The Connected Domination and Tree Domination of $P(n,k)$ for $k = 1, 2, \lfloor n/2 \rfloor$*

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Abstract

Let $\gamma_c(G)$ be the connected domination number of G and $\gamma_{tr}(G)$ be the tree domination number of G. In this paper, we study the generalized Petersen graphs $P(n,k)$, prove $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$ and show their exact values for $k = 1, 2, \lfloor n/2 \rfloor$.
Keywords: *Generalized Petersen Graph; Connected domination number; Tree domination number;*

1 Introduction

We only consider finite connected and undirected graphs without loops or multiple edges.

Let $G = (V(G), E(G))$ be a graph with $|V(G)| = p$ and $|E(G)| = q$. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V(G)$ is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinalities of minimal dominating sets.

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Sampathkumar and Walikar [2] defined a connected dominating set S to be a dominating set S whose induced subgraph $G[S]$ is connected. The minimum cardinality of a connected dominating set of G is the connected domination number $\gamma_c(G)$.

Chen et al.[1] etc. defined a tree dominating set S to be a dominating set S whose induced subgraph $G[S]$ is a tree. The minimum cardinality of a tree dominating set of G is the tree domination number $\gamma_{tr}(G)$. If there is no tree dominating set in G, then let $\gamma_{tr}(G) = 0$. They showed the exact values of the tree domination number for several classes of graphs, including $P_p, C_p, K_p, K_{1,p-1}, K_{r,s}$ and T, and gave several bounds for γ_{tr} and the relationship of γ_c and γ_{tr}.

Observation 1.1. If $\gamma_{tr}(G) > 0$, then $\gamma_c(G) \leq \gamma_{tr}(G)$.

Theorem 1.2. Let G be a connected graph with $\delta(G) \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{(\delta+1)p-2q-2}{\delta-1}$ and the bound is sharp.

Corollary 1.3. Let G be a connected k-regular graph and $k \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{p-2}{k-1}$ and the bound is sharp.

Theorem 1.4. Every connected graph G contains a spanning connected subgraph H such that $\gamma_{tr}(H) = \gamma_c(G)$.

The generalized Petersen graph $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 0 \leq i \leq n-1$, subscripts module $n\}$.

In this paper, we study the connected domination number and tree domination number of the generalized Petersen graph $P(n,k)$, prove $\gamma_c(P(n, k)) = \gamma_{tr}(P(n,k))$ and show their exact values for $k = 1, 2, \lfloor n/2 \rfloor$.

2 The relationship of $\gamma_c(P(n,k)$ and $\gamma_{tr}(P(n,k)$

Theorem 2.1. For $n = 2k$, $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k)) = n$.
Proof. Let $S = \{v_i : 0 \leq i \leq n-3\} \cup \{u_{k-1}, u_{k-2}\}$, then S is a tree dominating set of $P(n,k)$ with $|S| = n$. Hence, $0 < \gamma_{tr}(P(n,k)) \leq n$. Let S be a connected dominating set of $P(n,k)$, then for every $0 \leq i \leq n-1$, at least one vertex of $\{v_i, u_{i+k}\}$ has to belong to S. Hence $|S| \geq n$, i.e. $\gamma_c(P(n,k)) \geq n$. By Observation 1.1, we have $n \leq \gamma_c(P(n,k)) \leq \gamma_{tr}(P(n,k)) \leq n$, i.e. $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k)) = n$. □

Theorem 2.2. For $n = 2k+1$, $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k)) = n-1$.
Proof. Let $S = \{v_i : (0 \leq i \leq n-3)\} \cup \{u_{(n-3)/2}\}$, then S is a tree

dominating set of $P(n,(n-1)/2)$ with $|S| = n-1$. Hence $\gamma_{tr}(P(n,(n-1)/2)) \leq n-1$. By Corollary 1.3, $\gamma_{tr}(P(n,(n-1)/2)) \geq (2n-2)/(3-1) = n-1$. i.e. $\gamma_{tr}(P(n,(n-1)/2)) = n-1$. □

Lemma 2.3. For $n \neq 2k$, $n-1 \leq \gamma_{tr}(P(n,k)) \leq n$.
Proof. Let $S = \{v_i : 1 \leq i \leq n-1\} \cup \{u_k\}$, then S is a tree dominating set of $P(n,k)$ with $|S| = n$. Hence, $0 < \gamma_{tr}(P(n,k)) \leq n$. By Corollary 1.3, $\gamma_{tr}(P(n,k)) \geq (2n-2)/(3-1) = n-1$. □

Theorem 2.4. $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$.
Proof. If $n = 2k$, then by Theorem 2.1, $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$. Thus we only need to consider the case of $n \neq 2k$.

