to the assumption that $G$ has no odd cycle of length five, $G$ is bipartite, and thus $\chi(G) = 2$.

We shall prove (2) by induction on the number of vertices of $G$. Suppose there is a function $g(\omega)$ such that $\chi(H) \leq g(\omega(H))$ for every proper induced subgraph $H$ of $G$. The function $g$ will be defined later.

We may assume that $\alpha \geq 2$, for otherwise the theorem holds trivially. If for every edge $xy \in E(G)$, $N(x) \setminus (N(y) \cup \{y\}) = \emptyset$, $N(y) \setminus (N(x) \cup \{x\}) = \emptyset$

and $M(xy) = \emptyset$, then $G$ must be a complete graph. If it is not, then there are two non-adjacent vertices, say $u$ and $v$, in $G$. Since $G$ is connected, we can take a neighbor, say $x$, of $u$. We consider the edge $xu$. If $xv \in E(G)$, then $N(x) \setminus (N(u) \cup \{u\}) \neq \emptyset$ since it contains $v$. If $xv \notin E(G)$, then $M(xu) \neq \emptyset$ since it contains $v$. A contradiction. So in this case, the result trivially holds.

Now assume that there is an edge $xy$ in $G$ such that at least one of $N(x) \setminus (N(y) \cup \{y\})$, $N(y) \setminus (N(x) \cup \{x\})$ and $M(xy)$ is not empty. If $(N(x) \setminus (N(y) \cup \{y\})) \neq \emptyset$, put $A = (N(x) \setminus (N(y) \cup \{y\})) \cup M(xy)$ and $B = \{y\} \cup (N(y) \setminus N(x))$. Otherwise, put $A = (N(y) \setminus (N(x) \cup \{x\})) \cup M(xy)$ and $B = \{x\} \cup (N(x) \setminus N(y))$. Obviously, $A, B$ are not empty. Without loss of generality, it suffices to consider the case that $A = (N(x) \setminus (N(y) \cup \{y\})) \cup M(xy)$ and $B = \{y\} \cup (N(y) \setminus N(x))$.

**Claim 1.** $G[A]$ and $G[B]$ are cographs.

**Proof.** By contradiction, suppose $abcd$ is an induced subgraph of $G[A]$ which is isomorphic to $P_4$. Then $G[\{a, b, d, y\}]$ is a $2P_1 + P_2$, a contradiction. So $A$ is $P_4$-free. Similarly, $N(y) \setminus (N(x) \cup \{x\})$ is $P_4$-free, and since $B \setminus \{y\} \subseteq N(y)$, $G[B]$ is $P_4$-free. So $G[A]$ and $G[B]$ are cographs. □

Let $W'$ be a maximum clique in $G[A]$, and $|W'| = s$. For $i = 0, 1, \cdots s$, put $X_i = \{u \in N(xy) : |N(u) \cap W'| = i\}$.

**Claim 2.** $X_0 \cup X_1 \cdots \cup X_{s-2}$ is a clique.

**Proof.** Suppose, on the contrary, that $u$ and $v$ are two non-adjacent vertices in $G[X_0 \cup X_1 \cdots \cup X_{s-2}]$ with $u \in X_i$ and $v \in X_j$.

First assume that $i = j$. By the definition of $X_i$, both $u$ and $v$ have exactly $i$ neighbors in $W'$. If $u$ and $v$ are adjacent to same $i$ vertices of $W'$, then for $i \leq s - 2$, there are two vertices in $W'$ adjacent to neither $u$ nor $v$, which contradicts the assumption that $G$ is $(2P_1 + P_2)$-free. If $u, v$ are not adjacent to same $i$ vertices of $W'$, then, there exist two vertices $u'$ and $v'$ of $W'$ such that $u'$ is adjacent to $u$ and is not adjacent to $v$, and $v'$ is adjacent to $v$ and is not adjacent to $u$. However, $G[\{y, u, v, u', v'\}] \cong C_5$, a contradiction. So $G[X_i]$ is a clique for any $i \in \{0, 1, 2, \cdots, s-2\}$.

