On the diameter and inverse degree *

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Abstract

The inverse degree $r(G)$ of a finite graph $G = (V, E)$ is defined by $r(G) = \sum_{v\in V} \frac{1}{\deg(v)}$, where $\deg(v)$ is the degree of v in G. Erdös et al. proved that, if G is a connected graph of order n, then the diameter of G is less than $(6r(G) + o(1))\frac{\log n}{\log \log n}$. Dankelmann et al. improved this bound by a factor of approximately 2. We give the sharp upper bounds for trees and unicyclic graphs, which improves the above upper bounds.

Keywords: average distance; diameter; inverse degree; Graffiti.

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1 Introduction

Given a connected, simple and undirected graph $G = (V, E)$ of order n, let the *inverse degree* $r(G)$ of G be defined by $r(G) = \sum_{v\in V} \frac{1}{\deg(v)}$, where $\deg(v)$ is the degree of v in G. The distance between two vertices u and v in G, denoted by $d_G(u, v)$ (or $d(u, v)$ for short), is the length of a shortest path joining u and v in G. The diameter $diam(G)$ of G is the maximum distance $d(u, v)$ over all pairs of vertices u and v of G. The *average distance* $\mu(G)$, an interesting graph-theoretical invariant, is defined as the average value of the distances between all pairs of vertices of G, i.e.,

$$\mu(G) = \frac{\sum_{u,v\in V} d(u, v)}{\binom{n}{2}}.$$

A tree is a connected graph of order n and size $n - 1$, while a unicyclic graph is a connected graph of order n and size n.

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The concept of average distance, also called the *mean distance*, was introduced in graph theory by Doyle and Graver [5] as a measure of the "compactness" of a graph. It has already been used in architecture [13] as a tool for the evaluation of floor plans. Since then it has arisen also in the study of molecular structure (see, e.g., [18]), inter-computer connections [14] and telecommunications networks [17]. In a network model, the time delay or signal disgradation for sending a message from one point to another is often proportional to the number of edges a message must travel. The average distance can be used to indicate the average performance of a network, whereas the diameter is related to the worst-case performance.

Graffiti is a program designed to make conjectures about, but not limited to mathematics, in particular graph theory, which was written by Fajtlowicz from the mid-1980's. A numbered, annotated listing of several hundred of Graffiti's conjectures can be found in [10]. Graffiti has correctly conjectured a number of new bounds for several well studied graph invariants. A number of these bounds involve the average distance. For example, the inequality $\mu(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G, which was proved by Chung [2] and improved by Dankelmann [3]. A Graffiti conjecture involving two distance parameters, $rad(G) \leq \mu(G) + r(G)$, was disproved by Dankelmann et al. [6], where $rad(G)$ denotes the radius of G. See [1, 4, 15] for other problems and results.

There is a Graffiti conjecture $\mu(G) \leq r(G)$ (see [9, 12]). However, the conjecture was refuted by Erdös, Pach and Spencer in [8]. They proved that, if G is a connected graph of order n and $r(G) \geq 3$, then

$$\left(\frac{2}{3}\lfloor r/3 \rfloor + o(1)\right)\frac{\log n}{\log\log n} \leq \mu(n,r) \leq diam(n,r) \leq (6r + o(1))\frac{\log n}{\log\log n}$$

where $\mu(n,r) = \max\{\mu(G) : r(G) \leq r\}$ and $diam(n,r) = \max\{diam(G) : r(G) \leq r\}$. Dankelmann et al. [7] improved the the upper bound by a factor of 2,

$$diam(G) \leq (3r(G) + 2 + o(1))\frac{\log n}{\log\log n},$$

which is also an upper bound on the average distance since $\mu(G) \leq diam(G)$.

In this paper, we give sharp upper bounds for trees and unicyclic graphs. We show that for a tree T of order n

$$diam(T) \leq \frac{3n - 2r(T) + 1 - \sqrt{4r(T)^2 - (4n-4)r(T) + n^2 - 2n - 7}}{2},$$

while for a unicyclic graph G of order n

$$diam(G) \leq \frac{3n - 2r(G) - 1 - \sqrt{4r(G)^2 - (4n-12)r(G) + n^2 - 6n + 1}}{2}.$$

We also prove that the two upper bounds are sharp.

