On 2-domination and independence domination numbers of graphs

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Abstract

Let G be a simple graph, and let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if every vertex $v \in V(G) - D$ is adjacent to at least p vertices in D. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. A subset $I \subseteq V(G)$ is an independent dominating set of G if no two vertices in I are adjacent and if I is a dominating set in G. The minimum cardinality of an independent dominating set of G is called *independence domination number* $i(G)$.

In this paper we show that every block-cactus graph G satisfies the inequality $\gamma_2(G) \geq i(G)$ and if G has a block different from the cycle C_4, then $\gamma_2(G) \geq i(G) + 1$. In addition, we characterize all block-cactus graphs G with $\gamma_2(G) = i(G)$ and all trees T with $\gamma_2(T) = i(T) + 1$.

Keywords: Domination; 2-Domination; Independence Domination; Block-Cactus Graph

1. Terminology and Introduction

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and if $S \subseteq V(G)$ then $N(S) = N_G(S)$

denotes the set of vertices that are adjacent to the vertices of S in G. We call $d(v) = d_G(v) = |N(v)|$ the *degree* of v and by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. A vertex of degree one is called a *leaf* and we denote with L_u the set of leaves that are adjacent to the vertex $u \in V(G)$ and with $L(G)$ the set of leaves in G. For a subset $S \subseteq V(G)$ we define by $G[S]$ the subgraph induced by S. The *distance* $d_G(x, y)$ of two vertices x and y of a connected graph G is the length of a path of minimum length with end vertices x and y. The *diameter* $dm(G)$ of a graph G is the longest distance between every two vertices of G.

We write C_n for the *cycle* of length n, K_n for the *complete graph* of order n and $K_{p,q}$ for the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$. A bipartite graph is called *p-semiregular* if its vertex set can be partitioned in such a way that every vertex in one partite set has degree p. A *subdivided star* SS_t is obtained from a star $K_{1,t}$ by subdividing each edge by exactly one vertex. A tree is a *double star* if it contains exactly two vertices of degree at least two. A double star with respectively s and t vertices attached to these two support vertices is denoted by $S_{s,t}$. A *subdivided double star* $SS_{s,t}$ is obtained from a double star $S_{s,t}$ by subdividing each edge by exactly one vertex.

A *block* of a graph G is a maximal induced subgraph of G without cut vertices. A graph G is a *block graph* if every block of G is a complete graph and a *cactus graph* if every block of G is either a cycle or isomorphic to K_2. We call G a *block-cactus graph* if every block of G is either a complete graph or a cycle. If we substitute each edge in a non-trivial tree by two parallel edges and then subdivide each edge, then we speak of a *C_4-cactus*.

Let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if $|N_G(v) \cap D| \geq p$ for every $v \in V(G) - D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A p-dominating set of minimum cardinality of a graph G is called a $\gamma_p(G)$-set. If a dominating set D of a graph G is also independent, that is, no two vertices of D are adjacent, then D is called an *independent dominating set*. The cardinality of a minimum independent dominating set in G is denoted with $i(G)$ and is called the *independence domination number* of G. A set of vertices $U \subseteq V(G)$ is a *covering* of G if every edge of G is incident with at least one vertex of U. A covering of minimum cardinality is called a *minimum covering* and its cardinality is denoted by $\beta(G)$, the *covering number* of G.

In [1], [2], Fink and Jacobson introduced the concept of p-domination. In [10] a more general domination concept was introduced. For a given integer-valued function f defined on the vertices set $V(G)$ of a graph G, a subset D of $V(G)$ is an *f-dominating set* if each vertex $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in D. The *f-domination number* $\gamma_f(G)$ of G was defined in [12] as the minimum cardinality of an f-dominating set of G. The concept of f-domination appeared already in [8] in a slightly different way. If $V(G) = \{x_1, x_2, \ldots, x_n\}$

is the set of vertices of a graph G, let associate every vertex x_i with an integer b_i such that $0 \leq b_i \leq d(x_i)$ and denote $b = (b_1, b_2, \ldots, b_n)$. A set D of vertices in G is called, according to [8], a *b-dominating set* if each $x_i \in V(G) - D$ is adjacent to at least b_i vertices in D. The *minimum b-dominating number* of G is defined as the cardinality of a minimum b-dominating set of G. Thus, if we define $f(x_i) = b_i$ for $1 \leq i \leq n$, the f-domination and the b-domination concepts coincide.

In [13], the relation between $\gamma_f(G)$ and $i(G)$ in a simple graph G is analyzed. In this paper, we concentrate on the parameters $\gamma_2(G)$ and $i(G)$ in block-cactus graphs G.

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [6], [7].

2. Preliminary results

The following results play an important role in our investigations.

Theorem 2.1 (Fink, Jacobson [1] 1985) Let $p \geq 1$ be an integer. If T is a tree, then

$$\gamma_p(T) \geq \frac{(p-1)n(T)+1}{p}$$

and $\gamma_p(T) = ((p-1)n(T)+1)/p$ if and only if T is a p-semiregular tree or $n(T) = 1$.

Corollary 2.2 (Fink, Jacobson [1] 1985) If T is a tree, then

$$\gamma_2(T) \geq \frac{n(T)+1}{2}$$

and $\gamma_2(T) = \frac{n(T)+1}{2}$ if and only if T is the subdivision graph of another tree.

Recently, Volkmann characterized the trees T with $\gamma_p(T) = \left\lceil \frac{(p-1)n(T)+1}{p} \right\rceil$.