Now we consider the case $i \neq j$, and without loss of generality, let $i < j$. If $N(u) \cap W' \subset N(v) \cap W'$, and since $i < j \leq s-2$, then there exists an edge

$ab$ in $W'$ such that both $a$ and $b$ are adjacent to neither $u$ nor $v$. Hence, $G[\{a,b,u,v\}] \cong 2P_1 + P_2$, a contradiction. If $N(u) \cap W' \subsetneq N(v) \cap W'$, there exists an edge $ab$ such that $G[\{y,u,v,a,b\}] \cong C_5$, a contradiction. So $X_0 \cup X_1 \cdots \cup X_{s-2}$ is a clique. □

**Claim 3.** $G[X_{s-1}]$ is a perfect graph.

**Proof.** Obviously, $G[X_{s-1}]$ contains no induced 5-cycles by assumption, and no induced odd cycle of length greater than five, otherwise, it must contain an induced subgraph isomorphic to $2P_1 + P_2$. Next we prove that the complement of $G[X_{s-1}]$ does not contain an induced cycle of length at least five. Suppose, on the contrary, it does such one. Then clearly, $G[X_{s-1}]$ contains an induced subgraph $H$ isomorphic to $P_1 + P_2$, and let $V(H) = \{a,b,c\}$, $bc \in E(H)$. If $N(a) \cap W' \neq N(b) \cap W'$, and since $|N(a) \cap W'| = |N(b) \cap W'| = s-1$, $(N(a) \setminus N(b)) \cap W' \neq \emptyset$, $(N(b) \setminus N(a)) \cap W' \neq \emptyset$. Let $a' \in (N(a) \setminus N(b)) \cap W'$, $b' \in (N(b) \setminus N(a)) \cap W'$. Then $G[\{a,a',b',b,y\}] \cong C_5$. It is not possible. Hence, $N(a) \cap W' = N(b) \cap W'$. Similarly, $N(a) \cap W' = N(c) \cap W'$. However, if let $d \in W' \setminus N(a)$, $G[\{d,a,b,c\}] \cong 2P_1 + P_2$, a contradiction. So neither $G[X_{s-1}]$ nor its complement contains an odd cycle of length greater than three, $G[X_{s-1}]$ is a perfect graph by strong perfect graph theorem. □

Let $\omega(G) = \omega$. Since $y$ is not adjacent to any vertex in $A$, and by Claim 1, $\chi(G[A \cup \{y\}]) = \omega(G[A \cup \{y\}]) = s$. Now, we give two different colorings of $G$. First, we have

$$\chi(G) \leq g_1 = \chi(G[N(xy) \cup (N(y) \setminus N(x))]) + \chi(G[A \cup \{y\}]).$$

Note that $\omega(G[N(xy) \cup (N(y) \setminus N(x))]) \leq \omega - 1$, so by the induction hypothesis,

$$g_1 \leq g(\omega - 1) + s.$$

On the other hand, since $G[A]$ is perfect, $\chi(G[A]) = \omega(G[A]) \leq \omega$. Since $\omega(G[X_s]) \leq \omega(G) - s$,

$$\begin{aligned}\chi(G) \leq g_2 &= \chi(G[X_0 \cup \cdots \cup X_{s-2}]) + \chi(G[A]) + \chi(G[B]) \\ &\quad + \chi(G[X_s]) + \chi(G[X_{s-1}]) \\ &\leq g(\omega - s) + 4\omega.\end{aligned}$$

By setting $g(\omega) = 2\omega^{\frac{3}{2}}$, we have $min(g_1, g_2) \leq g(\omega) = 2\omega\sqrt{\omega}$. Indeed, if $s \leq 2\sqrt{\omega}$ then $g_1 \leq 2(\omega - 1)\sqrt{\omega} + 2\sqrt{\omega} \leq g(\omega)$ and if $s > 2\sqrt{\omega}$ then $g_2 \leq 2(\omega - s)\sqrt{\omega - s} + 4\omega \leq 2(\omega - 2\sqrt{\omega})\sqrt{\omega} + 4\omega \leq g(\omega)$. □

**Corollary 2.2.** *Suppose that the graph $G$ contains no induced 5-cycle. We have*

*(1) If $G$ contains no induced $kP_1 + P_2$ for an integer $k \geq 2$, then $\chi(G) \leq 2\omega^{k-1}\sqrt{\omega}$,*