2 Main results

Theorem 1 *Let T be a tree of order n. Then*

$$diam(T) \le \frac{3n - 2r(T) + 1 - \sqrt{4r(T)^2 - (4n-4)r(T) + n^2 - 2n - 7}}{2},$$

with equality if and only if $T \cong T(n, diam(T))$, where $T(n, diam(T))$ is a tree having a unique vertex with maximum degree $n + 1 - diam(T)$ and all other vertices with degree one or two.

Proof. Let T be a tree of order n and $diam(T) = d$. If $d = n - 1$, then $T \cong P_n$ and $r(P_n) = \frac{n+2}{2}$. It is easy to check that

$$diam(P_n) = \frac{3n - 2r(P_n) + 1 - \sqrt{4r(P_n)^2 - (4n-4)r(P_n) + n^2 - 2n - 7}}{2}.$$

In the following we suppose $T \ncong P_n$. Denote by $T(n, d)$ a tree having a unique vertex with maximum degree $n + 1 - d$ and all other vertices with degree one or two. If $P = u_0 u_1 \ldots u_d$ is a longest path of T, then the vertices u_0 and u_d must be leaves. Note that there are at most $n - d + 1$ leaves since $diam(T) = d$. Suppose the number of leaves in T is k ($k \ge 3$). First, we will show that

$$r(T) \le k + \frac{1}{k} + \frac{n-k-1}{2}$$

with equality if and only if $T \cong T(n, n-k+1)$.

We apply induction on n. It is easy to check that the assertion holds for smaller n. Suppose it holds for $n - 1$. It is well-known that every tree has at least two leaves. Let v be a leaf of T and u be the unique neighbor of v.

If $deg(u) = 2$, $T - v$ is a tree having $n - 1$ vertices and k leaves. Then we have

$$\begin{aligned} r(T) &= \frac{1}{2} + r(T-v) \\ &\le \frac{1}{2} + k + \frac{1}{k} + \frac{n-1-k-1}{2} = k + \frac{1}{k} + \frac{n-k-1}{2}. \end{aligned}$$

Equality holds if and only if $T - v \cong T(n-1, n-1-k-1)$, i.e., $T \cong T(n, n-k-1)$.

If $deg(u) = 3$, $T - v$ is a tree having $n - 1$ vertices and $k - 1$ leaves. Then for $k \ge 3$ we have

$$\begin{aligned} r(T) &= 1 + \frac{1}{3} - \frac{1}{2} + r(T-v) \\ &\le \frac{5}{6} + k - 1 + \frac{1}{k-1} + \frac{n-k-1}{2} \le k + \frac{1}{k} + \frac{n-k-1}{2}, \end{aligned}$$

where equality holds throughout if and only if $T - v \cong T(n-1, n-k-1)$ and $k = 3$. That is to say, in this case, equality holds throughout if and only if $T \cong T(n, n-k-1)$ and $k = 3$. For $k > 3$, $r(T) < r(T(n, n-k-1)) = k + \frac{1}{k} + \frac{n-k-1}{2}$.

If $deg(u) \geq 4$, we have $k \geq 4$ and $T - v$ is a tree having $n-1$ vertices and $k-1$ leaves. Then for $k \geq 4$ we have

$$\begin{aligned} r(T) &= 1 + \frac{1}{deg(u)} - \frac{1}{deg(u)-1} + r(T-v) \\ &\leq 1 - \frac{1}{deg(u)(deg(u)-1)} + k - 1 + \frac{1}{k-1} + \frac{n-k-1}{2} \\ &\leq k + \frac{1}{k} + \frac{n-k-1}{2}, \end{aligned}$$

where equality holds throughout if and only if $T - v \cong T(n-1, n-k-1)$ and $k = deg(u)$. It is easy to check that equality holds throughout if and only if $T \cong T(n, n-k-1)$ in this case.