Let $S = \{v_i : 1 \leq i \leq n-1\} \cup \{u_k\}$, then S is a connected dominating set of $P(n,k)$ with $|S| = n$. Hence, $0 < \gamma_c(P(n,k)) \leq n$. Let S^* be a connected dominating set of $P(n,k)$ with $|S^*| = \gamma_c(P(n,k))$. Let t be the number of edges in $P(n,k)$ having one vertex in S^* and the other in $V(P(n,k)) - S^*$, then,

$$\begin{aligned}
2(3n - |E(S^*)|) &= \textstyle\sum_{v_i \epsilon V-S^*} d(v_i) + t \geq 3(2n - |S^*|) + (2n - |S^*|) \\
&= 8n - 4|S^*|, \\
4|S^*| &\geq 2n + 2|E(S^*)|, \\
2|S^*| &\geq n + |E(S^*)|.
\end{aligned}$$

Since $P(n,k)[S^*]$ is connected, $E(S^*) \geq |S^*| - 1$. Hence, $\gamma_c(P(n,k)) = |S^*| \geq n-1$.

Case 1. Suppose $\gamma_c(P(n,k)) = n-1$, then $|V(P(n,k)) - S^*| = n+1$, and $t \geq n+1$. Hence,

$$\begin{aligned}
2(3n - |E(S^*)|) &= \textstyle\sum_{v_i \epsilon V-S^*} d(v_i) + t \\
&\geq 3(n+1) + (n+1) \\
&= 4n + 4, \\
2|E(S^*)| &\leq 2n - 4, \\
|E(S^*)| &\leq n - 2.
\end{aligned}$$

Since S^* is connected, we have $E(S^*) \geq n-2$. Hence, $|E(S^*)| = n-2$, and $P(n,k)[S^*]$ is a tree, and S^* is a tree dominating set of $P(n,k)$ with $|S^*| = n-1$, $\gamma_{tr}(P(n,k)) \leq n-1$. By Lemma 2.3, $n-1 \leq \gamma_{tr}(P(n,k))$, we have $\gamma_{tr}(P(n,k)) = n-1 = \gamma_c(P(n,k))$.

Case 2. Suppose $\gamma_c(P(n,k)) = n$, then by Observation 1.1 and Lemma 2.3, we have $n = \gamma_c(P(n,k)) \leq \gamma_{tr}(P(n,k)) \leq n$. Hence, $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$.

By Case 1 and Case 2, we have $\gamma_c(P(n,k)) = \gamma_{tr}(P(n,k))$. □

Let S be a minimum tree dominating set of $P(n,k)$. By Lemma 2.3 and Theorem 2.4, we have

Lemma 2.5. If $\gamma_{tr}(P(n,k))) = n-1$, then every vertex of $P(n,k) - S$ is dominated by exactly one vertex of S.
Proof. Since $\gamma_{tr}(P(n,k))) = n-1$, $|S| = n-1$ and $E(S) = n-2$. Let t be the number of edges in $P(n,k)$ having one vertex in S and the other in $V(P(n,k)) - S$, then

$$\begin{aligned} 2(3n - |E(S)|) &= \textstyle\sum_{v_i \epsilon V-S} d(v_i) + t \\ 2(3n-(n-2)) &= 3(n+1) + t \\ t &= n+1. \end{aligned}$$

Thus, every vertex of $P(n,k) - S$ is dominated by exactly one vertex of S. □

By Lemma 2.5, we have

Corollary 2.6. If $\gamma_{tr}(P(n,1))) = n-1$ and there exists one edge w_1w_2 in $P(n,1)[S]$, then for any edge $w_3w_4 \in E(P(n,1))$ with $w_3 \in N(w_2)$ and $w_4 \in N(w_1)$, $w_3 \notin S$ and $w_4 \notin S$. □

Corollary 2.7. If $\gamma_{tr}(P(n,2))) = n-1$ and there exists a path $w_1w_2w_3$ in $P(n,2)[S]$, then for any edge $w_4w_5 \in E(P(n,2))$ with $w_4 \in N(w_3)$ and $w_5 \in N(w_1)$, $w_4 \notin S$ and $w_5 \notin S$. □

3 The Exact Value of $\gamma_{tr}(P(n,1))(\gamma_c(P(n,1)))$

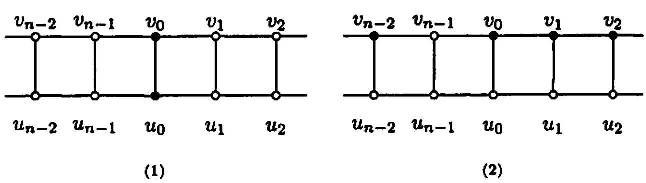

Figure 3.1.