Theorem 2.3 (Volkmann [11] 2007) If T is a tree of order $n = n(T)$, then $\gamma_p(T) = \left\lceil \frac{(p-1)n+1}{p} \right\rceil$ if and only if

(i) $n = pt + 1$ for an integer $t \geq 0$ and T is a p-semiregular tree or $n = 1$ or

(ii) $n = pt + r$ for integers $t \geq 0$ and $2 \leq r \leq p$ and T consists of r trees $T_1, T_2, \ldots, T_r$ which satisfy the conditions in (i) and $r - 1$ further edges such that the trees $T_1, T_2, \ldots, T_r$ together with these $r - 1$ edges result in a tree.

Corollary 2.4 (Volkmann [11] 2007) If T is a tree of order $n = n(T)$, then $\gamma_2(T) = \lceil \frac{n+1}{2} \rceil$ if and only if

(i) n is odd and T is the subdivision graph of another tree or

(ii) n is even and T consists of two subdivision trees $S(T_1)$ and $S(T_2)$ and a further edge, connecting $S(T_1)$ with $S(T_2)$.

Theorem 2.5 (Volkmann [11] 2007) A non-trivial tree T satisfies

$$\gamma_2(T) = \gamma(T) + 1$$

if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$.

Lemma 2.6 (Randerath, Volkmann [9] 1998) Let G be a connected C_4-cactus with the partite sets A and B. If $|A| \leq |B|$, then $|A| = \gamma(G) = \beta(G)$ and $|B| = 2|A| - 2$.

3. Main results

Theorem 3.1 If G is a connected block-cactus graph, then $\gamma_2(G) \geq i(G)$.

Proof. If G is a complete graph or a cycle, then it is easy to see that $\gamma_2(G) \geq i(G)$. Now suppose that G has a cut vertex. We will prove the statement by induction on the number of blocks in G. Let B be an end block of G with cut vertex u in G.

Case 1. Suppose that $B \cong K_2$. Let D be a $\gamma_2(G)$-set and let $V(B) = \{u, v\}$.

Case 1.1. Suppose that $u \notin D$. If $L_u \cup \{u\} = V(G)$, then G is a star and we are done. Let $|L_u \cup \{u\}| < n$. Then $D - L_u$ is a 2-dominating set of $G' := G - (L_u \cup \{u\})$ and so $\gamma_2(G') \leq \gamma_2(G) - |L_u|$. Clearly, $i(G) \leq i(G') + |L_u|$. By the induction hypothesis follows $\gamma_2 \geq i$ for every component of G' and thus $\gamma_2(G') \geq i(G')$. This implies

$$\gamma_2(G) \geq \gamma_2(G') + |L_u| \geq i(G') + |L_u| \geq i(G).$$

Case 1.2. Suppose that $u \in D$. Since $D - \{v\}$ is a 2-dominating set of $G'' := G - v$, we conclude $\gamma_2(G'') \leq \gamma_2(G) - 1$. Since $i(G) \leq i(G'') + 1$, we obtain by the induction hypothesis

$$\gamma_2(G) \geq \gamma_2(G'') + 1 \geq i(G'') + 1 \geq i(G).$$

Case 2. Assume that $B \cong K_p$ for an integer $p \geq 3$. Let D be a $\gamma_2(G)$-set. Without loss of generality, we can suppose that $u \in D$. Then $D \setminus (V(B) - \{u\})$

is a 2-dominating set of $G' := G - (V(B) - \{u\})$ and thus together with the induction hypothesis and the evident fact that $i(G') + 1 \geq i(G)$ we obtain

$$\gamma_2(G) \geq \gamma_2(G') + 1 \geq i(G') + 1 \geq i(G)$$

Case 3. Assume that B is isomorphic to a cycle of length $p \geq 3$. Let D be a $\gamma_2(G)$-set. Without loss of generality, we can suppose that $u \in D$. Then $D \setminus (V(B) - \{u\})$ is a 2-dominating set of $G' := G - (V(B) - \{u\})$ and so

$$\gamma_2(G') \leq \gamma_2(G) - \left\lceil \frac{n(B) - 2}{2} \right\rceil .$$

On the other hand, we observe that

$$i(G) \leq i(G') + \left\lceil \frac{n(B) - 1}{3} \right\rceil .$$

It follows by the induction hypothesis

$$\gamma_2(G) \geq \gamma_2(G') + \left\lceil \frac{n(B) - 2}{2} \right\rceil \geq i(G') + \left\lceil \frac{n(B) - 1}{3} \right\rceil \geq i(G)$$

and the statement is proved. □

Theorem 3.2 Let G be a connected block-cactus graph. If there is a block B of G, which is different from the cycle C_4, then $\gamma_2(G) \geq i(G) + 1$.

Proof. If G is a complete graph or a cycle different from C_4, then it is evident that $\gamma_2(G) \geq i(G) + 1$. Now suppose that G has a cut vertex. Let B be an end block of G with cut vertex u in G such that $G' := G - (V(B) - \{u\})$ has still a block different from the cycle C_4. Now we can proceed as in the proof of Theorem 3.1 with the only difference that by the induction hypothesis we have $\gamma_2(G') \geq i(G') + 1$. Thus in all three cases we obtain $\gamma_2(G) \geq i(G) + 1$ and the proof is complete. □

Corollary 3.3 Let G be a non-trivial block graph. Then $\gamma_2(G) \geq i(G) + 1$.

Corollary 3.4 Let G be a unicyclic graph. If $G \neq C_4$, then $\gamma_2(G) \geq i(G) + 1$.

Theorem 3.2 allows us to give a former result of Hansberg and Volkmann as a corollary.

Corollary 3.5 (Hansberg, Volkmann [3, 4]) If G is a non-trivial block graph or a unicyclic graph different from the cycle C_4, then $\gamma_2(G) \geq \gamma(G) + 1$.

