RECURRENCE RELATIONS OF THE SECOND ORDER AND INFINITE SERIES IDENTITIES

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ABSTRACT. For the sequence satisfying the recurrence relation of the second order, we establish a general summation theorem on the infinite series of the reciprocal product of its two consecutive terms. As examples, several infinite series identities are obtained on Fibonacci and Lucas numbers, hyperbolic sine and cosine functions, as well as the solutions of Pell equation.

1. INTRODUCTION AND PRELIMINARY

For the classical Fibonacci numbers $\{F_n\}$, Dean Clark [1, 1992] asked for evaluating the following reciprocal sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}}. \tag{1}$$

The authors observe that the summability of the last infinite series relies crucially on the fact the Fibonacci numbers satisfy a recurrence relation of the second order. In fact, there exist many combinatorial sequences which satisfy the recurrence relation of the second order. To be concrete, let $\{A_n\}$

2000 *Mathematics Subject Classification.* Primary 05A10, Secondary 05A19.
Key words and phrases. Fibonacci numbers; Lucas numbers; Recurrence relation.
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be such a sequence with the fixed initial values A_0 and A_1 as well as

$$A_{n+1} = bA_n + dA_{n-1} \quad \text{for} \quad n = 1, 2, \cdots. \tag{2}$$

For a nonnegative integer ε, the objective of the present paper is to evaluate the following general infinite series:

$$\sum_{n=\varepsilon}^{\infty} \frac{(-d)^n}{A_n A_{n+1}} \quad \text{and} \quad \sum_{n=\varepsilon}^{\infty} \frac{(-d)^n}{A_n A_{n+2}}. \tag{3}$$

Their closed formulae will be given as Theorem and Corollary in the next section. Interestingly enough, there is almost no other known infinite series identities of this kind in literature except for the example displayed in (1). On account of the common character that $\{A_n\}$ satisfies the recurrence relation of the second order, several new summation formulae are exemplified as consequences, in the third section, concerning the Fibonacci and Lucas numbers, hyperbolic sine and cosine functions, as well as the solutions of Pell equations.

2. The Main Results and Proofs

With b and d being the coefficients in recurrence relation (2), define two conjugate algebraic numbers α and γ by

$$\alpha = \frac{b + \sqrt{b^2 + 4d}}{2} \quad \text{and} \quad \gamma = \frac{b - \sqrt{b^2 + 4d}}{2}. \tag{4}$$

Then we are ready to state the main theorem of this paper.

Theorem. *Let b, d and γ be three real numbers with $b > 0$ and γ being defined by (4). For the sequence $\{A_n\}$ satisfying recurrence relation (2), there holds the following infinite series identity:*

$$\sum_{n=\varepsilon}^{\infty} \frac{(-d)^n}{A_n A_{n+1}} = \frac{(-d)^{\varepsilon}}{A_{\varepsilon} A_{\varepsilon+1} - A_{\varepsilon}^2 \gamma} \quad \textit{for} \quad \varepsilon = 0, 1, \cdots. \tag{5}$$

In particular, for $\varepsilon = 0$ and $\varepsilon = 1$, the identity (5) reduces respectively to the following:

$$\sum_{n=0}^{\infty} \frac{(-d)^n}{A_n A_{n+1}} = \frac{1}{A_0 A_1 - A_0^2 \gamma}, \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-d)^n}{A_n A_{n+1}} = \frac{d}{A_1^2 \gamma - A_1 A_2}; \tag{7}$$

which will be used frequently in this paper.