*(2) If $G$ contains no induced $kP_1 + P_3$ for an integer $k \geq 1$, then $\chi(G) \leq \omega^k\sqrt{\omega}$.*

**Proof.** The proof is made by induction on $\omega + k$.

First we prove (1). If $k = 2$, then $kP_1 + P_2 = 2P_1 + P_2$, and by Theorem 2.1, $\chi(G) \leq 2\omega\sqrt{\omega}$. Now assume that $k \geq 3$ and $G$ contains no induced 5-cycle and no $kP_1 + P_2$. Pick a vertex $x$ from $G$. Then it is clear that $G[M(x)]$ contains no induced 5-cycle and no $(k-1)P_1 + P_2$. So by induction hypothesis that $\chi(G[\{x\} \cup M(x)]) \leq 2\omega^{k-2}\sqrt{\omega}$. On the other hand, $\omega(G[N(x)]) \leq \omega - 1$, and thus $\chi(G[N(x)]) \leq 2(\omega - 1)^{k-1}\sqrt{\omega - 1}$. This gives

$$\begin{aligned}
\chi(G) &= \chi(G[N(x)] + \chi(G[\{x\} \cup M(x)]) \\
&\leq 2(\omega - 1)^{k-1}\sqrt{\omega - 1} + 2\omega^{k-2}\sqrt{\omega} \\
&\leq 2(\omega - 1)\omega^{k-2}\sqrt{\omega} + 2\omega^{k-2}\sqrt{\omega} \\
&\leq 2\omega^{k-1}\sqrt{\omega}.
\end{aligned}$$

Now we show (2). If $k = 1$, $kP_1 + P_3$ is known as the co-paw, and by [5], $\chi(G) \leq \omega\sqrt{\omega}$. Now assume $k \geq 2$, and $G$ contains no induced 5-cycle and no $kP_1 + P_3$. Then clearly, for any vertex $x \in V(G)$, $G[M(x)]$ contains no induced $(k-1)P_1 + P_3$ and no induced 5-cycle, and thus by induction hypothesis, $\chi(G) \leq \chi(G[N(x)]) + \omega^{k-1}\sqrt{\omega}$. On the other hand, since $\omega(G[N(x)]) \leq \omega - 1$, and by induction hypothesis, $\chi(G[N(x)]) \leq (\omega - 1)^k\sqrt{\omega - 1}$.

$$\begin{aligned}
\chi(G) &= \chi(G[N(x)] + \chi(G[\{x\} \cup M(x)]) \\
&\leq (\omega - 1)^k\sqrt{\omega - 1} + \omega^{k-1}\sqrt{\omega} \\
&\leq (\omega - 1)\omega^{k-1}\sqrt{\omega} + \omega^{k-1}\sqrt{\omega} \\
&\leq \omega^k\sqrt{\omega}.
\end{aligned}$$

The proof is complete. □

# 3 Divisibility

**Lemma 3.1.** [5] *Every $C_5$-free non-complete graph is strongly $\alpha$-divisible.*

Observe that for a subset $S$ of the vertices of a graph $G$, if there exists a subset $T \subseteq V(G)$ such that $S \cap T = \emptyset$ and each vertex of $T$ is adjacent to each vertex of $S$ in $G$, then $S$ does not contain any maximal clique of $G$. In the proof of the following theorem, we frequently use this fact.

**Theorem 3.2.** *Suppose that a graph $G$ contains no induced 5-cycle and no induced $2P_1 + P_2$. Then $G$ is strongly 2-divisible.*

**Proof.** We assume $G$ is connected, and $S$ is a maximum independent set of $G$. Take a vertex $x$ from $S$ and let $y$ be a neighbor of $x$. We only need prove that $G$ has a strong 2-division. Let us consider $M(xy)$.

If $M(xy) = \emptyset$, it is easy to see that $(A, B)$ is a strong 2-division of $G$, where $A = N(y) \setminus N(x)$ and $B = N(x)$.