Theorem 3.1. For $n \geq 4$, $\gamma_{tr}(P(n,1)) = \gamma_c(P(n,1)) = n$.
Proof. Let S be a minimum tree dominating set of $P(n,1)$. Assume that $|S| = n-1$. By symmetry, we assume that there is at least one vertex of $\{v_0, v_1, \ldots, v_{n-1}\}$ in S. Since $|S| = n-1$, there is at least one vertex of $\{v_0, v_1, \ldots, v_{n-1}\}$ not in S, without loss of generality, we may assume that $v_{n-1} \notin S$,$v_0 \in S$.

Case 1. Suppose $u_0 \in S$(see Figure 3.1(1)). Then, by Corollary 2.6, $v_1 \notin S$, $u_1 \notin S$ and $v_{n-1} \notin S$, $u_{n-1} \notin S$, a contradiction with $P(n,1)[S]$ being a tree.

Case 2. Suppose $u_0 \notin S$(see Figure 3.1(2)), then since $P(n,1)[S]$ is a tree and both vertices v_{n-1} and u_0 do not belong to S, we have $v_1 \in S$. By Corollary 2.6, $u_0 \notin S$, $u_1 \notin S$. Since $P(n,1)[S]$ is a tree and any one vertex of $\{v_{n-1}, u_0, u_1\}$ does not belong to S, we have $v_2 \in S$. By Corollary 2.6, $u_2 \notin S$. Since $P(n,1)[S]$ is a tree and any one vertex of $\{v_{n-1}, u_0, u_1, u_2\}$ does not belong to S, we have $v_3 \in S$. Continue in this way, we have $v_i \in S$ for $4 \leq i \leq n-2$, thus, v_{n-1} would be dominated by both v_{n-2} and v_0, a contradiction with Lemma 2.5.

By Cases 1-2, we have $|S| \neq n-1$. Furthermore, by Lemma 2.3, we have $\gamma_{tr}(P(n,1)) = n$. □

4 The Exact Value of $\gamma_{tr}(P(n,2))(\gamma_c(P(n,2)))$

In [3], Watkins showed: (1) $P(n,k) \cong P(n,n-k)$; (2) If $1 \leq k, m \leq n-1$ and $km \equiv 1(\mod n)$, then $P(n,m) \cong P(n,k)$. So, for odd n, we have $P(n,2) \cong P(n,(n+1)/2) \cong P(n,(n-1)/2)$. By Theorem 2.2, we can get

Theorem 4.1. For odd $n \geq 5$, $\gamma_{tr}(P(n,2)) = \gamma_c(P(n,2)) = n-1$. □

Let S be a minimum tree dominating set of $P(n,2)$.

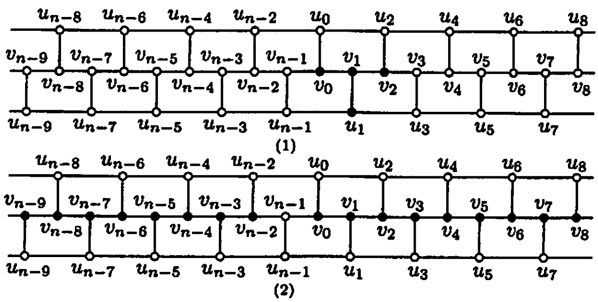

Figure 4.1.

Lemma 4.2. For even $n \geq 6$, if $\gamma_{tr}(P(n,2))) = n-1$, $v_{i-1} \notin S$, $v_i \in S$ and $v_{i+1} \in S$, then $v_{i+2} \notin S(0 \leq i \leq n-1$, subscripts module $n)$.
Proof. By contradiction. Without loss of generality, suppose that $v_{n-1} \notin S$, $v_0 \in S$ and $v_1 \in S$, and $v_2 \in S$. By Corollary 2.7, $u_0 \notin S$ and $u_2 \notin S$.

Since $P(n,2)[S]$ is a tree and any one vertex of $\{v_{n-1}, u_0, u_2\}$ does not belong to S, we have, at least one vertex of $\{u_1, v_3\}$ has to belong to S.

Case 1. Suppose $u_1 \in S$. Then, by Corollary 2.7, any one vertex of $\{v_{n-1}, u_{n-1}, u_0, u_2, v_3, u_3\}$ does not belong to S(see Figure 4.1(1)), a contradiction with $P(n,2)[S]$ being a tree with $|S| = n-1$.

Case 2. Suppose $v_3 \in S$. Then, by Corollary 2.7, $u_1 \notin S$ and $u_3 \notin S$(see Figure 4.1(2)). Since $P(n,2)[S]$ is a tree, $v_4 \in S$. By Corollary 2.7, $u_4 \notin S$. Since $P(n,2)[S]$ is a tree, $v_5 \in S$. Continue in this way, we have $v_j \in S$ for $0 \le j \le n-2$. Thus, v_{n-1} would be dominated by both v_{n-2} and v_0, a contradiction with Lemma 2.5.