Proof. This is evident if we regard the inequality $\gamma(G) \leq i(G)$ and Corollaries 3.3 and 3.4. □

In the same work, Hansberg and Volkmann characterized all block graphs and unicyclic graphs G that fulfill $\gamma_2(G) = \gamma(G) + 1$.

Additionally, we obtain directly from Theorem 3.2 the following result, which will be very usefull for characterizing all block-cactus graphs G with $\gamma_2(G) = i(G)$.

Corollary 3.6 If G is a block-cactus graph with $\gamma_2(G) = i(G)$, then G only consists of C_4-blocks.

Lemma 3.7 Let T be a tree of order $n = n(T)$. If $\gamma_2(T) = i(T) + 1$, then $\gamma_2(T) = \lceil \frac{n+1}{2} \rceil$.

Proof. Suppose that $\gamma_2(T) \geq \lceil \frac{n+1}{2} \rceil + 1$. Let A and B be bipartition sets of T. Then both sets A and B are independent and dominating and hence $i(T) \leq n/2$ holds. It follows

$$\gamma_2(T) \geq \left\lceil \frac{n+1}{2} \right\rceil + 1 > \frac{n}{2} + 1 \geq i(T) + 1,$$

which is a contradiction to our hypothesis. Therefore $\gamma_2(G) \leq \lceil \frac{n+1}{2} \rceil$ and, together with Corollary 2.2, we obtain $\gamma_2(T) = \lceil \frac{n+1}{2} \rceil$. □

Theorem 3.8 Let T be a non-trivial tree of order n. Then $\gamma_2(T) = i(G) + 1$ if and only if $\gamma_2(T) = \gamma(T) + 1$ or T is isomorphic to the graph illustrated in Figure 1.

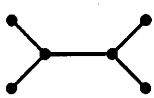


Figure 1.

Proof. If T is isomorphic to the graph in Figure 2, then $\gamma_2(T) = 4 = i(T) + 1$. If $\gamma_2(T) = \gamma(T) + 1$, then, since the inequality $i(T) \geq \gamma(T)$ is always valid and since $\gamma_2(T) \geq i(T) + 1$ in view of Corollary 3.3, we obtain $\gamma_2(T) = i(T) + 1$.

Conversely, assume that $\gamma_2(T) = i(G) + 1$. Hence, Lemma 3.7 leads to $\gamma_2(T) = \lceil \frac{n+1}{2} \rceil$. We distinguish two cases.

Case 1. Assume that n is odd. Then $\gamma_2(T) = (n+1)/2$ and by Corollary 2.4 it follows that T is the subdivision graph of another tree. If $dm(T) \leq 6$, we obtain that T is either a subdivided star SS_t or a subdivided double star $SS_{s,t}$, for which $\gamma_2(T) = \gamma(T) + 1$ hold. We will now prove by induction on n that we can never reach equality in $\gamma_2(T) \geq i(T) + 1$ for $dm(T) \geq 8$. Let z be the central vertex of T and let L_i be the set of leaves in T of distance i from z. If $dm(T) = 8$, then $N_T(L_4) \cup L_2 \cup \{z\}$ is an independent dominating set of G. Since $|L_4| \geq 2$, one can easily see that $|N_T(L_4) \cup L_2 \cup \{z\}| \leq \frac{n-3}{2}$ and hence $\gamma_2(T) = \frac{n+1}{2} > \frac{n-3}{2} + 1 \geq i(T) + 1$. Now suppose that $dm(T) \geq 10$. Let u be a leaf of T and v its support vertex, for which obviously $d_T(v) = 2$ is fulfilled. Let $T' := T - \{u, v\}$ and let I' be an $i(T')$-set. Then $I' \cup \{u\}$ is an independent dominating set of T and $i(T) \leq i(T') + 1$ follows. Since T' is again a subdivision graph and $dm(T') \geq 8$, by the induction hypothesis it follows that

$$i(T) + 1 \leq i(T') + 2 < \gamma_2(T') + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \gamma_2(T).$$

Therefore the only possible trees T of odd order with $\gamma_2(T) = i(G) + 1$ are those with $dm(T) \leq 6$.

Case 2. Assume that n is even. Then $\gamma_2(T) = (n+2)/2$ and from Corollary 2.4 we obtain that T consists of two subdivsion trees T_1 and T_2 of other two trees and T_1 and T_2 are connected by a further edge uv where $u \in V(T_1)$ and $v \in V(T_2)$. Additionally, $n(T_1)$ and $n(T_2)$ are both odd and, by Corollary 2.2, $\gamma_2(T_1) = (n(T_1) + 1)/2$ and $\gamma_2(T_2) = (n(T_2) + 1)/2$.

Case 2.1 Assume that $n(T_1) \geq 3$ and $n(T_2) \geq 3$. Let A_1 and A_2 be the smaller sets of the bipartition sets of T_1 and T_2, respectively. Then A_1 is an independent dominating set of T_1 and A_2 an independent dominating set of T_2. If $u \notin A_1$ or $v \notin A_2$, then $A_1 \cup A_2$ is an independent dominating set of T and thus

$$i(T) \leq |A_1| + |A_2| \leq \frac{n(T_1) - 1}{2} + \frac{n(T_2) - 1}{2} = \frac{n+2}{2} - 2 = \gamma_2(T) - 2,$$