Now suppose $|M(xy)| = 1$ and let $M(xy) = \{a\}$. We claim that at most one of $\{a\} \cup (N(x) \setminus N(y))$ and $\{a\} \cup (N(y) \setminus N(x))$ contains a maximal clique of $G$. Otherwise, let $X$ and $Y$ be maximal cliques of $\{a\} \cup (N(x) \setminus N(y))$ and $\{a\} \cup (N(y) \setminus N(x))$, respectively. Clearly, both $X$ and $Y$ contain $\{a\}$. Pick a vertex $x'$ from $X$. Then it must be adjacent to each vertex of $Y$, since a vertex $y' \in Y$ is not adjacent to $x'$, $G[\{x, x', a, y', y\}] \cong C_5$, a contradiction. So, without loss of generality, assume that $\{a\} \cup (N(x) \setminus N(y))$ does not contain a maximal clique of $G$. Then G has a strong 2-division $A, B$ with $A = \{a\} \cup (N(x) \setminus N(y))$, $B = N(y)$.

If $\alpha \leq 2$, G is strongly 2-divisible by Lemma 3.1. Next we consider the case when $\alpha \geq 3$ and $|M(xy)| \geq 2$.

**Claim 1.** The following statements are true.

(1) Each vertex of $(M(xy) \cup (N(y) \setminus N(x))) \setminus S$ is adjacent to each vertex of $S \setminus \{x\}$ in $G$.

(2) $M(xy)$ is a clique of $G$.

(3) Each vertex of $N(x) \setminus (\{y\} \cup N(y))$ is not adjacent to at most one vertex of $M(xy)$ in $G$.

**Proof.** We show (1) by contradiction. Suppose a vertex $u \in (M(xy) \cup (N(y) \setminus N(x))) \setminus S$ is not adjacent to a vertex, say $v$, of $S \setminus \{x\}$. Since $S$ is a maximum independent set of $G$, $v$ is adjacent to a vertex, say $w$, of $S$. It is obvious that $w \neq x$, and $v, w, x$ are not mutually adjacent since they are elements of $S$. Thus $G[\{u, v, w, x\}] \cong 2P_1 + P_2$, a contradiction. To see $M(xy)$ is a clique, if two vertices $u$ and $v$ of $M(xy)$ are not adjacent in $G$, then $G[\{u, v, x, y\}] \cong 2P_1 + P_2$, a contradiction. Now suppose that two vertices $u, v \in M(xy)$ which are not adjacent to a vertex $w \in N(x) \setminus (\{y\} \cup N(y))$. Then $G[\{u, v, w, y\}] \cong 2P_1 + P_2$, a contradiction. □

Accordingly, $S \subseteq N(y) \setminus N(x)$, or $|S \cap M(xy)| = 1$ and $S \setminus M(xy) \subseteq N(y) \setminus N(x)$.

**Claim 2.** If $S \subseteq N(y) \setminus N(x)$, then

(1) Each vertex of $N(x) \setminus N(y)$ is adjacent to each vertex of $S$ in $G$.

(2) $(A, B)$ is a strong 2-division of $G$, where $A = S \cup N(xy)$ and $B = M(xy) \cup (N(x) \setminus N(y)) \cup (N(y) \setminus S)$.

**Proof.** We prove (1) by contradiction. Suppose a vertex $u \in N(x) \setminus N(y)$ and $v \in S$ are not adjacent in $G$. Clearly, $u \neq y$ and $x \neq v$. By Claim 1, there exists a vertex, say $w$, of $M(xy)$, which is adjacent to both $u$ and $v$ in $G$. Then $G[\{x, y, v, w, u\}] \cong C_5$, a contradiction.

Let us consider (2). Since $A \subseteq N(y)$, it contains no maximal clique of $G$. By (1) of Claim 1 and (1) of Claim 2, each vertex of $B$ is adjacent to each vertex of $S \setminus \{x\}$, and thus $B$ does not contain any maximal clique of $G$ either. □

Now assume that $|S \cap M(xy)| = 1$ and $S \setminus M(xy) \subseteq N(y) \setminus N(x)$. Let $M(xy) \cap S = \{a\}$.

**Claim 3.** Set $C_1 = N(xy) \cap N(a)$, $C_2 = N(xy) \setminus C_1$. Let $D_1 = \{u \in C_1 \colon S \subseteq N(u)\}$. $D_2 = C_1 \setminus D_1$. If $|M(xy)| \geq 3$, then

(1) Each vertex of $N(x) \setminus N(y)$ is adjacent to each vertex of $S \setminus \{a\}$.