By Cases 1-2, $v_2 \notin S$. □

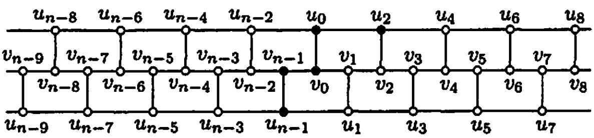

Figure 4.2.

Lemma 4.3. For even $n \ge 6$, if $\gamma_{tr}(P(n,2))) = n-1$, $u_{i-2} \notin S$, $u_i \in S$ and $u_{i+2} \in S$, then $u_{i+4} \in S$ ($0 \le i \le n-1$, subscripts module n).

Proof. By contradiction. Without loss of generality, suppose that $u_{n-2} \notin S$, $u_0 \in S$, $u_2 \in S$ and $u_4 \notin S$(see Figure 4.2). Since $P(n,2)[S]$ is a tree and both vertices u_{n-2} and u_4 do not belong to S, we have, at least one vertex of $\{v_0, v_2\}$ has to belong to S. By symmetry, we may assume $v_0 \in S$. By Corollary 2.7, $v_1 \notin S$, $v_2 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, u_4, v_1, v_2\}$ does not belong to S, we have $v_{n-1} \in S$. By Corollary 2.7, $v_{n-2} \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, u_4, v_1, v_2, v_{n-2}\}$ does not belong to S, we have $u_{n-1} \in S$. By Corollary 2.7, $u_1 \notin S$. By Lemma 2.5, any one vertex of $\{u_3, v_3, v_4\}$ does not belong to S. Thus, v_3 would not be dominated by S, a contradiction. □

Lemma 4.4. For even $n \ge 6$, if $\gamma_{tr}(P(n,2))) = n-1$, $u_{i-2} \notin S$, $u_i \in S$, $u_{i+2} \in S$ and $u_{i+4} \in S$, then $u_{i+6} \notin S$($0 \le i \le n-1$, subscripts module n).

Proof. By contradiction. Without loss of generality, suppose that $u_{n-2} \notin S$, $u_0 \in S$, $u_2 \in S$, $u_4 \in S$ and $u_6 \in S$. Since $P(n,2)[S]$ is a tree, we have, at least one vertex of $\{v_0, v_2, v_4, v_6, u_8\}$ has to belong to S.

Suppose $v_0 \in S$. Then, by Corollary 2.7, $v_1 \notin S$ and $v_2 \notin S$. By Lemma 2.5, $u_1 \notin S$ and $v_3 \notin S$. Since $N[v_3] \cap S \ne \emptyset$, at least one vertex of

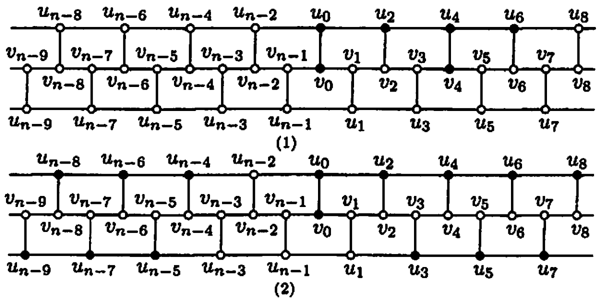

Figure 4.3.

$\{v_4, u_3\}$ has to belong to S. If $v_4 \in S$(see Figure 4.3(1)), then by Corollary 2.7, $v_5 \notin S$ and $v_6 \notin S$. By Lemma 2.5, $u_5 \notin S$. Thus, u_3 would not be dominated by S, a contradiction. Hence $v_4 \notin S$, so $u_3 \in S$(see Figure 4.3(2)). Since $u_1 \notin S$, By Lemma 2.5, $u_{n-1} \notin S$. Since $u_{n-2} \notin S$, By Lemma 2.5, $v_{n-2} \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_1 and v_3 are not belong to S, we have $u_5 \in S$. Since $v_4 \notin S$, by Lemma 2.5, $v_5 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_1, v_3, v_5\}$ does not belong to S, we have $u_7 \in S$. By Lemma 2.5, $v_6 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_{n-1}, v_1, v_2, v_4, v_6\}$ does not belong to S, we have $u_8 \in S$. Continue in this way, we have $u_j \in S$ for $7 \leq j \leq n-4$. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5. Hence $v_0 \notin S$.

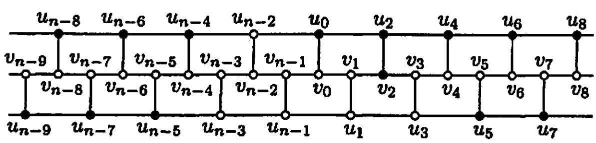

Figure 4.4.