Proof. From the recurrence relation (2), we can compute without difficulty (cf. Wilf [5] for example) the generating function of $\{A_n\}$. In fact, multiplying by x^{n+1} the both sides of (2) and summing over n from $1+\varepsilon$ to ∞, we can proceed as follows:

$$\sum_{n=1+\varepsilon}^{\infty} A_{n+1}x^{n+1} = b\sum_{n=1+\varepsilon}^{\infty} A_n x^{n+1} + d\sum_{n=1+\varepsilon}^{\infty} A_{n-1}x^{n+1}.$$

Defining $f(x) = \sum_{n=\varepsilon}^{\infty} A_n x^n$, we can reduce the above equation to the following

$$f(x) - A_\varepsilon x^\varepsilon - A_{\varepsilon+1}x^{\varepsilon+1} = bx\{f(x) - A_\varepsilon x^\varepsilon\} + dx^2 f(x).$$

Resolving this equation, we find that

$$f(x) = x^\varepsilon \frac{A_\varepsilon + x(A_{\varepsilon+1} - bA_\varepsilon)}{1 - bx - dx^2}.$$

Noting that $1 - bx - dx^2 = (1-\alpha x)(1-\gamma x)$ with α and γ defined in the theorem, we can decompose $f(x)$ into partial fractions:

$$f(x) = \frac{x^\varepsilon}{\alpha-\gamma}\left\{\frac{A_{\varepsilon+1} - A_\varepsilon\gamma}{1 - x\alpha} - \frac{A_{\varepsilon+1} - A_\varepsilon\alpha}{1 - x\gamma}\right\}.$$

Extracting the coefficients of x^n from the both sides of the equation just displayed, we get an explicit expression for A_n:

$$A_n = \frac{A_{\varepsilon+1} - A_\varepsilon\gamma}{\alpha-\gamma}\alpha^{n-\varepsilon} - \frac{A_{\varepsilon+1} - A_\varepsilon\alpha}{\alpha-\gamma}\gamma^{n-\varepsilon}.$$

Noting that

$$\alpha\times\gamma = \frac{b+\sqrt{b^2+4d}}{2}\times\frac{b-\sqrt{b^2+4d}}{2} = -d$$

and defining

$$T_n = \frac{\gamma^{n-\varepsilon}}{A_n} = \frac{\gamma^{n-\varepsilon}}{\frac{A_{\varepsilon+1}-A_\varepsilon\gamma}{\alpha-\gamma}\alpha^{n-\varepsilon} - \frac{A_{\varepsilon+1}-A_\varepsilon\alpha}{\alpha-\gamma}\gamma^{n-\varepsilon}},$$

we can easily calculate the following difference:

$$T_n - T_{n+1} = \frac{(A_{\varepsilon+1} - A_\varepsilon\gamma)(-d)^{n-\varepsilon}}{A_n A_{n+1}}.$$

By means of telescoping method, we then evaluate the infinite series in the theorem:

$$\begin{aligned}\sum_{n=\varepsilon}^{\infty}\frac{(-d)^n}{A_nA_{n+1}} &= \frac{(-d)^\varepsilon}{A_{\varepsilon+1}-A_\varepsilon\gamma}\lim_{m\to\infty}\sum_{n=\varepsilon}^{m}\{T_n-T_{n+1}\}\\ &= \frac{(-d)^\varepsilon}{A_{\varepsilon+1}-A_\varepsilon\gamma}\left\{\frac{1}{A_\varepsilon}-\lim_{m\to\infty}T_{m+1}\right\}\\ &= \frac{(-d)^\varepsilon}{A_\varepsilon A_{\varepsilon+1}-{A_\varepsilon}^2\gamma},\end{aligned}$$

where the limiting process has been justified by

$$\left|\frac{\alpha}{\gamma}\right| = \left|\frac{b+\sqrt{b^2+4d}}{b-\sqrt{b^2+4d}}\right| > 1 \quad \text{for} \quad b>0$$

and

$$\lim_{m\to\infty}T_m = \lim_{m\to\infty}\frac{\gamma^{m-\varepsilon}}{A_m} = \lim_{m\to\infty}\frac{1}{\frac{A_{\varepsilon+1}-A_\varepsilon\gamma}{\alpha-\gamma}\left(\frac{\alpha}{\gamma}\right)^{m-\varepsilon}-\frac{A_{\varepsilon+1}-A_\varepsilon\alpha}{\alpha-\gamma}} = 0.$$