which is a contradiction. Hence, let $u \in A_1$ and $v \in A_2$ and, since T_1 and T_2 are subdivision trees and A_1 and A_2 are the smaller partite sets of T_1 and T_2, $d_{T_1}(u) = 2$ and $d_{T_2}(v) = 2$. Then, if we regard $T_2 - v$, it consists of two sudivision trees T_2' and T_2''. Suppose that $n(T_2') \geq 3$ and $n(T_2'') \geq 3$ and let A_2' and A_2'' be the smaller partite sets of the bipartitions of T_2' and T_2'', respectively. Then $A_1 \cup A_2' \cup A_2''$ is an independent dominating set of T and thus

$$\begin{aligned} i(T) \leq |A_1| + |A_2'| + |A_2''| &\leq \frac{n(T_1) - 1}{2} + \frac{n(T_2') - 1}{2} + \frac{n(T_2'') - 1}{2} \\ &= \frac{n-4}{2} = \frac{n+2}{2} - 3 = \gamma_2(T) - 3, \end{aligned}$$

which is a contradiction. Now assume that $n(T_2') = 1$ or $n(T_2'') = 1$. Suppose that $n(T_2') = 1$ and $n(T_2'') \geq 3$. Let $V(T_2') = \{w\}$. Then, if A_2'' is again the smaller partite set of the bipartition of T_2'', $A_1 \cup A_2'' \cup \{w\}$ is a independent dominating set of T and thus

$$\begin{aligned} i(T) \leq |A_1| + |A_2'| + 1 &\leq \frac{n(T_1) - 1}{2} + \frac{n(T_2') - 1}{2} + 1 \\ &= \frac{n-4}{2} + 1 = \frac{n+2}{2} - 2 = \gamma_2(T) - 2, \end{aligned}$$

again a contradiction. It follows that $n(T_2') = n(T_2'') = 1$. Because of the symmetry, the same follows for T_1 and thus T is isomorphic to the graph illustrated in Figure 1.

Case 2.2 Assume that T_1 is the trivial graph. We distinguish now with respect to the diameter $dm(T_2)$ of T_2 four cases.

(i) If $dm(T_2) = 0$, then T_2 is the trivial graph and T consists only of the edge uv, that is, T is a subdivided star SS_1 without a leaf and $\gamma_2(T) = \gamma(T) + 1 = 2$.

(ii) Let $dm(T_2) = 2$. Then T_2 is a path of length 2. If uv would be incident with the central vertex of T_2, we would have $i(T) = 1$ and $\gamma_2(T) = 3$, which is not allowed. Therefore uv has to be incident with a leaf of T_2 and hence T is a path of length 3, that is, the subdivided star SS_2 without a leaf and $\gamma_2(T) = \gamma(T) + 1 = 3$.

(iii) Let $dm(T_2) = 4$. Then T_2 is a subdivided star SS_t for an integer $t \geq 2$. If uv would be incident with a leaf or with a vertex $x \neq z$ in T_2, where z is the central vertex of T_2, then $\gamma_2(T) = t + 2$ and $i(T) = t$ and the assumption $\gamma_2(T) = i(T) + 1$ would be contradicted. Therefore uv has to be incident with the central vertex z of T_2. In this case T is the subdivided star SS_{t+1} without a leaf and $\gamma_2(T) = \gamma(T) + 1 = t + 1$.

(iv) Suppose $dm(T_2) \geq 6$. We will show by induction on n that in such a case the assumption $\gamma_2(T) = i(T) + 1$ cannot be satisfied. Let $dm(T_2) = 6$. Then T_2 is a subdivided double star $SS_{s,t}$. By analyzing which vertices of $V(T_2)$ the edge uv could be incident with, one can easily show that $\gamma_2(T) = i(T) + 2$ holds always. Let now $dm(T_2) \geq 8$. Let x be a leaf in T_2 and y its support vertex such that $u \notin N_T(\{x, y\})$. Then $T_2 - \{x, y\}$ is again a subdivided graph of diameter at least 6 and by the induction hypothesis we know that in $T' = T - \{x, y\}$ the inequality $\gamma_2(T') \geq i(T') + 2$ holds. Thus if I' is a $i(T')$-set, then $I' \cup \{x\}$ is an independent dominating set of T and hence it is not difficult to see that

$$i(T) \leq i(T') + 1 \leq \gamma_2(T') - 1 \leq \gamma_2(T) - 2.$$

Because of the symmetry, we do not have to distinguish more cases and thus $\gamma_2(T) = \gamma(T) + 1$ or T is isomorphic to the graph in Figure 1. □

Corollary 3.9 Let T be a non-trivial tree of order n. Then $\gamma_2(T) = i(G) + 1$ if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$ or T is isomorphic to the graph showed in Figure 1.

Proof. This follows directly from Theorems 2.5 and 3.8. □

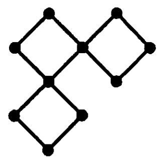
Theorem 3.10 Let G be a non-trivial connected block-cactus graph. Then $\gamma_2(G) = i(G)$ if and only if G is a C_4-cactus.

Proof. By Corollary 3.6, G is a block-cactus graph whose blocks are all C_4-cycles. If G consists of only one block, then $G \cong C_4$. If G has a cut vertex, then there is an end block C isomorphic to the cycle C_4. Let u be the cut vertex of C in G. Then it is easy to see for the graph $G' := G - (V(C) - \{u\})$ that

$$\gamma_2(G') \leq \gamma_2(G) - 1 = i(G) - 1 \leq i(G'),$$

and together with Theorem 3.1 we have that $\gamma_2(G') = i(G')$. Now consider the following graph F.