Suppose $v_2 \in S$. Then, by Corollary 2.7, $v_0 \notin S$, $v_1 \notin S$, $v_3 \notin S$ and $v_4 \notin S$. By Lemma 2.5, $u_1 \notin S$, $u_3 \notin S$(see Figure 4.4). Since $N[u_3] \cap S \neq \emptyset$, u_5 has to belong to S. By Lemma 2.5, $v_5 \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_3 and v_5 do not belong to S, we have $u_7 \in S$. By Lemma 2.5, $v_6 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_1, v_3, v_4, v_6\}$ does not belong to S, we have $u_8 \in S$. Continue in this way, we have $u_j \in S$ for $7 \leq j \leq n-4$. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5. Hence $v_2 \notin S$.

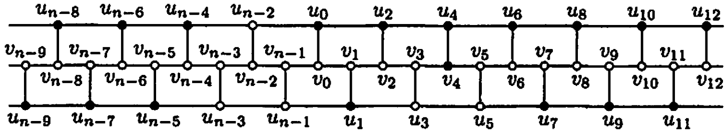

Figure 4.5.

Suppose $v_4 \in S$. Then, by Corollary 2.7, $v_2 \notin S$, $v_3 \notin S$, $v_5 \notin S$ and $v_6 \notin S$. By Lemma 2.5, $u_3 \notin S$ and $u_5 \notin S$(see Figure 4.5). Since $N[u_5] \cap S \neq \emptyset$, u_7 has to belong to S. Since $N[u_3] \cap S \neq \emptyset$, u_1 has to belong to S. By Lemma 2.5, $v_1 \notin S$ and $v_0 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, v_3, v_5, v_6\}$ does not belong to S, we have $u_8 \in S$. By Lemma 2.5, $v_7 \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_5 and v_7 do not belong to S, we have $u_9 \in S$. Continue in this way, we have $u_j \in S$ for $7 \leq j \leq n-4$. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5. Hence $v_4 \notin S$.

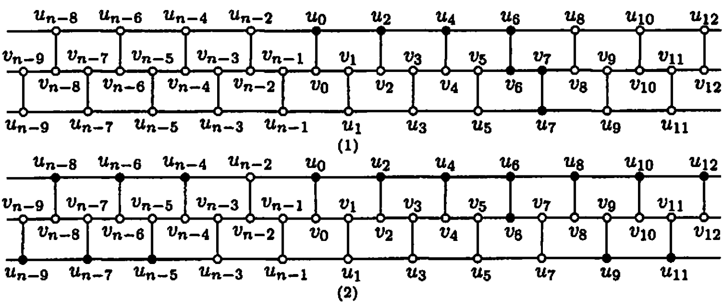

Figure 4.6.

Suppose $v_6 \in S$. Then, by Corollary 2.7, $v_4 \notin S$, $v_5 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, v_4, v_5\}$ does not belong to S, we have, at least one vertex of $\{v_7, u_8\}$ has to belong to S. If $v_7 \in S$, then by Corollary 2.7, $u_8 \notin S$ and $v_8 \notin S$. Since $P(n,2)[S]$ is a tree, we have $u_7 \in S$. By Corollary 2.7, $u_5 \notin S$. By Lemma 2.5, $v_3 \notin S$ and $u_3 \notin S$. Thus, v_3 would not be dominated by S, a contradiction(see Figure 4.6 (1)). Hence u_8 has to belong to S. By Lemma 2.5, $u_5 \notin S$ and $u_7 \notin S$. Since $N[u_7] \cap S \neq \emptyset$, u_9 has to belong to S. By Corollary 2.7, $v_8 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, v_4, v_5, v_7, v_8\}$ does not belong to S, u_{10} has to belong to S. By Lemma 2.5, $v_9 \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_7 and v_9 do not belong to S, we have $u_{11} \in S$. Continue in this way, we have $u_i \in S$ for $9 \leq i \leq n-4$.

Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5. Hence $v_6 \notin S$.

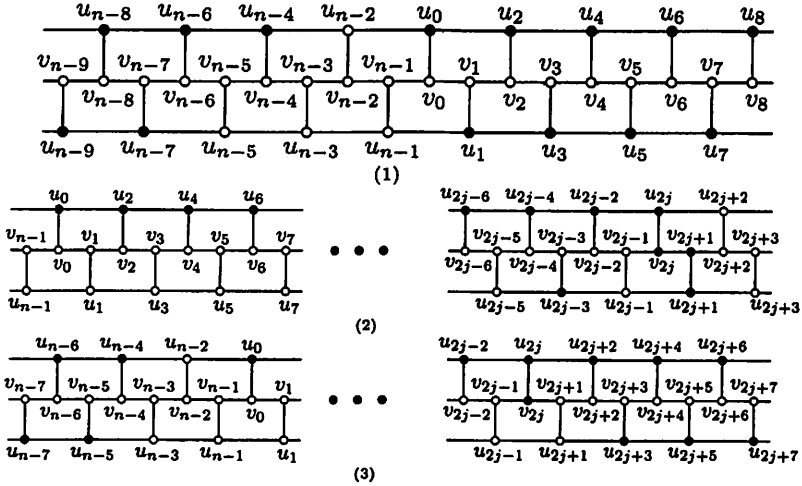

Figure 4.7.

Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, v_4, v_6\}$ does not belong to S, we have, $u_8 \in S$. If there is no $v_{2i} \in S(4 \leq i \leq (n-6)/2)$, then since $P(n,2)[S]$ is a tree, we have $u_i \in S$ for $9 \leq i \leq n-4$. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5(see Figure 4.7(1)). So, at least one vertex of $v_{2i} \in S(4 \leq i \leq (n-6)/2)$, say v_{2j}, has to belong to S. By Corollary 2.7, $v_{2j-2} \notin S$ and $v_{2j-1} \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, \cdots, v_{2j-2}, v_{2j-1}\}$ does not belong to S, we have, at least one vertex of $\{v_{2j+1}, u_{2j+2}\}$ has to belong to S. If $v_{2j+1} \in S$, then by Corollary 2.7, $u_{2j+2} \notin S$ and $v_{2j+2} \notin S$. Since $P(n,2)[S]$ is a tree, we have $u_{2j+1} \in S$. By Corollary 2.7, $u_{2j-1} \notin S$. By Lemma 2.5, $v_{2j-3} \notin S$ and $u_{2j-3} \notin S$. Thus, v_{2j-3} would not be dominated by S, a contradiction(see Figure 4.7 (2)). Hence u_{2j+2} has to belong to S. By Lemma 2.5, $u_{2j-1} \notin S$ and $u_{2j+1} \notin S$. Since $N[u_{2j+1}] \cap S \neq \emptyset$, u_{2j+3} has to belong to S. By Corollary 2.7, $v_{2j+2} \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, \cdots, v_{2j-2}, v_{2j-1}, v_{2j+1}, v_{2j+2}\}$ does not belong to S, u_{2j+4} has to belong to S. By Lemma 2.5, $v_{2j+3} \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_{2j+1} and v_{2j+3} do not belong to S, we have $u_{2j+5} \in S$. Continue in this way, we have $u_i \in S$ for $2j+2 \leq i \leq n-4$. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5. Hence $u_8 \notin S$(see Figure 4.7(3)).

Thus $u_6 \notin S$. □

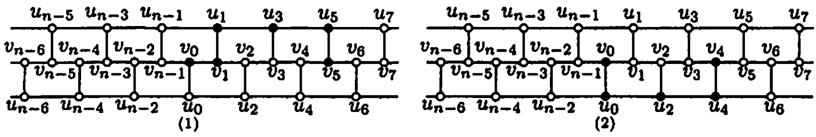

Figure 4.8.

Theorem 4.5. For even $n \geq 6$, $\gamma_{tr}(P(n,2)) = \gamma_c(P(n,2)) = n$.

Proof. By contradiction, suppose $\gamma_{tr}(P(n,2))) = n-1$. Since $P(n,2)[S]$ is a tree, at least one vertex of S is a leaf.

Case 1. Suppose that at least one vertex of $\{v_i : 0 \leq i \leq n-1\}$, say v_0, is a leaf of $P(n,2)[S]$. Then, by symmetry, we need only consider the cases for either $v_1 \in S$ or $u_0 \in S$.

Case 1.1. Suppose $v_1 \in S$. Since v_0 is a leaf of $P(n,2)[S]$, we have $v_{n-1} \notin S$ and $u_0 \notin S$. By Lemma 4.2, $v_2 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{v_{n-1}, u_0, v_2\}$ does not belong to S, we have $u_1 \in S$. By Lemma 2.5, $u_{n-1} \notin S$. Since $P(n,2)[S]$ is a tree, $u_3 \in S$. By Lemmas 4.3-4.4, $u_5 \in S$ and $u_7 \notin S$. By Corollary 2.7, $v_2 \notin S$ and $v_3 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{v_{n-1}, u_0, v_2, u_{n-1}, v_3, u_7\}$ does not belong to S, we have $v_5 \in S$. By Corollary 2.7, $v_4 \notin S$. By Lemma 2.5, $u_2 \notin S$ and $u_4 \notin S$. Thus, u_2 would not be dominated by S, a contradiction(see Figure 4.8(1)).