This completes the proof of the theorem. □

Corollary. *With the same parameters defined in the theorem, we have:*

$$\sum_{n=\varepsilon}^{\infty}\frac{(-d)^n}{A_nA_{n+2}} = \frac{(-d)^\varepsilon}{b}\left\{\frac{1}{A_\varepsilon A_{\varepsilon+1}-A_\varepsilon^2\gamma}-\frac{d}{A_{\varepsilon+1}A_{\varepsilon+2}-A_{\varepsilon+1}^2\gamma}\right\}.$$

Proof. From the recurrence relation satisfied by the sequence $\{A_n\}$, it is trivial to see that

$$\frac{bA_{n+1}}{A_nA_{n+2}} = \frac{1}{A_n}-\frac{d}{A_{n+2}},$$

which is equivalent to

$$\frac{1}{A_nA_{n+2}} = \frac{1}{b\,A_nA_{n+1}}-\frac{d}{b\,A_{n+1}A_{n+2}}.$$

In view of the theorem, the infinite series results in the following closed form:

$$\begin{aligned}\sum_{n=\varepsilon}^{\infty}\frac{(-d)^n}{A_nA_{n+2}} &= \frac{1}{b}\left\{\sum_{n=\varepsilon}^{\infty}\frac{(-d)^n}{A_nA_{n+1}}+\sum_{n=\varepsilon+1}^{\infty}\frac{(-d)^n}{A_nA_{n+1}}\right\}\\ &= \frac{1}{b}\left\{\frac{(-d)^\varepsilon}{A_\varepsilon A_{\varepsilon+1}-A_\varepsilon^2\gamma}+\frac{(-d)^{\varepsilon+1}}{A_{\varepsilon+1}A_{\varepsilon+2}-A_{\varepsilon+1}^2\gamma}\right\},\end{aligned}$$

which is essentially the same as the formula stated in the corollary. □

We remark that the right hand sides of the infinite series identities stated in the theorem and corollary depend only on b, d and $A_\varepsilon, A_{\varepsilon+1}$. In what follows, we shall collect some examples to show the usefulness of the theorem.

3. Examples

In order to exemplify the method, we show, by means of the theorem and corollary demonstrated in the last section, five classes of infinite series identities on Fibonacci and Lucas numbers, hyperbolic sine and cosine functions as well as the solutions of Pell equation.

3.1. Fibonacci Numbers. Define the sequence $\{A_n\}$ by $A_n := F_n$, where $\{F_n\}$ is the well-known Fibonacci sequence [2, P45], which satisfies the recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

with the initial condition

$$F_0 = F_1 = 1.$$

Now it is trivial to figure out three parameters $b = d = 1$ and $\gamma = \frac{1-\sqrt{5}}{2}$. According to the theorem, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1}{F_0 F_1 - F_0^2 \gamma} = \frac{1}{1 - \frac{1-\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{2}.$$

By means of the corollary, we can similarly evaluate infinite series (1):

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} &= \frac{d}{b}\left\{\frac{1}{F_1^2\gamma - F_1F_2} - \frac{d}{F_2^2\gamma - F_2F_3}\right\} \\ &= \left\{\frac{1}{6-2(1-\sqrt{5})} - \frac{1}{2 - \frac{1-\sqrt{5}}{2}}\right\} = \sqrt{5} - \frac{5}{2}.\end{aligned}$$

3.2. Lucas Numbers. Define the sequence $\{A_n\}$ by $A_n := L_n$, where $\{L_n\}$ is the Lucas sequence [3, P298] satisfying the recurrence relation:

$$L_{n+1} = L_n + L_{n-1}$$

with the initial condition

$$L_0 = 2 \quad \text{and} \quad L_1 = 1.$$

They have the same parameters $b = d = 1$ and $\gamma = \frac{1-\sqrt{5}}{2}$ as Fibonacci numbers. Then the theorem gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{L_n L_{n+1}} = \frac{1}{L_0 L_1 - L_0^2 \gamma} = \frac{1}{2 - 2(1-\sqrt{5})} = \frac{1}{2\sqrt{5}}.$$