F:



Here $\gamma_2(F) = 5 = i(F) + 1$ holds. Note that the block-cactus G is a C_4-cactus if and only if every block of G is a C_4-cycle and G does not contain the graph F as a subgraph. Hence, if G would not be a C_4-cactus, we could reduce G to the graph F by taking away C_4-end cycles one after the other. According to our previous analysis, every reduction G' of G should satisfy $\gamma_2(G') = i(G')$. Hence, $\gamma_2(F)$ has to be equal to $i(F)$, which is a contradiction. It is now evident that G has to be a C_4-cactus. □

As a Corollary to this theorem we obtain the characterization of all connected block-cactus graphs G with $\gamma_2(G) = \gamma(G)$, which we gave in a former article.

Corollary 3.11 (Hansberg, Volkmann [5]) Let G be a connected block-cactus graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is a C_4-cactus.

Observation 3.12 Let G be a connected C_4-cactus with partite sets A and B and $|A| \leq |B|$. Then the following properties are satisfied:

(i) A is a $\gamma(G)$-, a $\gamma_2(G)$-, an $i(G)$- and a $\beta(G)$-set.

(ii) If $n(G) \geq 7$, then A is the only $\gamma_2(G)$- and $\beta(G)$-set of G.

Proof. (i) By Lemma 2.6 we know that $|A| = \gamma(G) = \beta(G)$ and from Theorem 2.7 and Corollary 3.11 follows $\gamma_2(G) = i(G) = \gamma(G)$. This implies $|A| = \gamma_2(G) = \gamma(G) = i(G) = \beta(G)$. Moreover, since every vertex $x \in B$ has degree $d_G(x) \geq 2$ and $N_G(V(B)) = A$, A is a 2-dominating set in G and thus dominating and, for being a partite set, it is independent. Evidently A is also a covering of G and hence (i) follows.

(ii) We will prove the statement by induction on $n = n(G)$. If $n(G) = 7$, then we have a C_4-cactus which consists of two C_4-cycles that have exactly one vertex in common. Then $|A| = 3 < |B| = 4$ and A is the only $\gamma_2(G)$- and $\beta(G)$-set of G. Observe that all vertices $x \in V(G)$ with $d_G(x) > 2$ are contained in A.

Now suppose that $n(G) > 7$. Let C be an end block of G with cut vertex u in G. Then the graph $G' := G - (V(C) - \{u\})$ is again a C_4-cactus graph but with less vertices than G. If A' and B' are partite sets of G' with $|A'| \leq |B'|$, then it follows by the induction hypothesis that A' is the unique $\gamma_2(G')$- and $\beta(G')$-set of G' and that all vertices $x \in V(G')$ with $d_{G'}(x) > 2$ are contained in A'. It is now evident from the definition of a C_4-cactus that $u \in A'$, $|A| = |A'| + 1$ and $|B| = |B'| + 2$ and that A is both a 2-dominating set and a coverig of G. Since $\gamma_2(G) \geq \gamma_2(G') + 1$ and $\beta(G) \geq \beta(G') + 1$, it follows that A is both a $\gamma_2(G)$- and a $\beta(G)$-set of G. It is also the only $\gamma_2(G)$- and $\beta(G)$-set of G since otherwise would exist a $\gamma_2(G')$- and a $\beta(G')$-set different from A'. □

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REMARKS ON GROUP RINGS AND THE DAVENPORT CONSTANT

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ABSTRACT. Let $D(G)$ be the Davenport constant of a finite abelian group G, defined as the smallest positive integer d such that every sequence of d elements in G contains a nonempty subsequence with sum zero the identity of G. In this short note, we use group rings as a tool to characterize the Davenport constant.

1. INTRODUCTION

Let G be an additively written finite abelian group. Let $\mathcal{F}(G)$ be the free abelian monoid over G, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called sequences over G. Let $S = g_1 \cdot \ldots \cdot g_t \in \mathcal{F}(G)$. We call S a *zero-sum sequence* if $\sum_{i=1}^{t} g_i = 0$. We call S a *minimal zero-sum sequence* if S is a nonempty zero-sum sequence and contains no proper zero-sum subsequence. We call S a *zero-sumfree sequence* if S contains no nonempty zero-sum subsequence. The Davenport constant of G, denoted by $D(G)$, is defined to be the smallest positive integer d such that every sequence of d elements in G contains a nonempty zero-sum subsequence. The problem of finding $D(G)$ was proposed by H. Davenport in 1966, and he also pointed out that $D(G)$ is connected to algebraic number theory in the following way. Let K be an algebraic number field and G be its class group. Then $D(G)$ is the maximal number of prime ideals (counting multiplicity) that can occur in the decomposition of an irreducible integer in K. It plays an important role in unique factorization theory in algebraic number theory. Furthermore, the Davenport constant is also connected with graph theory, classical number theory and coding theory, and the study of $D(G)$ has attracted a great deal of attention (See for example, [1], [2], [4], [7], [9], [10], and [13]). The exact value of $D(G)$ has been determined only for a few classes of groups, such as finite abelian p-groups, abelian groups of rank not exceeding 2, and certain very special abelian groups of rank 3 (See for example, [3], [4], [6],[11], and [12]).

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In this paper, we use group rings as a tool to investigate $D(G)$. In Section 2, we give a new characterization of $\mathsf{D}(G)$. Perhaps, this characterization will be helpful in estimating $D(G)$.

Let $\mathbf{R}$ be a commutative ring with unity. The group algebra $\mathbf{R}G$ of group G over the ring $\mathbf{R}$ is a free $\mathbf{R}$-module with basis $\{X^g \mid g \in G\}$ (built with a symbol X).

Let $\mathsf{d}(G)$ denote the maximal length of a zero-sumfree sequence over G. Then $\mathsf{d}(G)+1$ is the Davenport constant of G.