(2) Each vertex of $C_2$ is adjacent to each vertex of $S \setminus \{x, a\}$

(3) Each vertex of $D_2$ is adjacent to each vertex of $M(xy)$.

(4) $(A, B)$ is a strong 2-division of $G$, where $A = S \cup D_2$ and $B = (N(x) \setminus N(y)) \cup (N(y) \setminus S) \cup (M(xy) \setminus \{a\}) \cup C_2 \cup D_1$.

**Proof.** We show (1) by contradiction. Suppose that a vertex $u \in N(x) \setminus (N(y) \cup \{y\})$ is not adjacent to a vertex $v \in S \setminus \{x, a\}$. Since $|M(xy)| \geq 3$ and Claim 1, there exists a vertex $w \in M(xy) \setminus \{a\}$, which is adjacent to $u$ and $v$. But, $G[\{x, u, w, v, y\}] \cong C_5$, a contradiction.

To show (2), suppose that a vertex $p \in C_2$ is not adjacent to a vertex $q \in S \setminus \{a, x\}$, then $G[\{a, q, x, p\}] \cong 2P_1 + P_2$, a contradiction.

Now we show (3) by contradiction. Suppose a vertex $u \in D_2$ is not adjacent to a vertex $w \in M(xy)$. By definition of $D_2$, there is a vertex $v \in S \setminus \{a, x\}$ which is not adjacent to $u$. Then $G[\{y, u, w, a, v\}] \cong C_5$, a contradiction.

Now let us prove (4). Note that all vertices of $B$ are adjacent to each vertex of $S \setminus \{x, a\}$. Hence $B$ contains no any maximal clique of $G$. $(S \setminus \{a\}) \cup D_2$ contains no maximal clique of $G$, since each vertex of $(S \setminus \{a\}) \cup D_2 \subseteq N(y)$. It remains to see that $\{a\} \cup D_2$ contains no maximal clique, since each vertex of $M(xy) \setminus \{a\}$ is adjacent to $\{a\} \cup D_2$. □

In what follows, assume that $|M(xy)| = 2$, and let $b$ be the other element of $M(xy)$ different from $a$.

**Claim 4.** Assume that $N(y) \setminus (S \cup N(x) \cup \{x\}) = \emptyset$. Let $B_1 = N(xy) \cap N(b)$ and $B_2 = N(xy) \setminus B_1$. Then $(A, B)$ is a strong 2-division of $G$, where $A = S \cup B_1$ and $B = \{b\} \cup (N(x) \setminus N(y)) \cup B_2$.

**Proof.** First, since $(S \setminus \{a\}) \cup B_1 \subseteq N(y)$ and $\{a\} \cup B_1 \subseteq N(b)$, $A$ contains no maximal clique of $G$. To prove $B$ contains no maximal clique of $G$, by the definition of $B_2$, it suffices to show that both $(N(x) \setminus N(y)) \cup B_2$ and $\{b\} \cup (N(x) \setminus N(y))$ does not contain any maximal cliques of $G$. It is easy to see that $(N(x) \setminus N(y)) \cup B_2$ contains no maximal clique of $G$, since $(N(x) \setminus N(y)) \cup B_2 \subseteq N(x)$. On the other hand, by Claim 1, $S \setminus \{x\} \subseteq N(b)$, and since $G$ contains no induced $C_5$, $N(b) \cap (N(x) \setminus N(y)) \subseteq N(z)$ for any vertex $z \in S \setminus \{a, x\}$. It follows that $\{b\} \cup (N(x) \setminus N(y))$ does not contain any maximal cliques of $G$. □

Now assume that $N(y) \setminus (S \cup N(x) \cup \{x\}) \neq \emptyset$ and we consider two cases based on $M(xy) = \{a, b\}$.

First assume that $\{a, b\}$ is a maximal clique of $G$. Let $A_1 = (N(x) \setminus (N(y) \cup \{y\})) \cap N(a)$, $A_2 = (N(x) \setminus (N(y) \cup \{y\})) \setminus A_1$.