Similarly, we can also evaluate, by means of the corollary, another infinite series related to Lucas numbers as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{L_n L_{n+2}} &= \frac{1}{b}\left\{\frac{1}{L_0 L_1 - L_0^2 \gamma} - \frac{d}{L_1 L_2 - L_1^2 \gamma}\right\} \\ &= \frac{1}{2 - 2(1-\sqrt{5})} - \frac{1}{3 - \frac{1-\sqrt{5}}{2}} = \frac{2-\sqrt{5}}{2\sqrt{5}}.\end{aligned}$$

3.3. **Hyperbolic Sine Function.** Let $\{A_n\}$ be given by $A_n = \sinh nz$, where the hyperbolic sine function [3, P271] reads as

$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

Then it is not hard to check the following recurrence relation

$$\sinh(n+1)z = (e^z + e^{-z}) \sinh nz - \sinh(n-1)z.$$

Noting further that $b = e^z + e^{-z}$, $d = -1$ and

$$\gamma = \begin{cases} e^{-z}, & \Re(z) > 0; \\ e^z, & \Re(z) < 0. \end{cases}$$

we derive from the theorem the following summation formula:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\sinh nz \, \sinh(n+1)z} &= \frac{1}{\sinh z \sinh 2z - \gamma \sinh^2 z} \\ &= \begin{cases} \frac{e^{-z}}{\sinh^2 z}, & \Re(z) > 0; \\ \frac{e^z}{\sinh^2 z}, & \Re(z) < 0. \end{cases}\end{aligned}$$

Analogously, the formula stated in the corollary gives us another hyperbolic function identity:

$$\sum_{n=1}^{\infty} \frac{1}{\sinh nz \, \sinh(n+2)z} = \begin{cases} \frac{1+2e^{-2z}}{(e^z+e^{-z})^2 \sinh^2 z}, & \Re(z) > 0; \\ \frac{1+2e^{2z}}{(e^z+e^{-z})^2 \sinh^2 z}, & \Re(z) < 0. \end{cases}$$

3.4. Hyperbolic Cosine Function. Similarly, let $\{A_n\}$ be the sequence defined by $A_n := \cosh nz$, where the hyperbolic cosine function [3, P272] is defined by

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

and satisfies the recurrence relation

$$\cosh(n+1)z = (e^z + e^{-z}) \cosh nz - \cosh(n-1)z.$$

With the same parameters b, d and γ determined previously, we get another hyperbolic function identity:

$$\sum_{n=0}^{\infty} \frac{1}{\cosh nz \, \cosh(n+1)z} = \frac{1}{\cosh z - \gamma} = \frac{2}{e^z + e^{-z} - 2\gamma}$$
$$= \begin{cases} \frac{2}{e^z - e^{-z}}, & \Re(z) > 0; \\ \frac{2}{e^{-z} - e^z}, & \Re(z) < 0. \end{cases}$$

In view of the corollary, the same reasoning leads us to the following hyperbolic function identity:

$$\sum_{n=0}^{\infty} \frac{1}{\cosh nz \, \cosh(n+2)z} = \begin{cases} \frac{2e^z + 6e^{-z}}{(e^z + e^{-z})^2 (e^z - e^{-z})}, & \Re(z) > 0; \\ \frac{2e^{-z} + 6e^z}{(e^z + e^{-z})^2 (e^{-z} - e^z)}, & \Re(z) < 0. \end{cases}$$

Recalling two hyperbolic trigonometric relations

$$\cosh(x+y)z + \cosh(x-y)z = 2 \cosh xz \cosh yz$$
$$\cosh(x+y)z - \cosh(x-y)z = 2 \sinh xz \sinh yz$$

we derive, as byproducts, two infinite series identities:

$$\sum_{n=0}^{\infty} \frac{1}{\cosh(2n+1)z + \cosh z} = \sum_{n=0}^{\infty} \frac{1}{2 \cosh nz \cosh(n+1)z}$$
$$= \begin{cases} \frac{1}{e^z - e^{-z}}, & \Re(z) > 0; \\ \frac{1}{e^{-z} - e^z}, & \Re(z) < 0. \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{1}{\cosh(2n+1)z - \cosh z} = \sum_{n=1}^{\infty} \frac{1}{2 \sinh nz \sinh(n+1)z}$$
$$= \begin{cases} \frac{e^{-z}}{2 \sinh^2 z}, & \Re(z) > 0; \\ \frac{e^z}{2 \sinh^2 z}, & \Re(z) < 0. \end{cases}$$