For a field $\mathbf{F}$, let $\mathsf{d}(G,\mathbf{F})$ denote the largest integer $l \in \mathbb{N}$ having the following property:

There is some sequence $S = g_1 \cdot \ldots \cdot g_l$ of length l over G such that

$$(X^{g_1} - a_1) \cdot \ldots \cdot (X^{g_l} - a_l) \neq 0 \in \mathbf{F}G \quad \text{for all} \quad a_1, \ldots, a_l \in \mathbf{F}^\times .$$

Then it is easy to see that $\mathsf{d}(G) \le \mathsf{d}(G,\mathbf{F})$. Define $\overline{d}(G) = \min_{\mathbf{F}}\{d(G,\mathbf{F})\}$. Clearly, $D(G) \le \overline{d}(G)+1$. For any finite abelian p-group G or finite cyclic group G, the equality $D(G) = \overline{d}(G)+1$ was confirmed by J.E. Olson [11], and by the first author and A. Gerlodinger [5], respectively. However, we do not know any other finite abelian group G for which $D(G) = \overline{d}(G)+1$ holds. In the final section 3, we will show that $D(G) = \overline{d}(G)+1$ holds for $G = C_2 \oplus C_{2n}$.

Our notations about group rings follow those of [8]. Throughout this paper, let G be a finite abelian group and let $\mathbf{F}$ be a field.

2. A new characterization of the Davenport constant $\mathsf{D}(G)$

It is well-known that the Davenport constant $\mathsf{D}(G)$ can be characterized by several equivalent conditions:

- $\mathsf{D}(G) = \max\{|S| \mid S \in \mathcal{F}(G)$ is a minimal zero-sum sequence$\}$.
- $\mathsf{D}(G)$ is the smallest integer l such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a non-empty zero-sum subsequence.
- $\mathsf{D}(G) = \mathsf{d}(G)+1$.

The equivalence of all these definitions is easy to check, details can be found in [6, Section 5.1]. It is the aim of this section to derive a further characterization of $\mathsf{D}(G)$ which could be useful when working with group algebras.

We fix the following notation. For an element $f \in \mathbf{F}G$ and all $g \in G$, let $\mathsf{c}_g(f) \in \mathbf{F}$ be defined by

$$f = \sum_{g \in G} \mathsf{c}_g(f) X^g .$$

Lemma 2.1. *Let* $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ *be a sequence and suppose that there exist* $a_1, \ldots, a_{l-1} \in \mathbf{F}$ *such that*

$$\mathrm{c}_0\Big(\prod_{i=1}^{l-1}(X^{g_i} - a_i)\Big) \neq (-1)^{l-1}\prod_{i=1}^{l-1} a_i .$$

Then there exist some $a_l \in \mathbf{F}$ *such that*

$$\mathrm{c}_0\Big(\prod_{i=1}^{l}(X^{g_i} - a_i)\Big) \neq (-1)^{l}\prod_{i=1}^{l} a_i .$$

Proof. Let $c = (-1)^{l-1}a_1a_2\cdots a_{l-1}$. Write

$$\prod_{i=1}^{l-1}(X^{g_i} - a_i) = \sum_{g\in G}\alpha_g X^g, \alpha_g \in \mathbf{F}.$$

Then $\alpha_0 \neq c$. Let $a_l \in \mathbf{F}$. Then
$c_0((X^{g_l} - a_l)\prod_{i=1}^{l-1}(X^{g_i} - a_i)) = c_0((X^{g_l} - a_l)\sum_{g\in G}\alpha_g X^g) = \alpha_{-g_l} - a_l\alpha_0$.
So, it suffices to choose a_l so that

$$\alpha_{-g_l} - a_l\alpha_0 \neq -a_l c.$$

This is equivalent to

$$a_l(\alpha_0 - c) \neq \alpha_{-g_l}.$$

Since $\alpha_0 - c \neq 0$, it is a unit in $\mathbf{F}$. Clearly we can choose a_l so that $a_l \neq \frac{\alpha_{-g_l}}{\alpha_0 - c}$. This completes the proof. □

Theorem 2.2. *Let* $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ *be a sequence.*
Then the following statements are equivalent:

(a) *There exist* $a_1, \ldots, a_l \in \mathbf{F}$ *such that* $\mathrm{c}_0(\prod_{i=1}^{l}(X^{g_i} - a_i)) \neq (-1)^l\prod_{i=1}^{l} a_i$.

(b) S *is not zero-sumfree.*

In particular, the Davenport constant $\mathsf{D}(G)$ *is the smallest integer* $l \in \mathbb{N}$ *having the following property:*

For every sequence $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ *of length* l *there exist* $a_1, \ldots, a_l \in \mathbf{F}$ *such that*

$$\mathrm{c}_0\Big(\prod_{i=1}^{l}(X^{g_i} - a_i)\Big) \neq (-1)^{l}\prod_{i=1}^{l} a_i .$$

Proof. It suffices to prove the equivalence of (a) and (b).

If there are l elements $a_1, a_2, \cdots, a_l \in \mathbf{F}$ (repetition allowed) such that the $c_0(\prod_{i=1}^{t}(X^{g_i} - a_i)) \neq (-1)^t a_1a_2\cdots a_l$, clearly S contains a nonempty zero-sum subsequence.