**Claim 5.** If $A_2 \neq \emptyset$, then the following statements hold:

(1) each vertex of $A_2$ is adjacent to each vertex of $S \setminus \{a\}$.

(2) each vertex of $N(xy)$ is adjacent to each vertex of $S \setminus \{a\}$.

(3) each vertex of $A_1$ is adjacent to each vertex $N(y) \setminus (S \cup N(x) \cup \{x\})$.

(4) each vertex of $A_2$ is adjacent to each vertex of $A_1$.

(5) $(A, B)$ is a strong 2-division of $G$ with $A = A_2 \cup (N(y) \setminus S) \cup \{y, b\}$ and $B = S \cup A_1$.

**Proof.** By contradiction, suppose a vertex $u \in A_2$ is not adjacent to a vertex $v \in S \setminus \{a\}$. Then $G[\{x, u, v, a\}] \cong 2P_1 + P_2$, a contradiction. This proves (1).

To show (2), suppose that a vertex $u \in N(xy)$ is not adjacent to a vertex $v \in S \setminus \{a, x\}$. Then $u$ must be adjacent to $a$, for otherwise, $G[\{u, a, v, y\}] \cong 2P_1 + P_2$. Moreover, since $\{a, b\}$ is a maximal clique, $u$ is not adjacent to $b$. So $G[\{y, u, v, a, b\}] \cong C_5$, a contradiction.

Now we prove (3). In fact, if a vertex $u \in A_1$ is not adjacent to a vertex $v \in N(y) \setminus (S \cup N(x) \cup \{x\})$, then by the definition of $A_1$, $ua \in E(G)$, and by Claim 1, $va \in E(G)$. Hence $G[\{u, v, x, y, a\}] \cong C_5$, a contradiction.

Suppose (4) is not true, and a vertex $u \in A_1$ is not adjacent to a vertex $v \in A_2$. Then, since $\{a, b\}$ is a maximal clique of $G$, $u$ is not adjacent to $b$ and by the definition $A_2$ and Claim 1, $v$ is not adjacent to $a$. In this case, $G[\{x, u, v, a, b\}] \cong C_5$, a contradiction.

Finally we prove (5). By Claim 1 and (1-2) of Claim 4, each vertex of $A$ is adjacent to each vertex of $S \setminus \{a, x\}$, $A$ does not contain any maximal clique of $G$. Since $S$ is an independent set, to see that $B$ has not maximal clique it suffices to show that $A_1 \cup \{z\}$ has not a maximal clique of $G$ for each $z \in S$. At first, by (4) of Claim 5, $\{x\} \cup A_1$ does not contain a maximal clique. Secondly, by (3), each vertex of $A_1 \cup (S \setminus \{x\})$ is adjacent to each vertex of $N(y) \setminus (S \cup N(x) \cup \{x\})$, $A_1 \cup (S \setminus \{x\})$ does not contain any maximal cliques of $G$. □

**Claim 6.** Let $C_1 = N(xy) \cap N(a)$, $C_2 = N(xy) \setminus C_1$. If $A_2 = \emptyset$, then the following statements holds:

(1) Each vertex of $C_2$ is adjacent to every vertex of $S \setminus \{a\}$.

(2) Each vertex of $C_1$ is adjacent of each vertex of $S \setminus \{a\}$.

(3) $(A, B)$ is a strong 2-division of $G$ with $A = (N(x) \setminus N(y)) \cup (N(y) \setminus$

$S) \cup \{b\} \cup C_1$ and $B = S \cup C_2$.

**Proof.** Suppose (1) is not true, and let a vertex $u \in C_2$ is not adjacent to a vertex $v \in S \setminus \{a\}$. Then $G[\{x, u, v, a\}] \cong 2P_1 + P_2$, a contradiction.

We show (2) by contradiction. Suppose a vertex $u \in C_1$ is not adjacent to a vertex $v \in S \setminus \{a\}$. By the definition of $C_1$ and the assumption that $\{a, b\}$ is a maximal clique of $G$, $ua \in E(G)$ and $ub \notin E(G)$. By Claim 1, $vb \in E(G)$. Hence $G[\{y, u, v, a, b\}] \cong C_5$, a contradiction. Now we conclude that each vertex of $(N(y) \setminus S) \cup \{y\} \cup C_1$ is adjacent to all of $S \setminus \{a\}$.