Considering all the above cases, we have proved the assertion above. Now we will prove the theorem. Notice that $k + \frac{1}{k} + \frac{n-k-1}{2}$ is a strictly increasing function for $k \geq 3$. Thus for a tree with $diam(T) = d$, we have

$$\begin{aligned} r(T) &\leq k + \frac{1}{k} + \frac{n-k-1}{2} \\ &\leq n - d + 1 + \frac{1}{n-d+1} + \frac{d-2}{2}. \end{aligned}$$

Now multiplying $2(n-d+1)$ to the two sides of the above inequality, we obtain

$$2(n-d+1)r(T) \leq 2(n-d+1)^2 + 2 + (n-d+1)(d-2).$$

By some simplifications, we obtain a quadratic inequality on d,

$$d^2 - (3n - 2r(T) + 1)d + 2n^2 + 2n + 2 - (2n+2)r(T) \geq 0.$$

We solve the inequality and give the following solution since the diameter $d \leq n-1$,

$$d \leq \frac{3n - 2r(T) + 1 - \sqrt{4r(T)^2 - (4n-4)r(T) + n^2 - 2n - 7}}{2}.$$

The proof is complete. ∎

For a unicyclic graph G, by a similar method to the proof of Theorem 1, we get

$$r(G) \leq n - d - 1 + \frac{1}{n-d+1} + \frac{d}{2}.$$

Then we obtain the following theorem.

Theorem 2 *Let G be a unicyclic graph of order n. Then*

$$diam(G) \leq \frac{3n - 2r(G) - 1 - \sqrt{4r(G)^2 - (4n - 12)r(G) + n^2 - 6n + 1}}{2},$$

with equality if and only if $T \cong G(n, diam(G))$, where $G(n, diam(G))$ is a unicyclic graph having a unique vertex with maximum degree $n + 1 - diam(G)$ and all other vertices with degree one or two.

3 Comparing of the upper bounds

Our two upper bounds are better than the following one given by Dankelmann et al. [7]:

$$d \leq (3r(G) + 2 + o(1))\frac{\log n}{\log \log n}.$$

In fact, we improve the above bound by a factor of approximately $\frac{4}{3} \cdot \frac{\log n}{\log \log n}$ (note that $\frac{\log n}{\log \log n} > 1$). Before proving it, we list the following two results proved by Li and Zhao [16], Zhang and Zhang [19], respectively. Let P_n be the path with n vertices, S_n the star with n vertices, C_n the cycle with n vertices and S_n^+ the graph obtained from S_n by joining two leaves with an edge.

Theorem 3 *(Li and Zhao [16]) For a tree T of order n, the inverse degree of T satisfies that*

$$\frac{n+2}{2} \leq r(T) \leq n - 1 + \frac{1}{n-1},$$

where the left inequality is an equality if and only if $T \cong P_n$, the right inequality is an equality if and only if $T \cong S_n$.

Theorem 4 *(Zhang and Zhang [19]) For a unicyclic graph G of order n, the inverse degree of G satisfies that*

$$\frac{n}{2} \leq r(G) \leq n - 2 + \frac{1}{n-1},$$

where the left inequality is an equality if and only if $T \cong C_n$, the right inequality is an equality if and only if $T \cong S_n^+$.

At first, for a tree T of order n, we will show that the following inequality holds for $\frac{n+2}{2} \leq r(T) \leq n - 1 + \frac{1}{n-1}$:

$$\frac{3n - 2r(T) + 1 - \sqrt{4r(T)^2 - (4n - 4)r(T) + n^2 - 2n - 7}}{2} < \frac{3}{4}(3r(T) + 2).$$

By some simplifications, we can transform the above inequality into the following one:

$$6n - 13r(T) - 4 < 2\sqrt{4r(T)^2 - (4n-4)r(T) + n^2 - 2n - 7}. \qquad (1)$$

When $\frac{n+2}{2} \leq r(T) \leq n - 1 + \frac{1}{n-1}$, we have $6n - 13r(T) - 4 < 0$ and $4r(T)^2 - (4n-4)r(T) + n^2 - 2n - 7 > 0$. Thus inequality (1) holds obviously, which implies that our bound is a better one.

By similar discussions as above, for a unicyclic graph G of order n, we can prove that the following inequality holds for $\frac{n}{2} \leq r(G) \leq n - 2 + \frac{1}{n-1}$:

$$\frac{3n - 2r(G) - 1 - \sqrt{4r(G)^2 - (4n-12)r(G) + n^2 - 6n + 1}}{2} < \frac{3}{4}(3r(G) + 2),$$

which also implies that our bound is a better one.

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