Next assume that S contains a nonempty zero-sum subsequence. Without loss of generality, we may assume that $T = g_1 \cdot \ldots \cdot g_k$ is a minimal zero-sum subsequence. Then $c_0(\prod_{j=1}^{k}(X^{g_j} - a_j)) = 1 + (-1)^k a_1 a_2 \cdots a_k \neq (-1)^k a_1 a_2 \cdots a_k$. By Lemma 2.1, we can find $a_{k+1}, \cdots, a_l$ inductively, such that $c_0(\prod_{i=1}^{l}(X^{g_i} - a_i)) \neq (-1)^t a_1 a_2 \cdots a_l$. □

3. On $d(G)$ and $\overline{d}(G)$

The question of when $D(G) = \overline{d}(G) + 1$ is investigated in this section. We are able to show that this equality holds when $G = C_2 \oplus C_{2n}$.

The following easy observation will be helpful in the proof of Theorem 3.3.

Lemma 3.1. *Let $S = g_1 \cdot \ldots \cdot g_t \in \mathcal{F}(G)$, and let $k_1, \cdots, k_t$ be some positive integers. If there exist $a_1, \cdots, a_t \in \mathbf{F}^\times$ such that $(X^{g_1} - a_1) \cdots (X^{g_t} - a_t) = 0 \in \mathbf{F}G$, then $(X^{k_1 g_1} - a_1^{k_1}) \cdots (X^{k_t g_t} - a_t^{k_t}) = 0 \in \mathbf{F}G$.*

For $n \in \mathbb{N}$, let $\mu_n(F) = \{\zeta \in F \mid \zeta^n = 1\} \subset F^\times$ denote the group of n-th roots of unity of F. Then $\mu_n(F)$ is a cyclic subgroup of $F^\times$. If $\exp(G) = n$, then $\mathrm{Hom}(G, F^\times) = \mathrm{Hom}(G, \mu_n(F))$, and F is called a *splitting field* of G if $|\mu_n(F)| = n$. Clearly, if F is a splitting field of G, then $\mathrm{char}(\mathbf{F}) \nmid \exp(G)$.

Lemma 3.2. *Let $S = g_1 \cdot \ldots \cdot g_t \in \mathcal{F}(G)$ be a sequence with $ord(g_1) \leq \cdots \leq ord(g_t)$. Let $\mathbf{F}$ a splitting field of G and let $m_i = ord(g_i)$ for $i = 1, \cdots, t$. If $[(1 - \frac{1}{m_1}) \cdots (1 - \frac{1}{m_{t-\ell}})|G|] \leq \ell$ holds for some non-negative integer $\ell < t$, then there exist nonzero elements $c_1, \cdots, c_t$ of $\mathbf{F}$ such that the product*

$$(X^{g_1} - c_1) \cdots (X^{g_t} - c_t) = 0 \in \mathbf{F}G.$$

Therefore, $\overline{d}(G) \leq t$.

Proof. This lemma follows immediately from Lemma 5.5.3 and Proposition 5.5.4.2 in [6]. □

Theorem 3.3. *Let $G = C_2 \oplus C_{2n}$ with $n \geq 2$ and suppose that $\mathbf{F}$ is a splitting field of G. Then the equality $d(G) = d(G, \mathbf{F})$ holds, and therefore, $D(G) = \overline{d}(G) + 1$.*

Proof. As observed in the introduction, we clearly have $\mathsf{d}(G) \leq \mathsf{d}(G, F)$. Thus it remains to prove the reverse inequality. Since $\mathsf{d}(G) = 2n$ (see [6, Theorem 5.8.3]), we have to show that $\mathsf{d}(G, F) \leq 2n$. Let $S = g_1 \cdot \ldots \cdot g_{2n+1} \in \mathcal{F}(G)$. We will prove that there are $a_1, \ldots, a_{2n+1} \in \mathbf{F}^\times$ such that

$$(X^{g_1} - a_1)(X^{g_2} - a_2) \cdots (X^{g_{2n+1}} - a_{2n+1}) = 0 \in \mathbf{F}G. \tag{3.1}$$

Then by definition of $d(G,F)$ it follows that $d(G,F) < |S| = 2n+1$, and we are done.

We now prove Equation (3.1). If S contains an element of order 2, we may assume that $ord(g_1) = 2$. Since $[(1-\frac{1}{ord(g_1)})|G|] = 2n$, (3.1) follows from Lemma 3.2 with $t = 2n+1$ and $\ell = 2n$. Thus we may assume that S contains no element of order 2.

Write $2n = 2^u v$ with $2 \not| v$. Let $G = C_2 \oplus C_{2n} = \langle x \rangle \oplus \langle y \rangle$ with $\langle x \rangle = C_2$ and $\langle y \rangle = C_{2n}$. Then every element in G is of the form ay or $x + ay$ with $a \geq 0$. Since S contains no element of order 2, for each $g \in S$ either $g = ay$ or $g = x + by$ with $b \in \{1, 2, \cdots, 2n-1\}$. If $g = ay$, then by Lemma 3.1 we can replace g by y. If $g = x + 2^c by$ with b odd, then $g = b(x + 2^c y)$, so by Lemma 3.1 we can replace g by $x + 2^c y$. Furthermore, if $c > u$ then $x + 2^c y = x + (2^c + 2n)y = x + (2^u(2^{c-u} + v))y = (2^{c-u} + v)(x + 2^u y)$ and we can replace $x + 2^c y$ by $x + 2^u y$. Thus we may assume that $g \in \{y, x+y, x+2y, x+4y, \cdots, x+2^u y\}$ holds for every $g \in S$. Let ξ be a primitive $2n$-th root of 1. Then $\{1, \xi, \xi^2, \cdots, \xi^{2n-1}\}$ are all $2n$-th roots of 1. Clearly $\{1, \xi, \xi^2, \cdots, \xi^{2n-1}\} = \{\pm 1, \pm\xi, \pm\xi^2, \cdots, \pm\xi^{n-1}\}$.