Now we show (3). Firstly, $B$ has not maximal clique of $G$, since $(S \setminus \{a\}) \cup C_2 \subseteq N(y)$. Next we prove that $A$ does not contain a maximal clique. It is easy to see that $((N(x) \setminus N(y)) \setminus \{y\}) \cup (N(y) \setminus S) \cup \{b\} \cup C_1 \subseteq N(a)$, $((N(x) \setminus N(y)) \setminus \{y\}) \cup (N(y) \setminus S) \cup \{b\} \cup C_1$ does not contain a maximal clique of $G$. It remains to show that $(N(y) \setminus S) \cup \{y\} \cup C_1$ not contain a maximal clique of $G$. It suffices to prove that each vertex of $C_1$ is adjacent of each vertex of $S \setminus \{a\}$. By (2), Accordingly, $A$ has not maximal clique. □

**Claim 7.** If $\{a, b\}$ is not a maximal clique of $G$, then the following statements holds.

(1) If a vertex $u \in N(x) \setminus (N(y) \cup \{y\})$ is adjacent to both $a$ and $b$, it is adjacent to all of $N(y) \setminus (N(x) \cup \{x\})$. Furthermore, $(A, B)$ is a strong 2-division of $G$, where $A = (N(y) \setminus N(x)) \cup \{a, b\}$ and $B = N(x)$.

(2) If each vertex of $N(x) \setminus (N(y) \cup \{y\})$ are adjacent to exactly one vertex of $\{a, b\}$, then $(A, B)$ is a strong 2-division of $G$ with $A = \{a, b\} \cup (N(x) \setminus N(y))$ and $B = N(y)$.

**Proof.** We prove (1) by contradiction. Suppose that there is a vertex $v \in N(y) \setminus (N(x) \cup \{x\})$ is not adjacent to $u$. We consider two cases. If $v \in S \setminus \{a, x\}$, then by Claim 1, $vb \in E(G)$. But, $G[\{x, y, u, v, b\}] \cong C_5$, a contradiction. If $v \in N(y) \setminus (N(x) \cup S)$, then by Claim 1, $va \in E(G)$. But, $G[\{x, y, u, v, a\}] \cong C_5$, a contradiction. It is clear that both $A$ and $B$ does not a maximal clique of $G$.

Let us show (2). Obviously, $B$ has not a maximal clique of $G$. Next we show that $A$ does not contain a maximal clique of $G$. Since $y$ is adjacent to neither $a$ nor $b$, it only need to prove that neither $N(x) \setminus N(y)$ nor $\{a, b\} \cup (N(x) \setminus (N(y) \cup \{y\}))$ contain a maximal clique of $G$. Since $N(x) \setminus$

$N(y) \subseteq N(x)$, $N(x) \setminus N(y)$ does not contain a maximal clique of $G$. To prove $N(x) \setminus (N(y) \cup \{y\}) \cup \{a, b\}$ has not maximal clique it suffices to prove both $N(x) \setminus (N(y) \cup \{y\}) \cup \{a\}$ and $N(x) \setminus (N(y) \cup \{y\}) \cup \{b\}$ does not contain a maximal clique of $G$, since each vertex of $N(x) \setminus (N(y) \cup \{y\})$ are adjacent to exactly one vertex of $\{a, b\}$. Suppose $N(x) \setminus (N(y) \cup \{y\}) \cup \{a\}$ contains a maximal clique $D$. Then for each vertex $z \in N(y) \setminus (N(x) \cup S)$, $D \subseteq N(z)$, otherwise, $G[\{x, y, d, a, z\}] \cong C_5$, where $d \in D \setminus \{a\}$.

Suppose $N(x) \setminus (N(y) \cup \{y\}) \cup \{b\}$ contains a maximal clique $D'$. Then for each vertex $w \in S \setminus \{x, a\}$, $D' \subseteq N(w)$, otherwise, $G[\{x, y, d', b, w\}] \cong C_5$, where $d' \in D' \setminus \{b\}$. Hence $A$ does not contain a maximal clique of $G$. The proof is complete. □

# References


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