Case 1.2. Suppose $u_0 \in S$. Since v_0 is a leaf of $P(n,2)[S]$, we have $v_{n-1} \notin S$ and $v_1 \notin S$. Since $P(n,2)[S]$ is a tree and both vertices v_{n-1} and v_1 do not belong to S, we have at least one vertex of $\{u_{n-2}, u_2\}$ has to belong to S. By symmetry, suppose that $u_2 \in S$. By Lemmas 4.3-4.4, $u_4 \in S$ and $u_6 \notin S$. By Corollary 2.7, $v_2 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{v_{n-1}, v_1, u_{n-2}, v_2, u_6\}$ does not belong to S, we have $v_4 \in S$. By Corollary 2.7, $v_3 \notin S$. By Lemma 2.5, $u_1 \notin S$, $u_3 \notin S$ and $u_{n-1} \notin S$. Thus, u_1 would not be dominated by S, a contradiction(see Figure 4.8(2)).

Case 2. Suppose that at least one vertex of $\{u_i : 0 \leq i \leq n-1\}$, say u_0, is a leaf of $P(n,2)[S]$. Then, by symmetry, we need only consider the cases for either $u_2 \in S$ or $v_0 \in S$.

Case 2.1. Suppose $u_2 \in S$. Since u_0 is a leaf of $P(n,2)[S]$, we have $v_0 \notin S$ and $u_{n-2} \notin S$. By Lemmas 4.3-4.4, $u_4 \in S$ and $u_6 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, u_6\}$ does not belong to S, we have, at least one vertex of $\{v_2, v_4\}$ has to belong to S. If $v_2 \in S$, then by Corollary 2.7, $v_1 \notin S$, $v_3 \notin S$ and $v_4 \notin S$, a contradiction with $P(n,2)[S]$

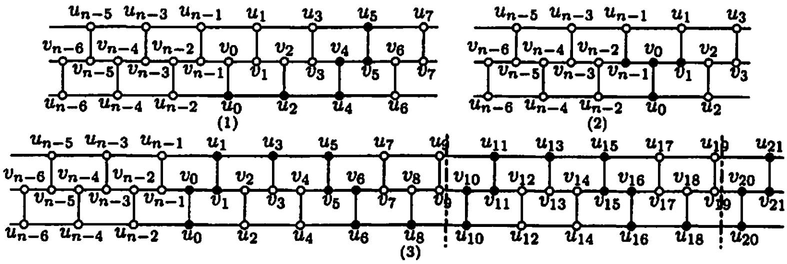

Figure 4.9.

being a tree with $|S| = n - 1$. Thus, v_4 has to belong to S. By Corollary 2.7, $v_3 \notin S$. By Lemma 4.2, $v_5 \in S$ and $v_6 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, v_0, v_2, v_3, u_6, v_6\}$ does not belong to S, we have $u_5 \in S$. By Corollary 2.7, $u_3 \notin S$. By Lemma 2.5, $u_1 \notin S$ and $v_1 \notin S$. Thus, v_1 would not be dominated by S, a contradiction(see Figure 4.9(1)).

Case 2.2. Suppose $v_0 \in S$. Since u_0 is a leaf of $P(n,2)[S]$, we have $u_2 \notin S$ and $u_{n-2} \notin S$. Since $P(n,2)[S]$ is a tree and both vertices u_2 and u_{n-2} do not belong to S, we have, at least one vertex of $\{v_{n-1}, v_1\}$ has to belong to S. By symmetry, suppose that $v_1 \in S$. If $v_{n-1} \in S$, then by Corollary 2.7, $v_{n-2} \notin S$, $u_{n-1} \notin S$, $u_1 \notin S$ and $v_2 \notin S$, a contradiction with $P(n,2)[S]$ being a tree with $|S| = n - 1$(see Figure 4.9(2)). Thus, $v_{n-1} \notin S$. By Corollary 2.7, $v_2 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, u_2, v_{n-1}, v_2\}$ does not belong to S, we have $u_1 \in S$. By Corollary 2.7, $u_{n-1} \notin S$. Since $P(n,2)[S]$ is a tree, and any one vertex of $\{u_{n-2}, u_2, v_{n-1}, v_2, u_{n-1}\}$ does not belong to S, we have $u_3 \in S$. By Corollary 2.7, $v_3 \notin S$. By Lemmas 4.3-4.4, $u_5 \in S$ and $u_7 \notin S$. Since $P(n,2)[S]$ is a tree and any one vertex of $\{u_{n-2}, u_2, v_{n-1}, v_2, v_3, u_7\}$ does not belong to S, we have $v_5 \in S$. By Corollary 2.7, $v_4 \notin S$. Since $P(n,2)[S]$ is a tree, $v_6 \in S$. By Lemma 4.2, $v_7 \notin S$. Since $P(n,2)[S]$ is a tree, $u_6 \in S$. By Corollary 2.7, $u_4 \notin S$. Since $P(n,2)[S]$ is a tree, $u_8 \in S$. By Lemmas 4.4-4.5, $u_{10} \in S$ and $u_{12} \notin S$. By Corollary 2.7, $v_8 \notin S$. Since $P(n,2)[S]$ is a tree, $v_{10} \in S$(see Figure 4.9(3)). Continue in this way, we have $\{u_{10i+0}, v_{10i+0}, v_{10i+1}, u_{10i+1}, u_{10i+3}, u_{10i+5}, v_{10i+5}, v_{10i+6}, u_{10i+6}, u_{10i+8}\} \subseteq S (0 \le i < \lfloor \frac{n}{10} \rfloor - 1)$.