The next 4 equations will be used later in the proof.

$$(3.2) \quad \prod_{i=0}^{n-1}(X^y - \xi^i)(X^y + \xi^i) = \prod_{i=0}^{2n-1}(X^y - \xi^i) = X^{2ny} - 1 = X^0 - 1 = 0.$$

$$(3.3) \qquad X^{2x} = X^0 = 1.$$

(3.4)
$$(X^x z^a - \eta^a)(X^x z^b + \eta^b) = (z^{a+b} - \eta^{a+b}) + \eta^b X^x z^a - \eta^a X^x z^b = (z-\eta)\alpha,$$

where $z, \eta, \alpha \in \mathbf{F}G$, $a, b \in N$ and $a < b$.

$$(3.5) \qquad (X^x z - \eta)(X^x z + \eta) = (z-\eta)(z+\eta),$$

where $z, \eta \in \mathbf{F}G$.

Next, we divide the terms of S into as many as possible disjoint pairs $(z_1, w_1), (z_2, w_2), \cdots, (z_q, w_q)$ such that each pair is of one of the following three forms: $(y, y), (x + 2^r y, x + 2^r y) (0 \leq r \leq u)$ and $(x + 2^s y, x + 2^t y) (1 \leq s < t \leq u)$.

Consider the remaining sequence obtained by deleting $(z_1, w_1, z_2, w_2, \cdots, z_q, w_q)$ from S, and clearly there are only two cases: (1) the remaining sequence is of the form $(y, x+y, x+2^f y)$ with $1 \leq f \leq u$, or (2) the remaining sequence contains only one term.

Case 1. If the remaining sequence is of the form $(y, x+y, x+2^f y)$, then there are $q = n-1$ pairs (z_i, w_i) of terms from S. We show that there exist $b_1, c_1, b_2, c_2, \cdots, b_{n-1}, c_{n-1} \in \mathbf{F}^\times$ such that $(X^y+1)(X^{x+y}-1)(X^{x+2^f y}+1)\prod_{i=1}^{n-1}(X^{z_i}-b_i)(X^{w_i}-c_i) = 0 \in \mathbf{F}G$. For each pair (z_i, w_i) with $i \in \{1, 2, \cdots, n-1\}$, we choose b_i, c_i in the following way:

(1) If $(z_i, w_i) = (y, y)$, then let $b_i = \xi^i$ and $c_i = -\xi^i = \xi^{n+i}$;

(2) If $(z_i, w_i) = (x+2^r y, x+2^r y)$, then let $b_i = (\xi^i)^{2^r}$ and $c_i = -(\xi^i)^{2^r}$. By (3.5), $(X^{z_i}-b_i)(X^{w_i}-c_i) = (X^{2^r y}-\xi^{i2^r})(X^{2^r y}+\xi^{i2^r}) = (X^y-\xi^i)(X^y+\xi^i)\alpha_i$ where $\alpha_i \in \mathbf{F}G$;

(3) If $(z_i, w_i) = (x+2^s y, x+2^t y)$ with $1 \le s < t \le u$, then let $b_i = (\xi^i)^{2^s}$ and $c_i = -(\xi^i)^{2^t}$. Thus by (3.4), $(X^{z_i}-b_i)(X^{w_i}-c_i) = (X^{2y}-(\xi^i)^2)\alpha_i = (X^y-\xi^i)(X^y+\xi^i)\alpha_i$ where $\alpha_i \in \mathbf{F}G$.

We just showed that for each above pair (z_i, w_i) with $i \in \{1, \cdots, n-1\}$ we can choose a pair (b_i, c_i) of elements in $\mathbf{F}^\times$ such that $(X^{z_i}-b_i)(X^{w_i}-c_i) = (X^y-\xi^i)(X^y+\xi^i)\alpha_i$ where $\alpha_i \in \mathbf{F}G$.

It follows from (3.4) that $(X^{x+y}-1)(X^{x+2^f y}+1) = (X^y-1)\beta$ where $\beta \in \mathbf{F}G$. We now have $(X^y+1)(X^{x+y}-1)(X^{x+2^f y}+1)\prod_{i=1}^{n-1}(X^{z_i}-b_i)(X^{w_i}-c_i) = (X^y+1)(X^y-1)\beta\prod_{i=1}^{n-1}(X^y-\xi^i)(X^y+\xi^i)\alpha_i = (\beta\prod_{i=1}^{n-1}\alpha_i)\prod_{i=0}^{n-1}(X^y-\xi^i)(X^y+\xi^i) = 0$.

Case 2. If the remaining sequence contains only one term, then there are $q = n$ pairs (z_i, w_i) of terms from S. As before, for each pair (z_i, w_i), we can find a pair (b_i, c_i) of elements in $\mathbf{F}^\times$ such that $(X^{z_i}-b_i)(X^{w_i}-c_i) = (X^y-\xi^i)(X^y+\xi^i)\alpha_i$, where $\alpha_i \in \mathbf{F}G$. Therefore, $\prod_{i=1}^{n}(X^{z_i}-b_i)(X^{w_i}-c_i) = (\prod_{i=1}^{n}\alpha_i)\prod_{i=1}^{n}(X^y-\xi^i)(X^y+\xi^i) = 0$.

In all the cases, we showed that (3.1) holds. This completes the proof. □

We are not aware of any example of a finite abelian group G for which the equality $d(G) = \mathsf{d}(G, \mathbf{F})$ fails to hold. We close this paper by making the following conjecture.

Conjecture 3.4. *For every finite abelian group* G *and every splitting field* $\mathbf{F}$ *of* G*, we have* $\mathsf{d}(G) = \mathsf{d}(G, \mathbf{F})$.

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