Two other infinite series identities with the summands given by the reciprocals $\cosh(2n+2)z \pm \cosh 2z$ can be established similarly as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{\cosh(2n+2)z + \cosh 2z} &= \sum_{n=0}^{\infty} \frac{1}{2\cosh nz \cosh(n+2)z} \\
&= \begin{cases} \frac{e^z+3e^{-z}}{(e^z+e^{-z})^2(e^z-e^{-z})}, & \Re(z) > 0; \\ \frac{e^{-z}+3e^z}{(e^z+e^{-z})^2(e^{-z}-e^z)}, & \Re(z) < 0. \end{cases} \\
\sum_{n=1}^{\infty} \frac{1}{\cosh(2n+2)z - \cosh 2z} &= \sum_{n=1}^{\infty} \frac{1}{2\sinh nz \sinh(n+2)z} \\
&= \begin{cases} \frac{1/2+e^{-2z}}{(e^z+e^{-z})^2 \sinh^2 z}, & \Re(z) > 0; \\ \frac{1/2+e^{2z}}{(e^z+e^{-z})^2 \sinh^2 z}, & \Re(z) < 0. \end{cases}
\end{aligned}$$

3.5. **Solutions of Pell Equation.** To make the paper self-contained, we review some basic fact about Pell equations. For the details, refer to [4, §7.8].

For non-perfect-square natural number D, the Pell equation

$$x^2 - Dy^2 = 1$$

admits infinite positive integer solutions $\{x_n, y_n\}_{n\geq 1}$. Let $(x,y) = (x_1, y_1)$ be the smallest one. Then $\{x_n, y_n\}_{n\geq 1}$ satisfies the crossing recurrence relations of the first order

$$x_{n+1} = x_1 x_n + D y_1 y_n, \tag{8a}$$

$$y_{n+1} = x_1 y_n + y_1 x_n. \tag{8b}$$

and the independent recurrence relations of the second order:

$$x_{n+1} = 2x_1 x_n - x_{n-1} : \qquad x_2 = x_1^2 + D y_1^2, \tag{9a}$$

$$y_{n+1} = 2x_1 y_n - y_{n-1} : \qquad y_2 = 2x_1 y_1. \tag{9b}$$

The explicit expressions read as

$$x_n = \frac{1}{2}\{(x_1 + y_1\sqrt{D})^n + (x_1 - y_1\sqrt{D})^n\}, \tag{10a}$$

$$y_n = \frac{1}{2\sqrt{D}}\{(x_1 + y_1\sqrt{D})^n - (x_1 - y_1\sqrt{D})^n\}. \tag{10b}$$

Both sequences have the same three parameters $b = 2x_1$, $d = -1$ and $\gamma = x_1 - \sqrt{x_1^2 - 1}$. Consequently we can evaluate the following two infinite series

$$\sum_{n=1}^{\infty}\frac{1}{x_nx_{n+1}} = \frac{1}{x_1x_2 - x_1^2\gamma} = \frac{1}{x_1(Dy_1^2 + x_1\sqrt{x_1^2-1})}, \tag{11a}$$

$$\sum_{n=1}^{\infty}\frac{1}{y_ny_{n+1}} = \frac{1}{y_1y_2 - y_1^2\gamma} = \frac{1}{y_1^2(x_1 + \sqrt{x_1^2-1})}. \tag{11b}$$

We now give a concrete example to show these formulae. For Pell equation with $D = 2$, it is not hard to check that $(x_1, y_1) = (3, 2)$ and figure out the three parameters $b = 6$, $d = -1$, $\gamma = 3 - 2\sqrt{2}$. The corresponding identities result in the following:

$$\sum_{n=1}^{\infty}\frac{1}{\{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\} \times \{(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}\}} = \frac{3\sqrt{2}-4}{48},$$

$$\sum_{n=1}^{\infty}\frac{1}{\{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\} \times \{(3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}\}} = \frac{3-2\sqrt{2}}{32}.$$

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