Let $a = \lfloor \frac{n}{10} \rfloor$, $b = n \bmod 10$ and $S_i = \{u_{10i+0}, v_{10i+0}, v_{10i+1}, u_{10i+1}, u_{10i+3}, u_{10i+5}, v_{10i+5}, v_{10i+6}, u_{10i+6}, u_{10i+8}\} (0 \le i < a - 1)$.

Case 2.2.1. Suppose $n \bmod 10 \equiv 0$. Then, $P(n,2)[S_i]$ would be a circle, a contradiction with $P(n,2)[S]$ being a tree(See Figure 4.10(1)).

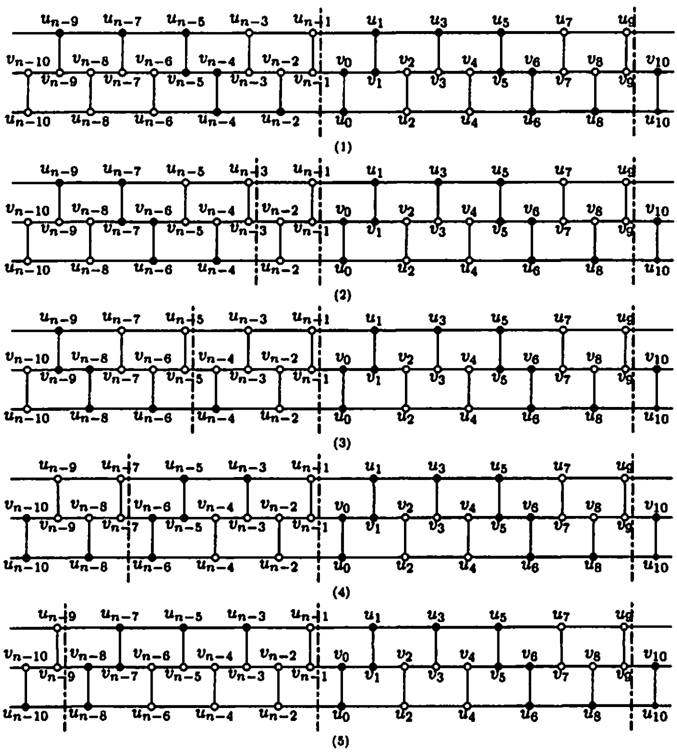

Figure 4.10.

Case 2.2.2. Suppose $n \bmod 10 \equiv 2$. Then, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5(See Figure 4.10(2)).

Case 2.2.3. Suppose $n \bmod 10 \equiv 4$. Then, by Lemma 4.3, u_{n-4} has to belong to S. Thus, u_{n-2} would be dominated by both u_{n-4} and u_0, a contradiction with Lemma 2.5(See Figure 4.10(3)).

Case 2.2.4. Suppose $n \bmod 10 \equiv 6$. Then, by Lemmas 4.3-4.4, $u_{n-6} \in S$ and $u_{n-4} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, v_{n-6} has to belong to S. By Corollary 2.7, $v_{n-7} \notin S$. By Lemma 4.2, $v_{n-5} \in S$ and $v_{n-4} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, u_{n-5} has to belong to S. By Corollary 2.7, $u_{n-7} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, u_{n-3} has to belong to S. Thus, u_{n-1} would be dominated by both u_{n-3} and u_1, a contradiction with Lemma 2.5(See Figure 4.10(4)).

Case 2.2.5. Suppose $n \bmod 10 \equiv 8$. Then, by Lemmas 4.3-4.4, $u_{n-6} \in S$ and $u_{n-4} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, v_{n-6} has to belong to S. By Corollary 2.7, $v_{n-7} \notin S$. By Lemma 4.2, $v_{n-5} \in S$ and $v_{n-4} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, u_{n-5} has to belong to S. By Corollary 2.7, $u_{n-7} \notin S$. Since $P(n,2)[S]$ is a tree with $|S| = n-1$, u_{n-3} has to belong to S. By Lemma 4.4, $u_{n-3} \in S$. Thus, u_{n-1} would be dominated by both u_{n-3} and u_1, a contradiction with Lemma 2.5(See Figure 4.10(5)).

By Cases 1-2 and Theorem 2.2, we have $\gamma_{tr}(P(n,2)) = \gamma_c(P(n,2)) = n$. □

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